Chapter 1

Introduction

1.1 A Glimpse of Geometry

Geometry is the branch of mathematics concerned with the properties and relation between points, lines, surfaces, solids and higher dimension analogue. It is possibly one of the oldest branches of mathematics. Initially Geometry originated in the practical science of measuring land, architecture and art.

Geometry (Greek \( \gamma\epsilon\omega\mu\epsilon\tau\rho\iota\alpha; \) geo = earth, metria = measure) arose as the field of knowledge dealing with spatial relationships. It is defined as:

“The science that treats of the shape and size of the things”.

Plato the eminent Philosopher was so much fascinated with the subject that he got following words inscribed on the main gate of his Platonic school called “Academy”,

“Let no man ignorant of Geometry can enter”.

1.1.1 Euclidean Geometry

In several ancient cultures there developed a form of geometry suited to the relationships between lengths, areas, and volumes of physical objects. This geometry was
codified in Euclid’s Elements about 300 BC on the basis of 5 axioms, or postulates, from which several hundred propositions and theorems were proved by deductive logical reasoning. In modern usage, a Euclidean space is any finite-dimensional vector space in which the distance between points is given by an extension of the usual formula for three-dimensional space. Euclid wrote a set of 13 books called the “elements” in 330-320 B.C. For more than 2000 years it is dominated all teaching of geometry.

1.1.2 Non-Euclidean geometry

In the beginning of 19th century, various mathematicians started working for alternate geometry by taking different set of postulates. Actually four postulates of Euclid common to all of them and it was same as taken by Euclid. The fifth well-known Euclid’s parallel postulate, which in its modern form, reads, given a line and a point not on the line, it is possible to drawn exactly one line through the given point parallel to the line. They tried without success that the alternatives were logically impossible. Instead, they discovered that consistent non-Euclidean geometries exist.

The fifth axiom says that a line and a point there is only one other line that can be draw through the point which is parallel to original line, but this axiom fails in non-Euclidian geometry. If we measure the shortest distance between points on earth the surface of earth is curved not flat. Technique from non-Euclidian geometry would be much more useful in this case.

1.1.3 Differential Geometry

The German mathematician Carl Friedrich Gauss (1777-1855), in connection with practical problems of surveying and geodesy, initiated the field of differential geometry. Using differential calculus, he characterized the intrinsic properties of curves and surfaces. For instance, he showed that the intrinsic curvature of a cylinder is the same as that of a plane, as can be seen by cutting a cylinder along its axis and flattening, but not the same as that of a sphere, which cannot be flattened without distortion.
These geometric concepts sometimes show a high level of abstraction and complexity. Geometry now uses methods of calculus and abstract algebra, so that many modern branches of the field are not easily recognizable as the descendants of early geometry. Riemannian geometry is the generalization of Euclidean geometry and Finsler geometry is generalization of Riemannian geometry with due restrictions. Riemannian geometry without the quadratic restrictions has been known as Finsler geometry. It is actually the geometry of a simple integral and is as old as the calculus of variation.

Riemannian geometry is not an abstract mathematical notion. Surface of a sphere is a concrete and simple illustration of Riemannian geometry. Here, by a straight line we mean the path of the shortest distance between two points on it. Such a curve on a surface is called a geodesic and it is a well known fact that the geodesics on the surface of a sphere are great circles. It is also obvious that any two great circles on a sphere meet in two points which are diametrically opposite to each other. And such conditions are not satisfied. Therefore the geometry of the surface of a sphere is Riemannian.

The earth being roughly spherical, one must use Riemannian geometry for measurements on the surface of earth. However, in our day to day surveying such as measurements of a sports field or of an agricultural piece of land or calculation of plinth area of a house we use Euclidean Geometry and not Riemannian geometry. We explain this by saying that in such surveying the portion of the earth under consideration is so small in comparison to the size of the earth that for all practical purposes we regard it as a flat region and therefore use Euclidean Geometry. Well, this is a general property of Riemannian geometry that in a region R of the space where Riemannian geometry holds, at every point we can choose a small neighborhood N, in which the Geometry becomes Euclidean. In other words Riemannian geometry is locally Euclidean. Fleni Klein gave the following names to the geometries discussed above. Unique parallel (Euclidean):
Parabolic Geometry Two parallels (Lobachevski): Hyperbolic Geometry No parallel: 
Elliptic Geometry (Riemannian or Klein) At present Riemannian geometry is regarded as generalization of Euclidean Geometry. Metric of Euclidean and Riemannian spaces are,

\[ ds^2 = \delta_{ij}dx^idx^j \quad (Euclidean) \]

where,

\[ \delta_{ij} = 1 \quad if, i = j \]
\[ \delta_{ij} = 0 \quad if, i \neq j \]

and \[ ds^2 = g_{ij}dx^idx^j \quad (Riemannian) \]

respectively.

The fact out of the several fundamentally different points of view only with regard to Finsler geometry two or more dominate. The view of H. Busemann has opened up new avenues of approach to Finsler geometry which are independent of methods of classical tensor analysis. We shall not discuss the description of Busemann approach. Although fundamental significance cannot be doubted.

It would be seen natural to further generalization of Riemannian metric \[ ds^2 = g_{ij}dx^idx^j \] that the distance \( ds \) between two neighboring points represented by the coordinate \( x^i \) and \( x^i + dx^i \), be defined by some function \( L(x^i, dx^i) \) as:

\[ ds = L(x^i, dx^i) , i=1,2,3,\ldots,n \]

Further, the function \( L \) is such that it is homogeneous of degree one in differential, convex and positive definite. Actually these three conditions have been imposed on our natural and native feeling about the distance \( ds \).

A few years later however a general development of Finsler geometry turned away from the basic aspects and the method of the theory as developed by Finsler. The later
development did not use the tensor calculus but the notion of calculus of variation. In the year 1925 the method of tensor calculus was applied to the theory independently but almost simultaneously by Synge, Taylor and Berwald. While the covariant derivative as introduced by Synge and Taylor coincides, the theory of Berwald shows a marked difference in the sense that in his geometry the lemma of Ricci is no longer valid.

Again the theory took a new and unexpected turn in 1934 when E. Cartan published his paper on Finsler spaces. He showed that it was indeed possible to defined connection coefficient and hence a covariant derivative such that Ricci’s lemma holds well. On this basis Cartan developed a theory of curvature and practically all subsequent investigation concerning the geometry of Finsler spaces where dominated by this approach. The new term facilitates the introduction of what Cartan calls the ‘Euclidian collection which forms a (2n-1) dimensional verity. The Cartan’s approach also depends on the introduction of one more term element of support, namely that at each point of Finsler space has a direction as well as position.

Later on it was felt that the introduction of an element of support was undesirable. From a geometrical point of view, while the natural link of the calculus of variation was weakened. This view was expressed independently by several authors, in particular by Wagner, Bussemann and H. Rund also. The rejection of the use of element of support, however desirable from a geometrical point of view, let to new difficulties, for instance, the natural orthogonality between two vectors is not in general symmetric while analytical difficulties was also enhanced, since the Ricci lemma can not be generalized as before.

1.1.4 Brief sketch of Finsler geometry

Finsler geometry as appears from the name that it has been originated by “P.Finsler” a French Mathematician in the year 1918[24]. A century has elapsed since 1918 this geometry does not depends such a foundation as Riemannian geometry and
it is indeed geometry of too complicated character.

In order to better understand we consider a history of Finsler space in dividing it into the following five periods:-

1854-1927 :

As appears from the name, Finsler geometry has started with Finsler’s famous dissertation submitted in 1918 but fact is otherwise. Geometricians believe that the creator of this geometry is L. Berwald 1925. The general believe, we guess, comes from the name, “Finsler geometry” which was first given by J. H. Taylor 1927.

It had formally been called “The theory of generalized metric spaces”. Finsler is only a proposer and had made little contribution expect the dissertation. We can not neglect B. Riemann. In his famous lecture (delivered on 18th June 1854) he has referred such generalized metric. Consequently, we will give the brief sketch about his famous lecture, whose title is given below:

“ On the hypothesis, which are basis of geometry ”

In the lecture, B. Riemann introduce the concept of n-dimensional differential manifold and to propose the study of manifold equipped with a, so-called Riemannian metric

\[ ds^2 = g_{ij}(x)dx^i dx^j \quad i, j = 1, 2, 3, ..., n \]

Before arriving on above metric he has also considered the metric

\[ ds^3 = g_{ijk}(x)dx^i dx^j dx^k \quad \text{and} \]

\[ ds^4 = g_{ijkl}(x)dx^i dx^j dx^k dx^l \]

where \( ds \) is the distance between two neighboring points \( x \) and \( x + dx \) in the manifold.

After discussing such generalized metric we can discuss it in a way similar to Riemann metrics but the computation for it will be very complicated and further it will be very difficult to give suitable geometrical meaning to give various differential
invariant. Consequently, he concludes that the theory of such generalized metrics will hardly contributes to the progress of geometry.

After Riemann lecture the geometry of generalized metric spaces had been forgotten for more than seventy years by geometricians, but there were several researcher who were concerned with the generalized metric spaces as G. M. Bliss, A. L. Underhill, G. Landsberg and so on. The title of the thesis is ‘About curves and sub-spaces in general spaces’.

It appears from the fact P. Finsler has not seen Riemann lecture, in which he has remark that the study of generalize geometry which is not contribute to any other.

The study of P. Finsler arose from his teacher supervisor Caratheodory’s idea to geometrize the variation calculus. All his content of his thesis consist of generalization of classical results in the theory of curves and surfaces in an ordinary space. He wrote in preface of his thesis:-

“(English translation) A treatment of geometry under the most general possible assumptions has a metric to let us know to what extent every theorem depends on special assumption of the merit or the number of dimension.”

Finsler has already introduce a metric tensor

\[ g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j} \]

and the C-tensor

\[ C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} \]

which are well known basic tensor of geometry. Which has been already introduced by Finsler in his dissertation. Actually, Finsler has confined himself to only those tensors which are zero in Riemannian geometry. Finsler had been a professor of the mathematical logic in Zurich University. He was a philosopher and relegationist rather than mathematician.

(2) 1927-1934 :
This period of differential geometry dominated by L. Berwald a professor of German University PRAGUE, till 1939. He was regarded as “non-Aryan” on the 26th of October, 1941. He was deported to outside Germany in the second world war. Since he did not return after the end of the war, people suppose that he is not alive any longer. These two papers on Finsler geometry which were fortunately brought out to U.S.A by Dr. H. Loewig, his pupil, and published in “Ann. of Math.” in 1941 and 1947. Theory of connection in Finsler space has been introduced by L. Berwald, was completely published in the above paper. The idea was an extremal curve of variation problem is given by

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x, \frac{dx^j}{ds} \frac{dx^k}{ds}) = 0$$

where $\gamma^i_{jk}(x, y)$ or the Christoffel symbol constructed from the fundamental tensor $g_{ij}(x, y)$ w.r. to $x^i$:

$$\gamma^j_{jk} = \frac{1}{2} g^{ir}(\frac{\partial g_{jr}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r})$$

(3) 1934-1952:

In the year 1935 E. Cartan comes out with definite establishment of Finsler geometry. Although various unsatisfactory point were left behind. In a short notes Cartan announced his system of axioms to decide a connection in a Finsler space. It was completely published in his famous monograph in the year 1936. “The spaces of Finsler News” Paris, 1934.2nd.

Consider a curve $C$, $x^i = x^i(t)$ with a parameter $t$ of an $n$-dimensional differential manifold $M$. Then we have a system of differential equation

$$\frac{dp}{dt} = \frac{dx^i}{dt} e_i \quad \frac{de_1}{dt} = \Gamma^k_{ij}(x(t)) \frac{dx^j}{dt} e_k$$

where $\Gamma^k_{ij}(x)$ functions a priori given in every co-ordinate neighborhood of $M$.

Furthermore, if a manifold $M$ with the affine connection $\Gamma = \{\Gamma^i_{jk}(x)\}$ is equipped with a Riemannian metric $ds^2 = g_{ij}(x)dx^i dx^j$ and the frame field $\{e_i(t)\}$ satisfies the equation $g_{ij} = e_i e_j$ along any curve, then $\{\Gamma^k_{ij}(x)\}$ is called an Euclidean connection. The condition for $\{\Gamma^k_{ij}(x)\}$ to be Cartesian is given by:-
\[ \frac{\partial g_{ij}}{\partial x^k} = T_{ijk} + \Gamma_{jik}, \quad \Gamma_{ijk} = g_{ir} \Gamma^r_{ik} \]

becomes

\[ \gamma_{ijk} = g_{jr} \gamma^r_{ik} = \Gamma_{ijk} - \frac{1}{2} (T_{ijk} - T_{kij} + T_{kij}) \]

Therefore, if we assume

\[ T_{ijk} = \Gamma_{ijk} - \Gamma_{kji} = 0 \]

Now, the problem is how to generalize \( e_1, e_2, e_3 \) to a Finsler space. In order to resolve the problem, Cartan introduced all together a new concept of line element space \( \bar{m} \) of a \( n \)-dimensional Finsler space \( M \). He considered \( n \)-dimensional Finsler space \( \bar{m} \) exist of connection of \( (x^i, y^i) \) Cartan’s called the pair \( (x,y) \), a supporting element of such a quantity and \( \bar{m} \) set of all supporting element.

(4) **1952-1963**

In this period of the history of Finsler geometry began in 1952, when H.Rund [54] introduced a new process of parallelism from the standpoint of Minkowskian geometry. Rund’s study was based on the notion of the tangent space \( T_nP \) at each point \( P \) of a Finsler space (Tangent space \( T_nP \) at \( P \) was defined a vector space generated by contravariant vectors emerging from \( P \)).

D.Laugwitz has published many papers in 1954, his early subject is the theory of connections and specially interested in Minkowski geometry. His pupil E.Heil mainly has studied the theory of convex bodies, but has written various interesting papers on Finsler geometry.

Moor has publishes many papers since 1950. In 1957 he treated three dimensional Finsler spaces by referring to a natural frame field which is introduced by generalizing Berwald’s idea to three dimensional case.

We must not forget a pole R.S.Ingarden work on Finsler geometry from the standpoint of application, monograph(1957), originally his thesis gives very interesting geometrical theory of problem of electron microscope. In his paper [1954] we find a really surprising theorem:-
“Any n-dimensional Finsler spaces can be isometrically imbedded in a 2n-dimensional Minkowski space”

(5) **1963-up to now**

In this period of history of development of Finsler geometry began in 1963, when H. Akbar Zahed [3] developed the modern theory of Finsler space based on the geometry of connections of fiber bundles. The reason of modernization is to establish a global distribution of connection in Finsler spaces and to re-examine the Cartan’s system of axioms, mathematician and physicist development Finsler geometry in 1970. The aim of this symposium was to find real models of Finsler spaces the contribution of Professor M. Matsumoto on the development of Finsler geometry with worth regards. He correlate his connections, Berwald connections and the Rund connections by the process called C-process an p-process. His various research papers (1992-96) on the theory of Finsler spaces with \((\alpha, \beta)\)– metric has great contribution in the development of Finsler spaces.

The study of Finsler spaces in India was started around 1963 under the leadership of Prof. R.S. Mishra, Prof. R.N. Sen, Prof. K.S. Amor, some important mathematicians in this field are as follows:-

Late Prof. M. Matsumoto, late Prof. H. Rund, late Prof. R.S. Mishra, late Prof. B.B. Sinha, Prof. R.B. Mishra, Prof. H. Shimada, Prof. P.L. Antonelli, Prof. R. Miron, Prof. C. Shibata, Prof. S. Numata, Prof. U.P. Singh, Prof. B.N. Prasad, Prof. P.N. Pandey, Prof. N. Narsinghmurty etc.

Now, we will discuss some preliminary concept of Finsler geometry which has been used in the present thesis.
1.2 The Homogeneous Lagrangian and Canonical Formalism

Homogeneous functions: A function \( f : V \rightarrow \mathbb{R} \) is said to be positively homogeneous function of degree \( r \in \mathbb{R} \) (or \( r \)-homogeneous) if

\[
f(\lambda v) = \lambda^r f(v), \quad \text{for all } v \in V, \lambda > 0.
\]

Co-ordinates: Let \( R \) be a region of \( n \)-dimensional space \( X_n \) which is covered completely by a co-ordinate system, such that any point \( P \) of \( R \) is represented by a set of \( n \)-real independent variables \( x^i (i = 1, 2, 3, \ldots, n) \), called the co-ordinates of the point. A transformation of co-ordinates is represented by a set of \( n \)-equations.

\[
\bar{x}^i = \bar{x}^i (x^1, x^2, \ldots, x^n) (i' = 1, 2, \ldots, n) \tag{1.2.1}
\]

which shows that the co-ordinates \( x^i \) of a point \( P \) of \( M_n \) are represented in the new co-ordinate system by new variables \( x^{i'} \). We assume that the functions \( x^{i'} \) of (1.3.1) are at least of class \( C^2 \) and

\[
det \left( \frac{\partial x^{i'}}{\partial x^i} \right) \neq 0 \tag{1.2.2}
\]

Curve: A set of points of \( R \), whose co-ordinates may be expressed as functions of a single parameter \( t \) is regarded as a curve of \( X_n \). Thus the equations

\[
x^i = x^i(t) \tag{1.2.3}
\]

define a curve \( C \) of \( X_n \). If the functions (1.3.3) are class \( C^1 \), we shall regard the entity whose components are given by

\[
\dot{x}^i = \frac{dx^i}{dt} = y^i \text{ (say)} \tag{1.2.4}
\]

as the tangent vector to \( C \).
Line-element: The combination \((x^i, y^i)\) of 2n elements is called the line element of the curve \(C\).

1.3 Variational problem and Euler-Lagrange’s equation

Let \(L : TM \to \mathbb{R}\) be a differentiable Lagrangian on the manifold \(M\) and \(C : t \in [0, 1] \to (x^i(t)) \in U \subset M\) be a regular curve (with a fixed parameter) having the image in a domain of local chart \(U\) of the manifold \(M\). The curve \(C\) can be extended to \(\pi^{-1}(U) \subset TM\) by

\[
C^* : t \in [0, 1] \to (x^i(t), \frac{dx^i}{dt}(t)) \in \pi^{-1}(U)
\]

Since the vector field \(\frac{dx^i}{dt}(t)\) does not vanish, the image of mapping \(C^*\) belongs to \(\overline{TM}\).

The integral of action of the functional

\[
I(C) = \int_0^1 L(x(t), \frac{dx^i}{dt}) dt
\]

The necessary condition for the action integral \(I(C)\) to be an extremal along the curve \(C(t) = x^i(t)\) is given by,

\[
E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{dt}
\]

This equation is called Euler-Lagrange’s equation and \(E_i(L)\) is called Euler Lagrange’s operator.

The Euler-Lagrange’s operator satisfied the following property:

(i) \(E_i(L)\) is differentiable covector field
(ii) \(E_i(L + L') = E_i(L) + E_i(L')\)
(iii) \(E_i(\alpha L) = \alpha E_i(L), \quad \alpha \in \mathbb{R}\)
1.4 Finsler Space and its Physical Aspects

Let the function $L(x^i, y^i)$ be defined for all the line elements in the region $\mathcal{R}$ and be a class of $C^5$ in all the $2n$ arguments. If we define the infinitesimal distance $ds$ between two points $P(x^i)$ and $Q(x^i + dx^i)$ of $\mathcal{R}$ by the relation:

$$ds = L(x^i, dx^i)$$  \hspace{1cm} (1.4.1)

then the space $\mathcal{X}_n$ equipped with the fundamental function $L$ defining the metric (1.4.1) is called a Finsler space [90], if $L(x^i, dx^i)$ satisfies the following conditions:

**Condition(A)** The function $L(x^i, y^i)$ is positively homogeneous of degree one in $y^i$ i.e.

$$L(x^i, py^i) = pL(x^i, y^i), \quad p > 0.$$ \hspace{1cm} (1.4.2)

It is the necessary and sufficient condition in order that the arc-length, $I = \int_{t_1}^{t_2} L(x^i, y^i)dt$, is independent of the choice of parameter $t$.

**Condition(B)** The function $L(x^i, y^i)$ is positive if not all $y^i$ vanish simultaneously, i.e.

$$L(x^i, y^i) > 0 \quad \text{with} \quad \Sigma_i (y^i)^2 > 0$$ \hspace{1cm} (1.4.3)

Thus, the distance between two distinct points is positive.

**Condition(C)** The quadratic form,

$$\partial_i \partial_j L^2(x^i, y^i) \xi^i \xi^j = \frac{\partial^2 L^2(x^i, y^i)}{\partial y^i \partial y^j} \xi^i \xi^j$$ \hspace{1cm} (1.4.4)

is assumed to be positive definite for all any variable $\xi^i$. That is, $L(x^i, y^i)$ is a convex function in $y^i$.

In this manner a metric is imposed on our $\mathcal{X}_n$. The space $\mathcal{X}_n$ is called a **Finsler space** if the fundamental function $L(x^i, y^i)$ defining the metric (1.4.1) satisfying the condition (A), (B) and (C).
From Euler’s theorem on homogeneous functions, we have
\[ \frac{\partial}{\partial y^i}L(x, y) y^i = L(x, y) \]  
(1.4.5)
and
\[ \frac{\partial^2}{\partial y^i \partial y^j}L(x, y) y^j = 0 \]  
(1.4.6)

The length integral \( I = \int_{t_1}^{t_2} L(x^i, y^i) dt \) give rise to the regular variational problem i.e. the Finslerian spray tensor field \( g_{ij} \) defined by:
\[ g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j} \]

Using the theory of quadratic forms and the Condition(C), we deduce from (1.4.5) that
\[ g(x, y) = |g_{ij}| > 0 \]  
(1.4.7)
for all line-elements \((x^i, y^i)\). The tensor field \( g_{ij} \) is called the fundamental tensor or metric tensor and \( g \) be it’s determinant.

If the function \( L \) is of form,
\[ L(x^i, dx^i) = \sqrt{g_{ij}(x^k)dx^i dx^j} \]  
(1.4.8)
where, the coefficients \( g_{ij}(x^k) \) are independent of \( dx^i \). The metric defined by this function is called Riemannian metric and the manifold \( M^n \) is called a Riemannian space. Throughout the present work, the n-dimensional Finsler space will be denoted by \( F^n \) or \((M^n, L)\), whereas n-dimensional Riemannian space will be denoted by \( R^n \).

In a perfectly homogeneous and isotropic medium, geometry is Euclidean, and shortest paths are straight lines. In a non-homogeneous space, geometry is Riemannian and the shortest paths are geodesics. If a medium is not only in homogeneous, but also non-isotropic , i.e. it has innate directional structure, the appropriate geometry is Finslerian [34] and the shortest paths are corresponding Finsler-geodesics. As a
consequence the fundamental metric tensor depends on both, position and direction. This is also a natural model for high angular resolution diffusion images. Finsler geometry has its genesis in integral of the form \( \int_{a}^{b} L(x, y) dt \), where \( x = (x^i), y = (y^i) = \frac{dx^i}{dt} \). Let us find out some contexts in which this integral arises.

(a). In non-isotropic medium (rays and wave fronts are not orthogonal to each other) the speed of light depends on its direction of travel. At each location \( x \), visualize \( y \) as an arrow that emanates from \( x \). We denote the time that light takes to trivial from \( x \) to the top of \( y \) call the result \( L(x, y) \). The integral \( \int_{a}^{b} L(x, y) dt \) represents total time that light takes to traverse in given path in this medium.

(b). It is well-known that the time taken by man in climbing up and going down on same length of the slope of mountain are distinct. It means time measure function \( L(x(t), y(t)) \) also depend on direction. The function \( L \) (fundamental function) together with slope of mountain \( S \) (Tangent bundle) is Finsler space.

(c). Cost of transportation function not only depends on distance but also on direction, except some other physical perturbation such as friction, air resistance. This function can be regarded as fundamental function of Finsler space.

(d). (Mathematical ecology) Suppose \( x \) stands for the state of coral reef, and \( y \) displacement vector from the state \( x \) to new state \( (x + dx, L(x, dx)) \) represents the energy one needs in order to develop from the state \( x \) to the neighboring state \( x + dx \). Hence the integral \( \int_{a}^{b} L(x, y) dt \) denotes the total energy cost of a given path of evolution.

So, from above we see that the ‘world is Finslerian’ and Finsler geometry has wide applications in theory of relativity, control theory, thermodynamics, optics, ecology and mathematical biology.

1.4.1 Tangent Space

Let us consider a change of local co-ordinates as represented by the equation (1.2.1). Along the curve (1.2.3) referred to an invariant parameter \( t \), the new compo-
The components of the tangent vector \( \bar{y}^i = \frac{d\bar{x}^i}{dt} \) are obtained by differentiating the relation

\[
\bar{x}^i = \bar{x}^i(x^i(t))
\]

with respect to \( t \), which gives

\[
\bar{y}^i = \frac{\partial \bar{x}^i}{\partial dt}
\]

Or, in terms of differentials

\[
d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^i} dx^i
\]

Here \( dx^i \) is interpreted as the components of a displacement in \( X_n \) from a point \( P(x^i) \) to a point \( Q(x^i + dx^i) \).

If the point \( P(x^i) \) is fixed, i.e., the coefficients \( \frac{\partial \bar{x}^i}{\partial x^i} \) of the transformation (1.4.11) are fixed, this relation represents a linear transformation of the \( dx^i \) onto the \( d\bar{x}^i \). The same is true for the variables \( y^i \) and \( \bar{y}^i \) in the transformation (1.4.10). Therefore, the \( n \)-entities of this kind may be taken to define the elements of an \( n \)-dimensional linear vector space. A system of \( n \)-quantities \( x^i \) whose transformation law under (1.4.9) is equivalent to that of the \( y^i \) is called a contravariant vector attached to the point \( P(x^i) \) of \( X_n \). Such contravariant vectors constitute the element of vector space. The totality of all contravariant vectors attached to \( P(x^i) \) of \( X_n \) is the tangent space denoted by \( T_n(P) \) or \( T_n(x^i) \).

### 1.4.2 Indicatrix

Let us consider the function \( L(x^i, y^i) \) defined for all line elements \( (x^i, y^i) \) over the region \( R \) of \( X_n \). The equation

\[
L(x^i, y^i) = 1, \quad (x^i \text{ fixed}, \ y^i \text{ variable})
\]

represents \((n - 1)\)-dimensional locus in \( T_n(P) \) i.e., a hypersurface. This hypersurface plays the role of unit sphere in geometry of the vector space \( T_n(P) \) and is called Indicatrix [90].
1.4.3 Geodesics

The geodesics of a Finsler space are the curves of minimum or maximum arc-length between any two points of the space. The differential equations of a geodesic in a Finsler space are given by \[32\],

\[
\frac{d^2 x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0,
\]

(1.4.12)

where \(s\) is the arc-length of the curve \(x^i = x^i(s)\) and \(2G^i = \gamma^i\frac{dx^i}{ds}\frac{dx^i}{ds}\).

1.4.4 Minkowskian space

A Finsler space \(F^n = (M^n, L(x, y))\) is called Minkowskian space if there exists a co-ordinate system \(x^i\) in which \(L\) is a function of \(y^i\) only [58].

A Finsler space is Minkowskian if and only if

a) \(C\Gamma : R^h_{ijk} = C^h_{ijk} = 0\).

b) \(R\Gamma : K^h_{ijk} = F^h_{ijk} = 0\).

c) \(B\Gamma : H^h_{ijk} = G^h_{ijk} = 0\).

Example: A Finsler space equipped with the metric \(L(y) = (y^1 y^2 \ldots y^n)^{\frac{1}{n}}\) is a Minkowskian space.

1.5 Metric Tensor

In a mathematical field of differential geometry a metric tensor is a type of function which takes as input a pair of tangent vector \(v\) and \(w\) at a point of a surface or higher dimensional differential manifold.

A metric tensor is called positive definite if it assigns a positive value \(g(u,v) > 0\) to every non-zero vector. A manifold equipped with a positive definite metric tensor is known as a Riemannian manifold. The metric tensor is an example of tensor field.
The component of a metric tensor in a coordinate basis take in the form of a symmetric matrix whose entries transform covariantly under the changes to the coordinate system. Thus a metric tensor is a covariant symmetric tensor.

Fundamental tensor $g_{ij}$ is defined as follows:

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2(x, y)}{\partial y^i \partial y^j} \tag{1.5.1}$$

Using the theory of quadratic form and the Condition (C) of the Finsler space we deduce that

$$g(x, y) = |g_{ij}(x, y)| > 0$$

It follows that inverse of matrix $g_{ij}$ exist. Thus if $g^{ij}$ denote the inverse of $g_{ij}$, then

$$g_{ij}(x, y)g^{jk}(x, y) = \delta^k_i$$

where $\delta^k_i$ is well known kronecker delta. Therefore, the tensor whose covariant and contravariant component are $g_{ij}(x, y)$ and $g^{ij}(x, y)$ is alled the metric tensor or the first fundamental tensor of the Finsler space $F^n$. Now, the torsion tensor $C_{ijk}$ is defined by

$$C_{ijk}(x, y) = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k} \tag{1.5.2}$$

is positively homogeneous of degree of -1 and is symmetric in all their indices. This tensor is called Cartan’s C - tensor and satisfies

$$(i) \ C_{ijk}(x, y)y^i = C_{ijk}(x, y)y^j = C_{ijk}(x, y)y^k = 0(ii) \ (\partial_h C_{ijk})y^i = (\partial_h C_{ijk})y^j = (\partial_h C_{ijk})y^k = 0$$

1.6 Magnitude and Orthogonality of a Vector

The metric tensor $g_{ij}(x, y)$ may be used in two different ways in defining the magnitude of a vector and also the angle between two vectors.
Let $X^i$ be a vector, then the scalar $X$ is given by

$$X^2 = g_{ij}(x, X)X^iX^j$$  \hspace{1cm} (1.6.1)$$
is called the magnitude of this vector.

If $Y^i$ is another vector, then the ratio

$$\cos(X, Y) = \frac{g_{ij}(x^i, X^j)X^iY^j}{L(x^i, X^i)L(x^j, Y^j)}$$  \hspace{1cm} (1.6.2)$$
is called the “Minkowskian Cosine” corresponding to the ordered pair of directions $X^i, Y^i$ (Rund [109]).

It is obvious from (1.6.2) that Minkowskian cosine is not symmetric in $X^i$ and $Y^i$.

Let $X^i$ be a vector and $y^i$ be an arbitrary fixed direction. The scalar $g_{ij}(x, y)X^iX^j$ is called the square of the magnitude of the vector $X^i$ for the pre assigned direction $y^i$.

If $y^j$ is another vector, then the ratio

$$\cos(X, Y) = \frac{g_{ij}(x, y)X^iY^j}{(g_{ij}(x, y)X^iX^j)^{\frac{1}{2}}(g_{ij}(x, y)Y^iY^j)^{\frac{1}{2}}}$$  \hspace{1cm} (1.6.3)$$
is called the cosine of $X^i, Y^i$ for the direction $y^i$.

It is to be noted that the concepts of magnitude of a vector and the cosine between two vectors given by (1.6.2) stands at each point of the space in a pre assigned direction $y^i$ which has been called the element of the support. Also, the cosine given by (1.6.3) is symmetric in $X^i$ and $Y^i$ (Synge[124]).

To distinguish between the two magnitudes we call the magnitude given by (1.6.1) as the Minkowskian magnitude of $X^i$ and that given by (1.6.2) the magnitude
of $X^i$.

The equations (1.6.2) and (1.6.3) are used to define the orthogonality of $F^n$. The vector $Y^i$ is said to be orthogonal with respect to $X^i$ if

$$g_{ij}(x, y)X^iX^j = 0 \quad (1.6.4)$$

Thus according to this definition of $Y^i$ is orthogonal with respect to $X^i$, then it is not necessary that $X^i$ is orthogonal to $Y^i$.

The vectors $X^i$ and $Y^i$ are said to be orthogonal (for a pre assigned $y^i$) if

$$g_{ij}(x, y)X^iY^j = 0 \quad (1.6.5)$$

This definition of orthogonality is symmetric in $X^i$ and $Y^i$.

### 1.7 Intrinsic fields of orthonormal frames

Berwald’s theory of two-dimensional Finsler space is based on the intrinsic field of orthonormal frame which consists of the normalized supporting element $l^i$ and unit vector orthonormal to $l^i$. Following this idea Moor introduced, in a three-dimensional Finsler space, the intrinsic field of orthonormal frame which consists of the normalized supporting element $l^i$, the normalized torsion vector $\frac{C^i}{C}$ and a unit vector orthogonal to them and developed a theory of three-dimensional Finsler space. Generalizing the Berwald’s and Moor’s ideas, Miron and Matsumoto developed a theory of intrinsic orthonormal frame fields on n-dimensional Finsler space as follows.

Let $L(x, y)$ be the fundamental function of an n-dimensional Finsler space and introduce Finsler tensor fields of $(0, 2\alpha - 1)$ type, $\alpha = 1, 2, \ldots, n$ by

$$L_{i_1i_2\ldots i_{(2\alpha-1)}} = \frac{1}{2\alpha} \partial^1_{i_1} \partial_2^1 \ldots \partial^1_{i_{(2\alpha-1)}} L^2$$

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Then we get a sequence of covariant vectors

\[ L_{\alpha}^i = L_{i1j2...j(2n-3)j(2n-2)} g^{(j1)} g^{(j2)} ... g^{(j2n-3)} j(2n-2) \]

**Definition 1.7.1.** If \((n - 1)\) covariant vectors \(L_{\alpha}^i, \alpha = 1, 2, \ldots, n - 1\) are linearly independent, the Finsler space is called strongly non-Riemannian.

Assume that above \(n\)-covectors \(L_{\alpha}^i\) are linearly independent and put \(e_1^i = \frac{L_1^i}{L_1} = i^i\). Here and in following we use raising and lowering of indices as \(L_{ij}^i = g^{ij} L_{1j}^1\). Further put \(N_{1ij} = g_{ij} - e_{1j} e_{1i}\), then the matrix \(N_1 = N_{1ij}\) is of rank \((n - 1)\). Second vector \(e_2^i\) is introduced by

\[ e_2^i = \frac{L_2^i}{L_2} \]

where \(L_2\) is the length of \(L_2^i\) relative to \(y^i\). Next we put \(N_{2ij} = N_{1ij} - e_{2j} e_{2i}\), \(E_{3}^i = N_{2ij} L_{3}^j\) and so third vector \(e_3^i\) is defined by

\[ e_3^i = \frac{E_{3}^i}{E_3} \]

where \(E_2\) is the length of \(E_{3}^i\) relative to \(y^i\). The repetition of above process gives a vector \(e_{r+1}\), \(r = 1, 2, \ldots, n - 1\) defined by

\[ e_{r+1}^i = \frac{(E_{r+1}^i)}{E_{r+1}} \]

where \(E_{r+1}^i = N_{r}^{i} L_{r+1}^{j} E_{r+1}\) is the length of \(E_{r+1}^i\) relative to \(y^i\) and \(N_{r}^{ij} = N_{r-1}^{ij} - e_{r+1}^i e_{2j}\).

**Definition 1.7.2.** The orthonormal frame \(\{e_\alpha\}, \alpha = 1, 2, \ldots, n\) as above defined in every co-ordinate neighborhood of a strongly non-Riemannian Finsler space is called the Miron Frame.

Consider the Miron frame \(\{e_\alpha\}\). If a tensor \(T_{j}^i\) of \((1, 1)\)-type is given then

\[ T_{j}^i = T_{\alpha \beta} e_{\alpha}^i e_{\beta}^j \]

where the scalars \(T_{\alpha \beta}\) are defined as
\[ T_{\alpha\beta} = T^i j e^i_{\alpha} e^j_{\alpha} \]

These scalars \( T_{\alpha\beta} \) are called the scalar components of \( T^i j \) with respect to Miron frame. Let \( H_{\alpha\beta\gamma} \) be scalar components of the h-covariant derivatives \( e^i_{\alpha} | j \) and \( \frac{V_{\alpha\beta\gamma}}{L} \) be scalar components of the v-covariant derivatives \( e^i_{\alpha} | j \) with respect to CT of the vector \( e^i_{\alpha} \) belonging to the Miron frame. Then

\[ e^i_{\alpha} | j = H_{\alpha\beta\gamma} e^i_{\beta} e^\gamma j \]

\[ e^i_{\alpha} | j = V_{\alpha\beta\gamma} e^i_{\beta} e^\gamma j \]

where the scalars \( H_{\alpha\beta\gamma} \) and \( V_{\alpha\beta\gamma} \) satisfying the following relations [53]:

\[ H_{1\beta\gamma} = 0H_{\alpha\beta\gamma} = -H_{\beta\alpha\gamma} \]

\[ V_{\alpha\beta\gamma} = \delta_{\beta\gamma} - \delta^1_{\beta} \delta^1_{\gamma}, V_{\alpha\beta\gamma} = -V_{\beta\alpha\gamma} \]

**Definition 1.7.3.** The scalars \( H_{\alpha\beta\gamma} \) and \( V_{\alpha\beta\gamma} \) are called connection scalars.

If \( \frac{C_{\alpha\beta\gamma}}{L} \) be the scalar components of the (h)hv-torsion tensor \( C^i_{jk} \) i.e.

\[ LC^i_{jk} = C_{\alpha\beta\gamma} e^i_{\alpha} e^\beta j e^\gamma k \]

then [66], we have

**Proposition 1.7.1.** (i) \( C_{1\beta\gamma} = 0 \)

(ii) \( C_{2\mu\mu} = LC, C_{3\mu\mu} = \ldots \ldots = C_{n\mu\mu} = 0 \)

for \( n \geq 3 \), where \( C \) is the length of \( C^i \).

Now we consider scalar components of covariant derivatives of a tensor field, for instance, \( T^i j \). Let \( T_{\alpha\beta,\gamma} \) and \( \frac{T_{\alpha\beta,\gamma}}{L} \) be the scalar components of h-and v-covariant derivatives with respect to CT respectively of a tensor \( T^i j \) i.e.,

\[ T^{i j}_{k} = T_{\alpha\beta,\gamma} e^i_{\alpha} e^\beta j e^\gamma k \quad (1.7.1) \]
and

\[ LT_j^i \mid k = T_{\alpha\beta\gamma} e^i_{\alpha} e^j_{\beta} e^k_{\gamma}, \quad (1.7.2) \]

then we have [53]

\[ T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta} e^k_{\gamma}) + T_{\mu\beta} H_{\mu\alpha\gamma} + T_{\alpha\mu} H_{\mu\beta\gamma} \quad (1.7.3) \]

and

\[ T_{\alpha\beta;\gamma} = L(\dot{\delta}_k T_{\alpha\beta} e^k_{\gamma}) + T_{\mu\beta} V_{\mu\alpha\gamma} + T_{\alpha\mu} V_{\mu\beta\gamma} \quad (1.7.4) \]

The scalar components \( T_{\alpha\beta,\gamma} \) and \( T_{\alpha\beta;\gamma} \) are called h- and v-scalar derivatives of \( T_{\alpha\beta} \) respectively.

### 1.7.1 Two-dimensional Finsler space

The Miron frame \( \{e_1, e_2\} \) is called the Berwald frame. The first vector \( e^i_1 \) is the normalized supporting element \( l^i = \frac{y^i}{L(x,y)} \) and the second vector \( e^i_2 = m^i \) is the unit vector orthogonal to \( l^i \). If \( C^i \) has non-zero length \( C \), the \( m^i = \pm \frac{C^i}{C} \). The connection scalars \( H_{\alpha\beta} \) and \( V_{\alpha\beta} \) of a two-dimensional Finsler space are such that [27] \( H_{\alpha\beta} = 0, \ V_{\alpha\beta} = \delta_{\alpha\beta} \), which implies

\[ l^i_{|j} = 0, \ m^i_{|j} = 0, \ Lm^i_{|j} = m^i m_j, \ Lm^i_{|j} = -m^i m_j \quad (1.7.5) \]

There is only one surviving scalar components of \( LC_{ijk} \) namely \( C_{222} \). If we put \( I = C_{222} \). Then

\[ LC_{ijk} = Im_im_jm_k \]

The scalar I is called the main scalar of a two-dimensional Finsler space.

**Proposition 1.7.2.** In a two-dimensional Finsler space

(i) The h-curvature tensor \( R_{hiijk} \) of CT is written as

\[ R_{hiijk} = R(l_h m_i - l_i m_h)(l_j m_k - l_k m_j) \]
(ii) The hv-curvature tensor $P_{hijk}$ of $C\Gamma$ is written as

$$P_{hijk} = I_1 (l_h m_i - l_i m_h) m_j m_k$$

(iii) The (v)hv-curvature tensor $P_{ijk}$ is written as

$$P_{ijk} = I_1 m_i m_j m_k$$

1.7.2 Main Scalar and its importance

The main scalar (I) of two-dimensional Finsler Space is given as

$$LC_{ijk} = I m_i m_j m_k$$

and main scalar of two-dimensional Finsler Space with $m^{th}$ root metric([78], equation (4.7)) is given by

$$\varepsilon I^2 = \frac{(m-2)^2 G^2}{(m-1) H^3}$$

where, H, I and G are scalars.

The main scalars H, I, J of three-dimensional Finsler space is defined by:

$$LC_{ijk} = H m_i m_j m_k - J (m_i m_j n_k + m_j m_k n_i + m_i m_j n_k) + I (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + J n_i n_j n_k$$

Since the scalars make clear the essential difference from the Riemannian structure, we have good reason to anticipate any physical meaning of those scalars.

1.7.3 Three-dimensional Finsler space

The Miron frame of a three-dimensional Finsler space is called the Moor-frame. The first vector $e_1$ of Moor-frame $e_1$, $e_2$, $e_3$ is the normalized supporting element $l^i$, the second vector $e_2$ is the normalized torsion vector $m^i = \frac{C^i}{C}$ and the third $e_3 = n^i$ is constructed by

$$n^i = \varepsilon^{ijk} e_1 j e_2 k$$
where $\epsilon^{ijk} = g^{-\frac{1}{2}} \delta_{123}^{ijk}$

Now, following two Finsler vector fields are defined [53]:

$$ h_i = h_\gamma e_{\gamma i} \quad \text{and} \quad v_i = v_\gamma e_{\gamma i} \quad (1.7.6) $$

Then the skew symmetric matrices $(H_{(\alpha)\beta\gamma})$ and $(V_{(\alpha)\beta\gamma})$, $\gamma$ being fixed, are respectively written as

$$ H_{(\alpha)\beta\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h_\gamma \\ 0 & h_\gamma & 0 \end{bmatrix}, \quad V_{(\alpha)\beta\gamma} = \begin{bmatrix} 0 & \delta_\gamma^2 & \delta_\gamma^3 \\ -\delta_\gamma^2 & 0 & v_\gamma \\ -\delta_\gamma^3 & -v_\gamma & 0 \end{bmatrix} $$

$$ \left\{ \begin{array}{l} l^i_{ij} = 0 \\ m^i_{ij} = n^i h_j \\ n^i_{ij} = -m^i h_j \end{array} \right\} $$

**Definition 1.7.4.** The Finsler vector fields $h_i$ and $v_i$ defined in (1.9.6) are called the $h$-and $v$-connection vectors of a three-dimensional Finsler space.

The (h)hv-torsion tensor of a three-dimensional Finsler space is given by [53]:

$$ LC_{ijk} = H m_i m_j m_k - J \otimes_{(ijk)} m_i m_j n_k + I \otimes_{(ijk)} m_i n_j n_k + J n_i n_j n_k \quad (1.7.8) $$

where, $\otimes_{(ijk)}$ stands for cyclic interchange of indices i,j,k and summation. The three scalar fields $H$, $I$ and $J$ of (1.9.8) are called the main scalars of a three-dimensional Finsler space and $\otimes_{(ijk)}$ represents cyclic sum of the terms obtained by cyclic permutation of i,j,k Ikeda[31] firstly solved the h-and v-connection vectors of a three-dimensional space in terms of main scalars explicitly.

### 1.7.4 Four-dimensional Finsler space
The Miron frame of a four dimensional Finsler space is \( \{e_1, e_2, e_3, e_4\} \).

The first vector of the frame is the normalized supporting element \( e_1 = \frac{1}{e} \).

The second vector of the frame \( e_2 = \frac{1}{m} \).

we take \( e_3 \) orthogonal to \( e_1 \) and \( e_2 \) and defined by \( e_3 = \frac{n}{i} \).

Now we have three linearly independent vector \( e_1, e_2, e_3 \) in \( F^4 \).

Then the vector
\[
e_4^i = \varepsilon^{ijkl} e_1^j e_2^k e_3^l, \quad \text{orthogonal to } e_\alpha^i, \quad \alpha = 1, 2, 3
\]

Let us put \( P_i = e_4^i \)

Thus \( e_\alpha^i, \quad \alpha = 1, 2, 3, 4 \) constitute an orthogonal frame.

The Finsler vector fields \( (h_j, K_j, J_j) \) and \( (U_j, V_j, W_j) \) given by \( H_{2j3}\beta = h_j, \quad H_{4j2}\beta = J_j, \quad H_{3j4}\beta = K_j \)

and \( V_{2j3}\gamma, \quad V_{2j4}\gamma = V_\gamma, \quad V_{3j4}\gamma = W_\gamma \) are called \((h, \text{res. to } V)\)-connection vector.

The scalars \( H_{2j3}\gamma, \quad H_{4j2}\gamma, \quad H_{3j4}\gamma \) are respectively are considered as the components \( h_\gamma, \quad J_\gamma, \quad K_\gamma \) of the \( h \)-connection vector with respect to the Moor frame, similarly the scalars \( V_{2j3}\gamma, \quad V_{2j4}\gamma \) respectively are considered as scalars component of \( U_\gamma, \quad V_\gamma, \quad W_\gamma \) of the \( V \)-connection vector with respect to generalise Moor frame of four dimention.

The skew symmetric matrices \( H_{\alpha j}\beta \gamma \) and \( V_{\alpha j}\beta \gamma \), \( \gamma \) being fixed are as given below

\[
H_{\alpha j}\beta \gamma =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -h_\gamma & J_\gamma \\
0 & h_\gamma & 0 & -K_\gamma \\
0 & -J_\gamma & K_\gamma & 0
\end{bmatrix},
\]

\[
V_{\alpha j}\beta \gamma =
\begin{bmatrix}
0 & \delta_2 \gamma & \delta_3 \gamma & \delta_4 \gamma \\
\delta_2 \gamma & 0 & U_\gamma & V_\gamma \\
-\delta_3 \gamma & -U_\gamma & 0 & W_\gamma \\
-\delta_4 \gamma & -V_\gamma & -W_\gamma & 0
\end{bmatrix}
\]

### 1.7.5 Tensor field \( T_{ijkl} \) and its importance

Tensor field \( T_{ijkl} \) was first introduced in 1972 by M.Matsumoto [71] and it is defined as:

\[
T_{ijkl} = LC_{ijk} | h + l_i C_{jk}h + l_j C_{ik}h + l_k C_{ij}h + l_h C_{ijk}
\]

It is evident that, in case of general dimension the above tensor \( T_{ijkl} \) is completely
symmetric. In two dimension its form will be

\[ LT_{ijklh} = I_{2} m_{i} m_{j} m_{k} m_{l} \quad (\because I_{2} = \frac{\partial I}{\partial y^{i}} m^{i}) \]

H.Kawaguchi [43] noticed the importance of \( T_{ijklh} \), when he considered a theory of transformation of Finsler spaces from stand point of Landsberg spaces. He adopt the letter ‘T’ to denote the tensor. Hasiguchi proved that \( T_{ijklh} = 0 \) is necessary and sufficient condition for a Landsberg space of general dimension to be such that it is still Landsberg under any conformal change of metric.

### 1.8 Connection and Covariant Differentiation

The Finsler connection \( F_{i}^{j} \) is a triad \((F_{i}^{j}, N_{i}^{k}, C_{j}^{i})\) of a v-connection \( F_{jk}^{i} \), a non-linear connection \( N_{i}^{j} \) and a vertical connection \( C_{jk}^{i} \) ([17],[62]). If a Finsler connection is given the h and v covariant derivative of any tensor field \( K_{ij}^{k} \) are defined by

\[ K_{ij}^{k} = \frac{\delta K_{ij}^{k}}{\delta x^{k}} + K_{jm}^{m} F_{mk}^{i} - K_{im}^{i} F_{jk}^{m} \quad (1.8.1) \]

\[ K_{ij}^{k} = \frac{\delta K_{ij}^{k}}{\delta x^{k}} + K_{jm}^{m} C_{mk}^{i} - K_{im}^{i} C_{jk}^{m} \quad (1.8.2) \]

where, \( \frac{\delta}{\delta x^{i}} = \frac{\partial}{\partial x^{i}} - N_{j}^{i} \frac{\partial}{\partial y^{j}} \)

For any Finsler connection \((F_{ij}^{i}, N_{i}^{j}, C_{jk}^{i})\) we have five torsion tensors and three curvature tensors, which are given by

I. (h) h-torsion: \( T_{ijk}^{i} = F_{ijk}^{i} - F_{kij}^{i} \)

II. (v) v-torsion: \( S_{ijk}^{i} = C_{ijk}^{i} - C_{jki}^{i} \)

III. (h) hv- torsion: \( C_{ijk}^{i} \) as the vertical connection \( C_{ijk}^{i} \)

IV. (v) h-torsion: \( R_{ijk}^{i} = \frac{\delta N_{k}^{i}}{\delta x^{i}} - \frac{\delta N_{j}^{i}}{\delta x^{j}} \)

V. (v) hv-torsion: \( P_{ijk}^{i} = \partial_{j} N_{k}^{i} - F_{kij}^{i} \)
VI. h-curvature: 
\[ R^i_{hjk} = \frac{\delta F^i_{hk}}{\delta x^j} - \frac{\delta F^i_{hj}}{\delta x^k} + F^m_{hk} F^i_{mj} - F^m_{hk} F^i_{mj} + C^i_{hm} R^i_{jk} \]

VII. hv-curvature: 
\[ P^i_{hjk} = \dot{\partial} k F^i_{hj} C^i_{hm} C^i_{hk} + C^i_{hm} F^m_{jk} \]

VIII. v-curvature 
\[ S^i_{hjk} = \dot{\partial} k C^i_{hj} - \dot{\partial} j C^i_{hk} + C^m_{hj} C^i_{mk} - C^m_{hk} C^i_{mj} \]

The deflection tensor field \( D^i_k \) of a Finsler connection \( FT \) is given by
\[ D^i_k = y^j F^i_{jk} - N^i_k \]

When a Finsler metric is given, various Finsler connections are determined from the metric. The well-known examples are Cartan’s connection, Rund’s connection, Berwald’s connection.

1.8.1 Cartan’s Connection

E. Cartan (1934) introduced a system of axioms to give uniquely a Finsler connection from the fundamental function \( L(x, y) \). His axioms are rather artificial and introduced after foreseeing the desirable results. According to M. Matsumoto (1966) the Cartan’s axioms are equivalent to the following natural elegant ones:

(i) \( g_{ij} = 0 \)  \( (\text{ii}) \) \( g_{ij} |_k = 0 \)  \( (\text{iii}) \) \( F^i_{jk} = F^i_{kj} \)  \( (\text{iv}) \) \( C^i_{jk} = C^i_{kj} \)  \( (\text{v}) \) \( D^i_k = 0 \)

Cartan’s connection[17] is denoted by \( CT = (\Gamma^i_{jk}, G^i_j, C^i_{jk}) \) and is given by
\[
\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left( \frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right)
\]
\[
G^i_j = \gamma^i_{jk} y^j - 2 C^i_{km} G^m_j
\]
\[
g^{jk} C^i_{jk} = C^i = \frac{1}{2} g^{jk} \frac{\partial g_{jk}}{\partial y^h}
\]

1.8.2 Rund’s Connection

The Rund’s connection \( (F^i_{jk}, N^i_k, C^i_{jk}) \) is uniquely determined from the fundamental function by the following axioms:

(i) \( g_{ij} |_k = 0 \)  \( (\text{ii}) \) \( F^i_{jk} = F^i_{kj} \)  \( (\text{iii}) \) \( C^i_{jk} = 0 \)  \( (\text{iv}) \) \( D^i_k = 0 \)
Thus Rund’s connection of a Finsler space \( F^n = (M^n, L) \) is a Finsler connection which is obtained from Cartan’s connection by the C-process. The C-process is characterized by expelling the torsion tensor \( C_{jk}^i \). Thus the first two connection coefficients of the Rund’s connection \( R\Gamma \) are the same with those of the Cartan’s connection \( C\Gamma \), while the third is equal to zero. Therefore the Rund’s connection \( R\Gamma \) of the Finsler space \( F^n \) is given by

\[
R\Gamma = (\Gamma_{ijk}, G_j^i, 0)
\]

1.8.3 Berwald’s connection

The Berwald’s connection \((F_{jk}^i, N_k^i, C_{jk}^i)\) is uniquely determined from the fundamental function by the following axioms:

\[
\begin{align*}
(i) \quad & L_{ji} = 0 \\
(ii) \quad & F_{jk}^i = F_{kj}^i \\
(iii) \quad & C_{jk}^i = 0 \\
(iv) \quad & D_k^i = 0 \\
(v) \quad & P_{jk}^i = \dot{\partial}_k N_j^i - F_{jk}^i = 0
\end{align*}
\]

Thus the Berwald’s connection of a Finsler space \( F^n = (M^n, L) \) is a Finsler connection which is obtained from Rund’s connection by the \( P^1 \)-process. The \( P^1 \)-process is characterized by expelling the torsion tensor \( P_{jk}^i \). The Berwald’s connection of Finsler space \( F^n \) is denoted by,

\[
B\Gamma = (G_{jk}^i, G_k^i, 0) \text{ where } G_{jk}^i = \dot{\partial}_j G_k^i
\]

1.8.4 Hashiguchi’s connection

The Hashiguchi’s connection \((F_{jk}^i, N_k^i, C_{jk}^i)\) is uniquely determined from the fundamental function by the following axioms:

\[
\begin{align*}
(i) \quad & L_{ji} = 0 \\
(ii) \quad & g_{ij} \big|_k = 0 \\
(iii) \quad & F_{jk}^i = F_{kj}^i \\
(iv) \quad & C_{jk}^i = C_{kj}^i \\
(v) \quad & D_k^i = 0 \\
(vi) \quad & P_{jk}^i = \dot{\partial}_k N_j^i - F_{jk}^i = 0
\end{align*}
\]

Thus the Hashiguchi’s connection of a Finsler space \( F^n = (M^n, L) \) is a Finsler connection which is obtained from Cartan’s connection by the \( P^1 \)-process. The Hashiguchi’s
connection of $F^n$ is denoted by $H \Gamma = (G^i_{jk}, G^i_k, C^i_{jk})$

1.9 Geodesic and paths in a Finsler space

The geodesics of a Finsler space are the curves of minimum or maximum arc-length between any two points of the space. The differential equation of a geodesic in a Finsler space is given by

$$\frac{d^2 x^i}{ds^2} + G^i(x, \frac{dx}{ds}) = 0,$$

where $s$ is the arc length of the curve $x^i = x^i(s)$ and

$$2G^i = \gamma^i_{jk}y^jy^k;$$

or

$$2G^i = g^{ir}(y^j \dot{\partial}_r \partial_j F - \partial_r F), \quad F = \frac{L^2}{2}. \quad (1.9.3)$$

Let $M^n$ be a manifold with a Finsler connection $F \Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$. A curve $C$ of the tangent bundle $T(M^n)$ over $M^n$ is called an $h$-path, if $C$ is the projection of an integral curve of an $h$-basic vector field $B^h(v)$ corresponding to a fixed $v \epsilon V^n$.

The differential equation of a path with respect to a Finsler connection $F \Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ is defined by,

$$\frac{d^2 x^i}{ds^2} + N^i_0(x, \frac{dx}{ds}) = 0. \quad (1.9.4)$$

The differential equation of an $h$-path is

$$\frac{dy^i}{dt} + N^i_j(x(t), y(t)) \frac{dx^j}{dt} = 0,$$

$$\frac{d^2 y^i}{dt^2} + F^i_{jk}(x(t), y(t)) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \quad (1.9.5)$$
1.10 Special Finsler Spaces

In Riemannian geometry we have many interesting theorems such that if a Riemannian space is assumed to have special geometrical properties, or to satisfy special tensor equations, or to admit special tensor fields, then the space reduces to one of well-known space forms, for instance, Euclidean space, spheres, topological spheres, projective spaces and so on.

On the other hand, in Finsler geometry we have special Finsler spaces, namely, Riemannian spaces and Minkowskian spaces, but there are various kinds of Riemannian spaces and Minkowskian spaces. As a consequence we have an important problem to classify all the Minkowskian spaces. It is easy to write down concrete forms of fundamental functions $L(x, y)$ which are interesting as a function, for instance, a Randers metric, Kropina metric, generalized Kropina metric, Matsumoto metric and cubic metric.

It is essential for the progress of Finsler geometry to find Finsler spaces, which are quite analogous to Riemannian spaces, but not Riemannian and Minkowskian spaces, which are analogous to flat spaces, but not flat. In the present section, we are mainly concerned with special tensor equations satisfied by torsion, curvature and other important tensors. In the following, we give some definitions of special Finsler spaces and their corresponding result.

Here, we want to discuss some special Finsler spaces which have great important in application.
1.10.1 Riemannian space

A Finsler space $F^n = (M^n, L(x, y))$ is said to be a Riemannian space, if its fundamental function $L(x, y)$ is written as

$$L(x, y) = g_{ij}(x)y^iy^j$$

Among Finsler spaces, the class of all the Riemannian spaces is characterized by $C_{ijk} = 0$ i.e. vertical connection $\Gamma^v$ of the Cartan’s connection $CT$ is flat.

1.10.2 Locally Minkowskian space

A Finsler space $F^n = (M^n, L(x, y))$ is called locally Minkoskian space, If there exit a co-ordinate system $(x^i)$ in which $L$ is a function of $y^i$ in which $L$ is a function of $y^i$ only [27].

A Finsler space is Locally Minkowskian space if and only if

a) $CT : R^h_{ijk} = C^h_{ijk} = 0$

b) $RT : K^h_{ijk} = F^h_{ijk} = 0$

c) $BT : H^h_{ijk} = G^h_{ijk} = 0$

1.10.3 Berwald space

If the connection coefficient $G^i_{jk}$ of the Berwald’s connection $BT$ given by,

$$G^i_{jk} = \hat{\partial}_j G^i_k$$

are function of position alone, the space is called a Berwald space [27].

A Finsler space is Berwald space if and only if

a) $CT : C^h_{ijk} = 0$

b) $RT : F^h_{ijk} = 0$

c) $BT : G^h_{ijk} = 0$
Example:

\[ L(x, y) = (a_i(x)y^i)^r(b_j(x)y^j)^s, \quad r + s = 1, \quad (1.10.1) \]

or, \[ L(x, y) = \alpha + \beta, \quad \alpha = (g_{ij}(x)dx^i dx^j)^{1/2}, \quad \beta = b'(x)y_i, \]

are Berwald space.

### 1.10.4 Landsberg space

A Finsler space is called a Landsberg space if the Berwald connection \( B\Gamma \) is h-metrical i.e. \( g_{ij(k)} = 0 \).

In terms of the Cartan’s connection \( C\Gamma \), a Landsberg space is characterized by

- **a)** \( P^i_{jk} = 0 \) or
- **b)** \( P^h_{ijk} = 0 \)

**Example:** A Finsler space equipped with the metrics,

\[ L(x, y) = \alpha + \beta, \quad \alpha = (g_{ij}(x)dx^i dx^j)^{1/2}, \quad \beta = b'(x)y_i \quad (1.10.2) \]

is a Landsberg space.

### 1.10.5 C-reducible Finsler space

A Finsler space of dimension \( n \), more than two, is called C-reducible if \( C_{ijk} \) is written in the form [61]:

\[ C_{ijk} = \frac{1}{(n + 1)}\Pi_{(ijk)}(h_{ij}C_k), \]

where \( C_i = C_{ijk}g^{jk} \) is the torsion vector, \( h_{ij} \) is the angular metric tensor given by \( h_{ij} = g_{ij} - l_i l_j \) and \( \Pi_{(ijk)} \) is the sum of cyclic permutation in \( i, j, k \).

**Example:** A Finsler space is C-reducible if and only if its metric function \( L \) is given by \( L = \alpha + \beta \) or \( L = \frac{\alpha^2}{\beta} \), where \( \alpha \) and \( \beta \) are given in (1.7.2).
1.10.6 Quasi C-reducible Finsler space

A Finsler space of dimension n, more than two, is called quasi C-reducible if there exists a symmetric Finsler tensor field $A_{ij}$, satisfying $A_{i0} = 0$, in terms of which $C_{ijk}$ is written in the form [61]:

$$C_{ijk} = \Pi_{(ijk)}(A_{ij}C_k)$$

1.10.7 Semi C-reducible Finsler space

A Finsler space is of dimension $n \geq 2$ is called semi C-reducible if $C_{ijk}$ is written in the form

$$C_{ijk} = \frac{p}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ik}C_j) + \frac{q}{C^2}C_iC_jC_k, \quad (1.10.3)$$

where $C^2 = g^{ij}C_iC_j$ and $p + q = 1$.

**Example**: A Finsler space of dimension $n > 2$ with $(\alpha, \beta)$ metric is semi C-reducible provided the torsion vector $C_i$ do not vanish.

1.10.8 P-reducible Finsler space

A Finsler space of dimension n, more than two, is called P-reducible if (v)hv-torsion tensor $P_{ijk}$ of $C\Gamma$ is written in the form [53],

$$P_{ijk} = \frac{1}{(n+1)}\Pi_{(ijk)}(h_{ij}C_{k0})$$

**Note**: Every C-reducible Finsler space is P-reducible.

1.10.9 C2-like Finsler space

A Finsler space is called C2-like Finsler space [53], if

$$C_{ijk} = \frac{1}{C^2}C_iC_jC_k$$
1.10.10 C3-like Finsler space

A Finsler space is called C3-like Finsler space [53], if

\[ C_{ijk} = S \circ_{ijk} (h_{ij}a_k + C_iC_jb_k) \]

where \( \circ_{ijk} \) represents the sum of cyclic permutation in i,j and k; S is a scalar function of position alone and \( a_k, b_k \) are components of arbitrary indicatory tensors.

**Example:** Every semi C-reducible Finsler space is C3-like.

1.10.11 S3-like Finsler space

A Finsler space \( F^n \) with fundamental function \( L(x, y) \) is called S3-like Finsler space [53], if v-curvature tensor \( S_{hijk} \) of CT is written in the form,

\[ L^2 S_{hijk} = S(h_{hj}h_{ik} + h_{hk}h_{ij}) \]

where S is a scalar and function of position alone.

**Example:** A Finsler space equipped with the metric \( L(y) = (y^1y^2......y^n)^\frac{1}{2} \) is a S3-like Finsler space.

1.10.12 S4-like Finsler space

A Finsler space \( F^n \) is called S4-like Finsler space [58], if v-curvature tensor \( S_{hijk} \) of CT is written in the form

\[ S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk} \]

where \( M_{ij} \) are components of a symmetric covariant tensor of second order and are \((-2)p\)-homogeneous in \( y^i \) satisfying \( M_{0j} = 0 \).

**Example:** A Finsler space of dimension \( n > 4 \) with \((\alpha, \beta)\) metric is S4-like provided the torsion vector \( C_i \) do not vanish.
1.10.13 R3-like Finsler space

A Finsler space of dimension more than three is called R3-like Finsler space [70] if h-curvature tensor $R_{hijk}$ of $C\Gamma$ is written in the forms

$$R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk}$$

where $L_{ij}$ are components of a covariant tensor of second order.

1.11 Finsler space with $(\alpha, \beta)$-metric

In the paper [34] concerned with the unified field theory of gravitation and electromagnetism Randers wrote, Perhaps the most characteristic property of the physical world is the unidirectional of time like interval. Since there is no obvious reason why this asymmetry should disappear in the mathematical description, it is of interest to consider the possibility of a metric with asymmetrical property. It is known that many reasons speak for the necessity of a quadratic induction. The only way of introducing an asymmetry while retaining the quadratic indicatrix, is to displace the center of the indicatrix. In other words we adopt as indicatrix an eccentric quadratic hypersurface. This involves the definition of a vector at each point of space determining the displacement of the center of the indicatrix. The formula for the length $ds$ of a line element $dx^i$ must necessarily be homogeneous of first degree in $dx^i$. The simplest eccentric line element possessing this property and of course being invariant is

$$ds = \sqrt{a_{ij}(x)dx^idx^j} + b_i(x)dx^i$$  \hspace{1cm} (1.11.1)

where $a_{ij}$ is the fundamental tensor of the Riemannian affine connection and $b_i$ is a covariant vector determining the displacement of the center of the indicatrix.

After sixteen years, in the monograph [34] concerned with electron microscope Ingarden wrote:
In arbitrary curvilinear co-ordinate systems the Lagrangian function of electron of electron optics may be written in the form,

\[ L(x, x') = imc\sqrt{\gamma_{ij}(x)x^i x'^j} + e_k A_i(x^i, x'^i) \]

where \( \gamma_{ij} \) is an isotropic tensor reducing in Lorentz systems to the constant unit tensor \( \delta_{ij} \). According to their physical interpretation, we shall call \( \gamma_{ij} \) the gravitational tensor and \( A_i \) the electromagnetic vector.

The special kind of Finsler space with the metric (1.8.1) we shall call a Randers space, since Randers [89] seems to have been the first to consider this kind of space, although he regarded them not as Finsler spaces but as “affinely connected Riemannian spaces” which is rather confusing notion. Randers could not use, therefore the methods of Finsler’s geometry and tried to reduce the study of (1.8.1) to a sort of 5-dimensional Kalza-Klein geometry, where Riemannian method plus a method of special projecting of tensors are used. Spaces with metric of the form (1.8.1) were also considered by Stephenson and Kilmister in 1953, but in investigations of these spaces they simply use pure Riemannian methods, which are obviously erroneous.” On the other hand, in 1959-1961 Kropina considered protectively flat Finsler spaces equipped with the metrics

\[ L(x, y) = \frac{a_{ij}(x)y^i y^j}{b_k(x)y^k} \quad (1.11.2) \]

\[ L(x, y) = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{2}} \quad (1.11.3) \]

Generalizing these special Finsler metrics of Randers type (1.8.1) and Kropina type (1.8.2), Matsumoto defined in 1972, the notion of \((\alpha, \beta)\)-metric as follows:-

**Definition 1.11.1.** A Finsler metric \( L(x, y) \) is called an \((\alpha, \beta)\)-metric, when \( L \) is positively homogeneous function \( L(\alpha, \beta) \) of first degree in two variables.

\[ \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j} \] and \( \beta(x, y) = b_i(x)y^i \)
It is usual to suppose that $\alpha$ is a Riemannian metric, i.e. non-degenerate (regular) and positive definite, but there are some cases for applications where these restrictions are relaxed. Further, we shall have to confine our discussions to suitable domain of $(x, y)$ on account of special form of the function $L(\alpha, \beta)$.

**Definition 1.11.2.** The $(\alpha, \beta)$-metric $L = \alpha + \beta$ [53] is called a Randers metric and Finsler space with this metric is called a Randers space.

In 1980 Hashiguchi and Ichijyo [28] gave the following interesting remark on Randers metric.

**Proposition 1.11.1.** A Randers metric $L = \alpha + \beta$ is positive valued, if and only if $a_{ij} - b_i b_j$ is positive-definite, provided that $a_{ij}$ is positive-definite.

**Definition 1.11.3.** The $\alpha, \beta$-metric $L = \alpha^2 \beta$ [53] is called Kropina metric and Finsler space with this metric is called a Kropina space.

Wrona has given the interesting example of Kropina metric. For a Kropina space the direction $y^i$ belonging to the hyperplane $\beta(x, y) = b_i(x) y^i = 0$ of the tangent space at any point $x$ must be obviously excluded. The indicatrix is to extend asymptotically along this hyperplane. Therefore a Kropina metric is never positive definite.

Although Kropina herself seems to have played attention to such a metric from a pure mathematical standpoint, there are close relation between this kind of metric and Lagrangian function of analytic dynamics.

**Definition 1.11.4.** The $(\alpha, \beta)$-metric $L = \alpha^{(m+1)} \beta^{(-m)} (m \neq 0, -1)$ [53] is called a generalized $m$-Kropina metric.

The Finsler metric given by (1.8.3) is called a cubic metric and was considered by Waganer (1935) and also by Kropina. It is regarded as a direct generalization of Riemannian metric in a sense. In the astronomy we measure the distance in a time, in
particular, in the light year. When we take a second as the unit, the unit surface (indicatrix) is a sphere with radius of 300,000 km. To each point of our space is associated such a sphere, this defines the distance (measured in a time) and the geometry of our space is the simplest one, namely, the Euclidean geometry. Next, when a ray of light is considered as the shortest line in the gravitational field, the geometry of our space is Riemannian geometry. Furthermore, in an isotropic medium the speed of the light depends on its direction, and the unit surface is not any longer a sphere.

Now, on the slope of the earth surface we sometimes measure the distance in a time namely, the time required such as seen on a guidepost. Then the unit curve (indicatrix) taken a minute as the unit, will be general closed curve without center, because we can walk only a shortest distance in an uphill road than in downhill road. This defines a general geometry (Finsler geometry), although it is not exact. The shortest line along which we can reach the goal, for instance, the top of a mountain as soon as possible will be a complicated curve.

The exact formulation given by Matsumoto is as follows:

**Proposition 1.11.2.** A slope, the graph of a function \( z = f(x, y) \), \([58]\) of the earth surface is regarded as a two-dimensional Finsler space with fundamental function,

\[
L(x, y, \dot{x}, \dot{y}) = \frac{\alpha^2}{v\alpha - w\beta}
\]

where \( v \) and \( w \) are non-zero constants and

\[
\alpha^2 = \dot{x}^2 + \dot{y}^2 + (\dot{x}f_x + \dot{y}f_y)^2
\]

\[
\beta = \dot{x}f_x + \dot{y}f_y
\]

This \( \alpha \) is the usual induced Riemannian metric and \( \beta \) is a derived form

\[
\beta(x, dx) = \alpha f(x, y)
\]
The two constants \( v \) and \( w \) are such that one can walk \( v \) meters per minute on the horizontal plane and \( 2w \) is equal to the acceleration of falling. Aikou, Hashiguchi and Yamauchi generalized and normalized the above metric as follows:

**Definition 1.11.5.** An \( n \)-dimensional \((\alpha, \beta)\)-metric \( L = \frac{\alpha^2}{(\alpha - \beta)} \) [2] is called a slope metric or Matsumoto metric and a Finsler space equipped with this metric is called a Matsumoto metric.

### 1.12 Finsler space with \( m^{th} \)- root metric

The \( m^{th} \)- root Finsler metric \( L(x, y) \) of an \( n \)-dimensional differential manifold \( M^n \) is first defined by H.Shimada[8] as,

\[
L^m(x, y) = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}
\]  

(1.12.1)

where the coefficient \( a_{i_1i_2...i_m}(x) \) are component of a symmetric tensor field covariant of order \( m \). Consequently the second root metric and we shall restrict \( m > 2 \).

If we take \( m=3 \) in (1.8.1) then it becomes

\[
\gamma^3 = L^3 = a_{ijk}(x)y^iy^jy^k,
\]

called cubic metric

and if we take \( m=4 \) in equation (1.8.1) then we obtained

\[
\delta^4 = L^4 = a_{ijkl}(x)y^hy^iy^jy^k,
\]

called quartic metric.

we shall sketch some fundamental part of the theory of Finsler spaces \( F^n = (M^n, L) \) with \( m^{th} \)- root metric \( L(x, y) \) for the later use.

Let us first define the tensor \( a_i(x, y), a_{ij}(x, y), a_{ijk}(x, y) \), as follows:

\[
\begin{align*}
(1) & \quad L^{m-1}a_i = a_{ij_1...j_{m-1}}y^{j_1}...y^{j_{m-1}} \\
(2) & \quad L^{m-2}a_{ij} = a_{ijk_1...k_{m-2}}y^{k_1}...y^{k_{m-2}} \\
(3) & \quad L^{m-3}a_{ijk} = a_{ijkl_1...l_{m-3}}y^{l_1}...y^{l_{m-3}}
\end{align*}
\]  

(1.12.2)
Then the normalized supporting element \( l_i = \frac{\partial L}{\partial y^i} \)

The angular metric tensor \( h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j} \)

The fundamental tensor \( g_{ij} = \dot{\partial}_i \dot{\partial}_j \left( \frac{L^2}{2} \right) \) and the C-tensor \( C_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k \left( \frac{L^2}{4} \right) \) of \( F^n \) is written as

\[
\begin{align*}
  l_i &= a_i, \quad h_{ij} = (m-1)(a_{ij} - a_ia_j), \quad (1.12.3) \\
  g_{ij} &= (m-1)a_{ij} - (m-2)a_ia_j, \\
  2LC_{ijk} &= (m-1)(m-2)(a_{ijk} - a_{ij}a_k - a_{jk}a_i - a_{ki}a_j + 2a_ia_ja_k).
\end{align*}
\]

### 1.13 Finsler space with \((\gamma, \beta)\)-metric

The concept of \((\gamma, \beta)\)-metric is given by Pandey and Chaubey [100], in the year 2011 on the line of \((\alpha, \beta)\)-metric. They obtained various important results for this Finsler spaces in their chapters. The definition given by them runs as given below:

A Finsler metric \( L(x, y) \) is called a \((\gamma, \beta)\) metric, when \( L \) is positively homogeneous function \( L(\gamma, \beta) \) of first degree in two variables \( \gamma \) and \( \beta \) where \( \gamma^3 = a_{ijk}(x)y^iy^jy^k \) is cubic metric and \( \beta = b_i(x)y^i \) is a one form metric.

### 1.14 Finsler space with \((\delta, \beta)\)-metric

In an \( n \)-dimensional Finsler space, a Finsler metric is defined as a \((\delta, \beta)\)- metric, when it is positively homogeneous function \( L(\delta, \beta) \) of degree one in two variables \( \delta \) and \( \beta \) as

\[
L \equiv L(\delta, \beta) \quad (1.14.1)
\]

where, \( \delta = \left\{ a_{ijkl}(x)y^iy^jy^ky^l \right\}^{\frac{1}{4}} \) is quartic metric and \( a_{ijkl}(x) \) are components of a symmetric tensor field \((0,4)\)- type depending on the position \( x \) alone and \( \beta = b_i(x)y^i \) is one form metric.