Chapter 3

Hypersurface of a Special Finsler Space with metric \( L = \beta + \frac{\alpha^3 + \beta^3}{\alpha(\alpha - \beta)} \)

3.1 Introduction

We consider an n-dimensional Finsler space \( F^n = (M^n, L) \) i.e., a pair consisting of an n-dimensional differentiable manifold \( M^n \) equipped with a fundamental function \( L(x,y) \). The concept of \((\alpha, \beta)\) metric has been first introduced by M. Matsumoto, in the year 1992 as a function of \( L(\alpha, \beta) \). After M. Matsumoto ([62]) many authors ([33],[51],[55],[66],[90],[119],[115]) have studied its special forms like Rander’s metric \((\alpha + \beta)\), Kropina metric \((\alpha^2)\) and generalized Kropina metric \(\frac{\alpha^{n+1}}{\beta^m} (m \neq 0, -1)\) which contribute a lot, in the growth of Finsler geometry. The Finsler metric \( L(\alpha, \beta) \) is positively homogeneous function of first degree in two variables \( \alpha = a_{ij}(x)y^iy^j \) and \( \beta = b_i(x)y^i \), where \( \alpha \) is a Riemannian metric i.e. non-degenerate (regular) and positive definite, \( \beta \) is a 1-form on \( M^n \).

3.2 Preliminaries

Now we consider a Finsler \((\alpha, \beta)-\)metric Space \( F^n = \{M^n, L(\alpha, \beta)\} \) with the metric
\[
L(\alpha, \beta) = \beta + \frac{\alpha^3 + \beta^3}{\alpha(\alpha - \beta)}
\]
Differentiating Partially above with respect to \( \alpha \) and \( \beta \) we have,
\[
L_\alpha = \frac{\alpha^4 + \beta^4 - 2\alpha\beta(\alpha^2 + \beta^2)}{\alpha^2(\alpha - \beta)^2}, \quad L_\beta = -\frac{2\beta}{\alpha} + \frac{2\alpha^2}{(\alpha - \beta)^2},
\]
\[
L_{\alpha\alpha} = -\frac{2\beta^2}{\alpha^3} + \frac{4\beta^2}{(\alpha - \beta)^2}, \quad L_{\beta\beta} = -\frac{2}{\alpha} + \frac{4\alpha^2}{(\alpha - \beta)^3}, \quad L_{\alpha\beta} = \frac{2\beta(\alpha - \beta)^3 - 4\alpha^3\beta}{\alpha^2(\alpha - \beta)^3},
\]
where \( L_\alpha = \frac{\partial L}{\partial \alpha} \), \( L_\beta = \frac{\partial L}{\partial \beta} \), \( L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha} \), \( L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta} \), \( L_{\alpha\beta} = \frac{\partial L_{\alpha\beta}}{\partial \beta} \).
In the Finsler space $F^n = \{ M^n, L(\alpha, \beta) \}$ the normalized supporting element $l_i = \partial_i L$ and angular metric tensor $h_{ij}[62]$, of the Metric $L(\alpha, \beta)$ is

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1} (b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j$$

where $Y_i = a_{ij} y^j$. For the fundamental function (3.2.1) above constants are

$$p = LL_\alpha \alpha^{-1} \quad (3.2.2)$$

$$q_0 = LL_\beta, \quad q_{-1} = LL_\alpha \alpha^{-1}$$

$$q_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1})$$

The fundamental metric tensor $g_{ij} = \frac{1}{2} \partial_i \partial_j L^2$ and its reciprocal tensor $g^{ij}$ for $L = L(\alpha, \beta)$ are given by ([62],[66])

$$g_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1} (b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j$$

where,

$$p_0 = q_0 + L^2_\beta$$

$$p_{-1} = q_{-1} + L^{-1} p L_\beta$$

$$p_{-2} = q_{-2} + p^2 L^{-2}$$

The reciprocal tensor $g^{ij}$ of $g_{ij}$ is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1} (b^i y^j + b^j y^i) - s_{-2} y^i y^j$$

where $b^i = a^{ij} b_j$ and $b^2 = a_{ij} b^i b^j$

$$s_0 = \frac{1}{\tau p} \{ pp_0 + (p_0 p_{-2} - p_{-1}^2) \alpha^2 \}, \quad (3.2.6)$$

$$s_{-1} = \frac{1}{\tau p} \{ pp_{-1} + (p_0 p_{-2} - p_{-1}^2) \beta \},$$

$$s_{-2} = \frac{1}{\tau p} \{ pp_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2 \},$$

$$\tau = p(p + p_0 b^2 + p_{-1} \beta) + (p_0 p_{-2} - p_{-1}^2) (\alpha^2 b^2 - \beta^2)$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2} \partial_k g_{ij}$ is given by [115]

$$2 p C_{ijk} = p_{-1} (h_{ij} m_k + h_{ik} m_j + h_{kj} m_i) + \gamma_1 m_i m_j m_k$$

where,

$$\gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3 p_{-1} q_0, \quad \gamma_1 = b_i - \alpha^{-2} \beta Y_i$$

where $m_i$ is a non-vanishing covariant vector orthogonal to the supporting element $y^i$. 57
Let $\{^i_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space $R^n$ and $\nabla_k$ be the covariant derivative with respect to $x^k$ relative to this christoffel symbol. Now we define,

$$2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji} \quad (3.2.9)$$

where $b_{ij} = \nabla_j b_i$.

Let $CT = (\Gamma^{\alpha}_{jk}, \Gamma^i_{0k}, \Gamma^i_{jk})$ be the cartan connection of $F^n$. The difference tensor $D^i_{jk} = \Gamma^i_{jk} - \{^i_{jk}\}$ of the special Finsler space $F^n$ is given by

$$D^i_{jk} = B^i E_{jk} + F^i B_j + F^i_{jk} B_k + B^i_{jk} b_{0k} + B^i_k b_{0j} - b_{0m} g^{jm} B^i_{jk} \quad (3.2.10)$$

where

$$B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^i_k = g^{kj} F_{ji} \quad (3.2.11)$$

$B_{ij} = \frac{1}{2} \{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}$, $B^i_k = g^{kj} B_{ji}$

$$A^m_k = B^m_{kj} E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k$$

$$\lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i$$

where '0' denote contraction with $y^i$ except for the quantities $p_0, q_0$ and $s_o$.

### 3.3 Induced Cartan Connection

If $F^{n-1}$ be a hypersurface of $F^n$ given by the equation $x^i = x^i(u^\alpha)$ (where $\alpha = 1, 2, 3... (n - 1)$). Then the element of support $y^i$ of $F^n$ is to be taken tangential to $F^{n-1}$, [67]

$$y^i = B^i_\alpha (u^\alpha) \quad (3.3.1)$$

Now, the metric tensor $g_{\alpha \beta}$ and hv-tensor $C_{\alpha \beta \gamma}$ of $F^{n-1}$ are given by

$$g_{\alpha \beta} = g_{ij} B^i_\alpha B^j_\beta, \quad C_{\alpha \beta \gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma$$

and at each point $(u^\alpha)$ of $F^{n-1}$, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u, v), y(u, v)\} B^i_\alpha N^j = 0, \quad g_{ij}\{x(u, v), y(u, v)\} N^i N^j = 1$$

Angular metric tensor $h_{\alpha \beta}$ of the hypersurface are given by

$$h_{\alpha \beta} = h_{ij} B^i_\alpha B^j_\beta, \quad h_{ij} B^i_\alpha N^j = 0, \quad h_{ij} N^i N^j = 1 \quad (3.3.2)$$

$(B^i_\alpha, N_i)$ inverse of $(B^i_\alpha, N^i)$ is given by

$$B^i_\alpha = g^{\alpha \beta} g_{ij} B^j_\beta, \quad B^i_\alpha B^j_\beta = \delta^\beta_\beta, \quad B^i_\alpha N^j = 0, \quad B^i_\alpha N_i = 0$$

$$N_i = g_{ij} N^j, \quad B^k_i = g^{kj} B_{ji}, \quad B^i_\alpha B^j_\beta + N^i N_j = \delta^i_j$$

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The induced connection $ICT = (\Gamma^\alpha_{\beta\gamma}, G^\alpha_{\beta j}, C^\alpha_{\beta j})$ of $F^{n-1}$ from the Cartan’s connection $CT = (\Gamma^i_{jk}, \Gamma^i_{0k}, C^i_{jk})$ is given by [67].

\[
\Gamma^\alpha_{\beta\gamma} = B^i_{\alpha}(B^j_{\beta\gamma} + \Gamma^i_{jk}B^j_{\beta\gamma}) + M^\alpha_{\beta}H^\gamma,
\]

\[
G^\alpha_{\beta j} = B^i_{\alpha}(B^j_{0\beta} + \Gamma^i_{0j}B^j_{\beta}), \quad C^\alpha_{\beta j} = B^i_{1}(C^i_{jk}B^j_{\beta}B^k_{\gamma})
\]

where,

\[
M_{\beta\gamma} = N_i C^i_{jk}B^j_{\beta}B^k_{\gamma}, \quad M^\alpha_{\beta} = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^i_{0j}B^j_{\beta})
\]

and,

\[
B^i_{\beta\gamma} = \frac{\partial B^i_{\beta}}{\partial x^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha
\]

The quantities $M_{\beta\gamma}$ and $H_\beta$ are called the second fundamental v-tensor and normal curvature vector respectively [67]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [67]

\[
H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^i_{jk}B^j_{\beta}B^k_{\gamma}) + M_{\beta}H^\gamma \tag{3.3.3}
\]

where,

\[
M_{\beta} = N_i C^i_{jk}B^j_{\beta}N^k \tag{3.3.4}
\]

The relative h and v-covariant derivatives of projection factor $B^i_{\alpha}$ with respect to $ICT$ are given by

\[
B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad B^i_{\alpha|\beta} = M_{\alpha\beta}N^i
\]

It is obvious form the equation (3.3) that $H_{\beta\gamma}$ is generally not symmetric and

\[
H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta}H^\gamma - M_{\gamma}H^\beta \tag{3.3.5}
\]

The above equation yield

\[
H_{0\gamma} = H_{\gamma}, \quad H_{\gamma 0} = H_{\gamma} + M_{\gamma}H_0 \tag{3.3.6}
\]

We shall use following lemmas which are due to Matsumoto [67] in the coming section.

**Lemma 3.3.1.** The normal curvature $H_\beta = H_{\beta}v^\beta$ vanishes if and only if the normal curvature vector $H_\beta$ vanishes.

**Lemma 3.3.2.** A hypersurface $F^{n-1}$ is a hyperplane of the first kind with respect to connection $CT$ if and only if $H_\alpha = 0$.

**Lemma 3.3.3.** A hypersurface $F^{n-1}$ is a hyperplane of the second kind with respect to connection $CT$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

**Lemma 3.3.4.** A hypersurface $F^{n-1}$ is a hyperplane of the third kind with respect to connection $CT$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$. 

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3.4 Hypersurface $F^{n-1}(c)$ of a special Finsler space

Let us consider a Finsler space with the metric $L = \beta + \frac{\alpha^3 + \beta^3}{\alpha (\alpha - \beta)}$, where, vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{n-1}(c)$ given by equation $b(x) = c$, a constant [115].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0$$

$$\frac{\partial b(x)}{\partial x^i} \frac{\partial x^i}{\partial u^\alpha} = 0$$

$$b_i B^i_\alpha = 0$$

Above shows that $b_i(x)$ are covariant component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad \text{and} \quad b_i y^i = 0 \quad \text{i.e.} \quad \beta = 0 \quad (3.4.1)$$

and induced metric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \quad (3.4.2)$$

which is a Riemannian metric.

Writing $\beta = 0$ in the equations (3.2.2), (3.2.3) and (3.2.5) we get

$$p = 1, \quad q_0 = 2, \quad q_{-1} = 0 \quad q_{-2} = -\alpha^{-2} \quad (3.4.3)$$

$$p_0 = 6 \quad p_{-1} = 2\alpha^{-1} \quad p_{-2} = 0 \quad \tau = 1 + 2b^2, \quad s_0 = \frac{2}{1 + 2b^2} \quad s_{-1} = \frac{2}{\alpha(1 + 2b^2)} \quad s_{-2} = \frac{4b^2}{\alpha^2(1 + 2b^2)}$$

from (3.2.5) we get,

$$g^{ij} = a^{ij} - \frac{2}{1 + 2b^2} b^i b^j - \frac{2}{\alpha(1 + 2b^2)} (b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2(1 + 2b^2)} y^i y^j \quad (3.4.4)$$

multiplying (3.4.4) by $b_i b_j$ and using (3.4.1), we get

$$g^{ij} b_i b_j = \frac{b^2}{1 + 2b^2}$$

So, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1 + 2b^2}} N_i, \quad b^2 = \alpha^{ij} b_i b_j \quad (3.4.5)$$

where, $b$ is the length of the vector $b^i$.
Again, from (3.4.4) and (3.4.5), we get
\[ b^i = a^{ij}b_j = \sqrt{b^2(1 + 2b^2)}N^i + \frac{2b_i^2}{\alpha^2}y^i \]  
(3.4.6)

Thus, we have

**Theorem 3.4.1.** In a special Finsler hypersurface \( F^{n-1}(c) \), the induced Riemannian metric is given by (3.4.2) and the scalar function \( b(x) \) is given by (3.4.5) and (3.4.6).

Now, the angular metric tensor \( h_{ij} \) and the fundamental metric tensor \( g_{ij} \) of \( F^n \) are given by
\[ h_{ij} = a_{ij} + 2b_i b_j - \frac{1}{\alpha^2} Y_i Y_j \quad \text{and} \quad g_{ij} = a_{ij} + 6b_i b_j + \frac{2}{\alpha} (b_i Y_j + b_j Y_i) \]  
(3.4.7)

From equation (3.4.1), (3.4.7) and (3.3.2) it follows that if \( h_{\alpha\beta}^{(a)} \) denote the angular metric tensor of the Riemannian \( a_{ij}(x) \) then, we have along \( F^{n-1}(c) \), \( h_{\alpha\beta} = h_{\alpha\beta}^{(a)} \).

Thus along \( F^{n-1}(c) \),
\[ \frac{\partial p_0}{\partial \beta} = \frac{24}{\alpha} \]

From equation (3.2.6), we get
\[ \gamma_1 = \frac{12}{\alpha}, \quad m_i = b_i \]

then hv-torsion tensor becomes
\[ C_{ijk} = \frac{1}{\alpha} \left( h_{ij} b_k + h_{jk} b_i + h_{ki} b_j \right) + \frac{6}{\alpha} b_i b_j b_k \]  
(3.4.8)

in the special Finsler hypersurface \( F^{n-1}(c) \). Due to fact from (3.3.2), (3.3.3), (3.3.5), (3.4.1) and (3.4.8), we have
\[ M_{\alpha\beta} = \frac{1}{\alpha} \sqrt{\frac{b_i^2}{(1 + 2b^2)}} h_{\alpha\beta} \quad \text{and} \quad M_{\alpha} = 0 \]  
(3.4.9)

Therefore from equation (3.3.6) it follows that \( H_{\alpha\beta} \) is symmetric. Thus we have

**Theorem 3.4.2.** The second fundamental v-tensor of the special Finsler hypersurface \( F^{n-1}(c) \) is given by (3.4.9) and the second fundamental h-tensor \( H_{\alpha\beta} \) is symmetric.

Now from (3.4.1) we have \( b_i B_{\alpha}^i = 0 \). Then, we have
\[ b_{i|\beta} B_{\alpha}^i + b_i B_{\alpha|i|\beta} = 0 \]

Therefore, from (3.3.5) and using \( b_{i|\beta} = b_{ij} B_{\beta}^j + b_{i|j} N^j H_{\beta} \), we have
\[ b_{i|j} B_{\alpha}^i B_{\beta}^j + b_{i|j} B_{\alpha}^i N^j H_{\beta} + b_\iota H_{\alpha\beta} N^i = 0 \]  
(3.4.10)

since \( b_{i|j} = -b_k C_{ij}^k \), we get
\[ b_{ij}B^i_\alpha N^j = 0 \]

Therefore from equation (3.4.10) we have,
\[ \sqrt{\frac{b^2}{1 + 2b^2}} H_{\alpha\beta} + b_{ij} B^i_\alpha B^j_\beta = 0 \tag{3.4.11} \]

because \( b_{ij} \) is symmetric. Now contracting (3.4.11) with \( v^\beta \) and using (3.3.1) we get
\[ \sqrt{\frac{b^2}{1 + 2b^2}} H_\alpha + b_{ij} B^i_\alpha y^j = 0 \tag{3.4.12} \]

Again contracting by \( v^\alpha \) equation (3.4.12) and using (3.3.1), we have
\[ \sqrt{\frac{b^2}{1 + 2b^2}} H_0 + b_{ij} y^i y^j = 0 \tag{3.4.13} \]

From lemma (3.3.1) and (3.3.2), it is clear that the hypersurface \( F_{n-1}^m \) is a hyperplane of first kind if and only if \( H_0 = 0 \). Thus from (3.4.13) it is obvious that \( F_{n-1}^m \) is a hyperplane of first kind if and only if \( b_{ij} y^i y^j = 0 \). This \( b_{ij} \) being the covariant derivative with respect to \( C^{\Gamma} \) of \( F_{n-1}^m \) defined on \( y^i \), but \( b_{ij} = \nabla_j b_i \) is the covariant derivative with respect to Riemannian connection \( \{i_{jk}\} \) constructed from \( a_{ij}(x) \). Hence \( b_{ij} \) does not depend on \( y^i \). We shall consider the difference \( b_{ij} - b_{ij} \) where \( b_{ij} = \nabla_j b_i \) in the following. The difference tensor \( D^i_{jk} = \Gamma^i_{jk} - \{i_{jk}\} \) is given by (3.2.10). Since \( b_{ij} \) is a gradient vector, then from (3.2.9) we have
\[ E_{ij} = b_{ij} \quad F_{ij} = 0 \quad \text{and} \quad F^i_j = 0 \]

Thus (3.2.10) reduces to
\[ D^i_{jk} = B^i_j + B^i_k b_{0j} + B^i_k b_{0j} - b_{0m} y^m B_{jk} - C^i_{jm} A^m_k - C^j_{km} A^m_s + \lambda^i (C^j_{km} C^m_s + C^i_{km} C^m_s - C^i_{jk} C^m_{ms}) \tag{3.4.14} \]

where
\[ B_i = 6b_i + 2\alpha^{-1} Y_i, \quad B^i = (\frac{2}{1 + 2b^2}) b^i + \frac{2}{\alpha(1 + b^2)} b^i \tag{3.4.15} \]
\[ \lambda^m = B^m b_{00}, \quad B_{ij} = \frac{1}{\alpha} (a_{ij} - \frac{Y_i Y_j}{\alpha^2}) + \frac{6}{\alpha} b_i b_j, \]
\[ B^i_j = \frac{1}{\alpha} \left[ \delta^i_j - 2(1 - 2b^2) \alpha^2 (1 + 2b^2)^{-1} Y_j y^i + \frac{4}{\alpha(1 + 2b^2)} b_j b^i - \frac{2(\alpha + 6b^2)}{\alpha^2 (1 + 2b^2)} b_j y^i - \frac{2}{\alpha(1 + 2b^2)} Y_j b^i \right] \]
\[ A^m_k = B^m_k b_{00} + B^m b_{k0} \]

In view of (3.4.3) and (3.4.4), the relation in (3.2.11) becomes (3.4.15) we have \( B^i_0 = 0, B_{00} = 0 \). Using above virtue in (3.4.5), we get \( A^m_0 = B^m b_{00} \).
Now contracting (3.4.14) by $y^k$, we get
\[ D^{ij}_{j0} = B^i b_{j0} + B^i b_{00} - B^m C^i_{jm} b_{00} \]
Again contracting the above equation with respect to $y^j$ we have
\[ D^i_{00} = B^i b_{00} = \left\{ \frac{2}{1 + 2b^2} b^j + \frac{2}{\alpha(1 + 2b^2)} y^j \right\} b_{00} \]

Paying attention to (3.4.1), along $F^{n-1}$, we get
\[ b^i D^{ij}_{00} = 2b^2 b^j - (1 + 2b^2) b^j b_{00} - (1 + 2b^2) b^i b^m C^{ij}_{jm} b_{00} \quad (3.4.16) \]

Now we contract (3.4.16) by $y^j$ we have
\[ b^i D^{ij}_{00} = \frac{2}{1 + 2b^2} b_{00} \quad (3.4.17) \]

From (3.3.3), (3.4.5), (3.4.6), (3.4.9) and $M = 0$, we have
\[ b^i b^m C^{ij}_{jm} b^i = b^2 M = 0 \]
Thus the relation $b_{ij} = b_{ij} - b^r D^r_{ij}$ the equation (3.4.16) and (3.4.17) gives
\[ b_{ij} y^i y^j = b_{00} - b^r D^r_{00} = \frac{1}{1 + 2b^2} b_{00} \]

Consequently (3.4.12) and (3.4.13) may be written as
\[ bH^\alpha + \frac{1}{1 + 2b^2} b_{00} B^i_{\alpha} = 0, \quad (3.4.18) \]
\[ bH_0 + \frac{1}{1 + 2b^2} b_{00} = 0 \]
Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b^i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b^i y^i b^j y^j$ for some $c_j(x)$. Thus, we can write,
\[ 2b_{ij} = b_i c_j + b_j c_i \quad (3.4.19) \]

Now from (3.4.1) and (3.4.19) we get
\[ b_{00} = 0, \quad b_{ij} B^i_{\alpha} B^j_{\beta} = 0, \quad b_{ij} B^i_{\alpha} y^j = 0 \]
Hence from (3.4.18) we get $H^\alpha = 0$, again from (3.4.19) and (3.4.15) we get $b_{00} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^j B^j_{\alpha} = 0$ and $B_{ij} B^i_{\alpha} B^j_{\beta} = \frac{1}{\alpha^2} h_{\alpha\beta}$.

Now we use equation (3.3.3), (3.3.4), (3.4.5), (3.4.6), (3.4.9) and (3.4.14) then we have
\[ b^r D^r_{ij} B^i_{\alpha} B^j_{\beta} = -\frac{c_0 b^2}{2\alpha(1 + 2b^2)} h_{\alpha\beta} \quad (3.4.20) \]
Thus the equation (3.4.11) reduces to

$$\sqrt{\frac{b^2}{(1+2b^2)}} H_{\alpha\beta} + \frac{b^2 c_0}{2\alpha(1+2b^2)} h_{\alpha\beta} = 0 \quad (3.4.21)$$

Hence the hypersurface $F^{n-1}_{(c)}$ is umbilic.

**Theorem 3.4.3.** The necessary and sufficient condition for $F^{n-1}_{(c)}$ to be a hyperplane of first kind is (3.4.19). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.

Now from lemma (3.3.3), $F^{n-1}_{(c)}$ is a hyperplane of second kind if and only if $H_{\alpha} = 0$ and $H_{\alpha\beta} = 0$. Thus from (3.4.20), we get

$$c_0 = c_i(x)y^i = 0$$

Therefore there exist a function $\psi(x)$ such that

$$c_i(x) = \psi(x)b_i(x)$$

Therefore (3.4.19) we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x)b_i b_j$$

**Theorem 3.4.4.** The necessary and sufficient condition for a hypersurface $F^{n-1}_{(c)}$ to be a hyperplane of second kind is (3.4.21).

Again lemma (3.4.4), together with (3.4.9) and $M_{\alpha} = 0$ shows that $F^{n-1}_{(c)}$ does not become a hyperplane of third kind.

**Theorem 3.4.5.** The hypersurface $F^{n-1}_{(c)}$ is not a hyperplane of the third kind.