Chapter-IV

Magnetic field dependence of the penetration depth in superconductors

The penetration depth for a superconductor is calculated. For large viscosity it depends on the square root of the field. When the viscosity is zero, it becomes linear in magnetic field. The magnetic field dependence for a viscous medium agrees with that measured in ErNi$_2$B$_2$C and that for a nonviscous medium agrees with that in NbSe$_2$ and YBa$_2$Cu$_3$O$_7$. When quantized phase of the Josephson current is considered, the penetration depth shows oscillations as a function of magnetic field. Such an oscillatory dependence is indicated in the experimental measurements of muon spin rotation of CeRu$_2$. The Josephson voltage depends linearly on the vortex velocity which becomes complex so that there is a possibility to switch from one velocity to another. Voltage measurements along the c-axis in Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$ appear to jump from one velocity to another just as found by theoretical calculations.
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Introduction

According to the BCS theory [1] the diamagnetic current is determined by the vector potential of the electromagnetic wave, so that the superfluid density depends on the wave vectors. The summation over the wave vector can be changed to integration with the use of density of states. Therefore, the density of states plays an important role in determining the magnetic field dependence of the superfluid density. Since the diamagnetic current,

\[
J = -\frac{e}{4\pi\lambda_L^2} A
\]  

(1)
can be written as

\[
J = -\frac{e^2}{mc} \sum_{k,q,\sigma} c_{k+q,\sigma}^\dagger c_{k,\sigma} e^{-iq.x} A(x)
\]  

(2)
we can obtain the London penetration depth from the superfluid density,

\[
\frac{1}{\lambda_L^2} = \frac{4\pi e^2}{mc^2} \sum_{k,q,\sigma} c_{k+q,\sigma}^\dagger c_{k,\sigma} e^{-iq.x}
\]  

(3)

Here \( J \) is the diamagnetic current. \( A \) is the vector potential of the electromagnetic wave and \( \lambda_L \) is the London penetration depth. The charge of the electron is \( e \) and \( m \) is the mass of the electron. The velocity of light is \( c \). The creation operator for an electron of wave vector \( k \) and spin \( \sigma \) is \( c_{k,\sigma}^\dagger \). The hermitian conjugate of which is the annihilation operator \( c_{k,\sigma} \). We multiply the right hand side of the above result by density of states to replace the summation \( \sum_k \) to integration \( \int \cdots dk \). Won and Maki [2] have shown that the density of states for a ground state of \( d \)-wave symmetry varies with the square root of the magnetic induction,

\[
\frac{N(B,0)}{N(0)} \propto \left( \frac{B}{Hc^2} \right)^{1/2}
\]  

(4)
so that we predict, \( \lambda_{L,d} \propto B^{-1/4} \). In the case of \( s \)-wave symmetry we expect that the density of states varies linearly with magnetic field. Therefore, the London penetration depth varies as the inverse square root of the magnetic field, \( \lambda_{L,s} \propto B^{-1/2} \). The operator
part of the integral, $c_{k+q,o}^+ c_{k,o}$ gives rise to a factor which depends on the Fermi distribution, $(1 - f_{k+q,o}) f_{k,o}$, and hence involves magnetic field through the single-particle energy. At low temperatures, the effect of this factor on the magnetic field dependence of the London penetration depth is quite small. Actually none of the experimental measurements agree with such a dependence on magnetic field. Therefore, it is clear that some more elementary approach is needed than is described by the BCS theory with diamagnetic current and suitable density of states.

Yip and Sauls [3] have suggested that the pair breaking effect of the magnetic field reduces the current so that the penetration depth as a function of field and temperature is given by,

$$\frac{1}{\lambda(T, H)} = \frac{1}{\lambda(T)} \left[ 1 - \frac{1}{3} \frac{H^2}{H_o^2} \right]$$

which diverges at $H = H_o (3/\alpha)^{1/2}$ where $H_o = \frac{3}{4} cv_c/e\lambda(T)$. Here $c$ is the velocity of light and $v_c$ is the critical velocity of the vortex. Apparently such a divergence has not been found in experimental measurements. In the case of anisotropic Fermi surface due to $d$-wave symmetry of the gap, the effective penetration depth is shown [3] to become,

$$\frac{1}{\lambda(T, H)} = \frac{1}{\lambda(T)} \left( 1 - \frac{2}{3} \frac{H}{H_o} \right)$$

which diverges at $H = 3H_o/2$. This divergence is also not found experimentally. It will be quite reasonable if such a divergence occurred at $H_{c1}$ or $H_{c2}$ which are the lower and the upper critical fields of the superconductor. Franz et al [4] have considered the effect of the gap on the penetration depth and have also found the vortices as a function of field but the value of $\lambda(T, H)$ was not evaluated analytically. Similarly, Affleck et al [5] have considered the vortex lattice structure using a generalized London free energy. Although, the $s$ and the $d$ wave amplitudes affected the value of the penetration depth, the magnetic field did not have a pronounced effect. It is found by Kosztin and Leggett [6] that the
penetration depth of a pure $d$-wave superconductor is proportional to $T^2$ due to nonlocal electrodynamics. An effort has been made by Coffey and Clem [7] to understand the temperature and field dependence of the penetration depth using vortex viscosity and the earlier result of Campbell [8] which used vortex oscillations.

In this chapter, we show that under certain conditions the London penetration depth depends on the square root of the applied magnetic field which becomes linear upon expansion. The tunneling along the $c$-axis gives rise to two resolvable components of the velocity of the vortex. The phase factor gives rise to oscillations in the measurement of the London penetration depth as a function of magnetic field. We find that the theoretical predictions are in accord with the experimental measurements in ErNi$_2$B$_2$C, NbSe$_2$, YBa$_2$Cu$_3$O$_{7-\delta}$, Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$, and CeRu$_2$.

Vortex velocity

There are vortices in the type-II superconductors. These vortices oscillate and have viscous forces [9]. The mass of the ordinary fluxoid is $M$ so that it is subject to a force $M \frac{dv}{dt}$. The forces are balanced by the current $J$, so that,

$$M \frac{dv}{dt} + \eta v + kx = \frac{1}{c} J \phi_0$$

(7)

where $\eta$ is the viscosity and $k$ is the force constant. The vortex velocity is dependent on time, $v = v_0 e^{-iw t}$ which we substitute in (7) and solve to find,

$$v = \frac{J \phi_0}{c(\eta - i\omega M + (ik/\omega))}$$

(8)

The vortex moving with velocity $v$ in a magnetic field $B$ produces the electric field,

$$E_x = -\frac{1}{c} v B$$

(9)
The London penetration depth is defined by the relation,

\[ \frac{dJ}{dt} = \frac{c^2}{4\pi \lambda_L^2} (E + E_\omega) \]  

where \( E + \nabla \varphi = -\partial A/\partial t \). Substituting (8) in (9) and the resulting relation into (10) and taking \( J = J_0 e^{-i\omega t} \), we find

\[ J = E \left[ \phi_0 B \frac{\phi_0 B}{c^2 \{ \eta - i\omega M + (ik/\omega) \} - \frac{4\pi i\omega \lambda_L^2}{c^2}} \right]^{-1} \]  

which can be used to define the resistivity or the conductivity, \( J/E = 1/\rho \) or

\[ \rho = \frac{\phi_0 B}{c^2 \{ \eta - i\omega M + (ik/\omega) \} - \frac{4\pi i\omega \lambda_L^2}{c^2}} \]  

Separating the real and imaginary parts we find that the complex resistivity is given by

\[ \rho = \frac{\phi_0 B \eta}{c^2 (\eta^2 + m^2)} - i \left[ \frac{m(\phi_0 B/c^2)}{\eta^2 + m^2} + \frac{4\pi \omega \lambda_L^2}{c^2} \right] \]  

where \( m = \frac{\eta}{\omega} - \omega M \). From the beginning of the problem, we could have written \( n\phi_0 \) in eq.(7) where \( n \) is an integer. This means that the resistivity oscillates and the values of \( n \) give quantized resistivity. Some time ago the flux was quantized \([10]\) in the units of \((n + \frac{1}{2})\phi_0 \) so that the minimum flux becomes \( \phi_0/2 \). However, recently we have found \([11]\) that in paramagnetic Meissner effect the minimum flux observed is \( \phi_0/4 \). Therefore, in a device the resistivity may be quantized in units of \( (p/q)\phi_0 \) from the first term in (13) where \( p \) and \( q \) are any integers. The last term in (13) depends on the penetration depth as if the heat loss which is \( J^2 I m \rho \) is caused by the penetration of field. Here \( I m \rho \) is the imaginary part of \( \rho \). The term previous to the one dependent on \( \lambda_L^2 \) actually depends on \( m \) so that part of the heat loss is caused by the oscillations of the vortex but the term dependent on mass causes retardation and opposes this effect.

We define an effective London penetration depth

\[ \lambda_{eff}^2 = \lambda^2(0) \left[ 1 + \frac{i\phi_0 B}{4\pi \omega \{ \eta - i\omega M + (ik/\omega) \} \lambda_L^2(0)} \right] \]
which can be written as

\[ \lambda_{\text{eff}} = \lambda_L(o) \left[ 1 + \frac{\phi_0 B}{4\pi \omega \lambda_L^2(o)(m - i\eta)} \right]^{1/2} \] (15)

where the complexity is caused by the application of the magnetic field. We separate the real and imaginary parts to define the absolute value of the London penetration depth as

\[ \lambda_{\text{eff}} = \lambda_L(o) \left[ \left\{ 1 + \frac{\phi_0 B m}{4\pi \omega \lambda_L^2(o)(m^2 + \eta^2)} \right\}^2 + \left\{ \frac{\phi_0 B \eta}{4\pi \omega \lambda_L^2(o)(m^2 + \eta^2)} \right\}^2 \right]^{1/4} \] (16)

This expression is of general nature. It can give rise to the square root of the magnetic field. When binomially expanded, it gives a linear function of magnetic field. The full phase factor introduced in \( \phi_0 \) due to Josephson tunneling can bend the \( \lambda(H) \) curve from the linear behaviour and introduction of quantized flux, \( n\phi_0 \), instead of \( \phi_0 \), leads to quantized resistivity. We consider several special cases.

**Case I:** At large fields

\[ \lambda_{\text{eff}} = B^{1/2}(\phi_0/4\pi \omega)^{1/2} \left[ \left( \frac{k}{\omega} - \omega M \right)^2 + \eta^2 \right]^{-1/4} \] (17)

and for large \( \eta \),

\[ \lambda_{\text{eff}} = B^{1/2}(\phi_0/4\pi \omega)^{1/2} \eta^{-1/2} \] (18)

which shows that the effective penetration depth depends on the square root of the magnetic field.

**Case II.** For small fields, we obtain the binomial expansion,

\[ \lambda_{\text{eff}} = \lambda(o) \left[ 1 + \frac{\phi_0 B}{8\pi \omega \lambda_L^2(m - i\eta)} \right] \] (19)

which for small viscosity, \( \eta = 0 \), becomes,

\[ \lambda_{\text{eff}} = \lambda(o) \left[ 1 + \frac{\phi_0 B}{8\pi \omega \lambda_L^2 m} \right] \] (20)
For zero vortex mass, \( M = 0 \), \( m = k/\omega \),

\[
\lambda_{\text{eff}} = \lambda(0) \left[ 1 + \frac{\phi_0 B c_2}{8\pi \lambda_2^2 k} \left( \frac{B}{B_{c2}} \right) \right] \tag{21}
\]

Introducing dimensionless variable \( h = B/B_{c2} \) and

\[
\beta = \frac{\phi_0 B c_2}{8\pi k \lambda_2^2(0)} \tag{22}
\]

we find that the penetration depth depends linearly on the magnetic field

\[
\lambda_{\text{eff}} = \lambda(0)[1 + \beta h] \tag{23}
\]

**Case III.** In the superconductors with weak links, the critical current is replaced by the Josephson current with quantized phase so that on the right hand side of eq. (1), we can add a phase factor. For small magnetic fields, we expand the cosine term, retaining only two terms,

\[
\frac{J \phi_0}{c} \cos \left( \frac{2\pi}{\phi_0} \phi_2 \right) \approx \frac{J \phi_0}{c} \left[ 1 - \frac{2\pi^2 \phi_0^2}{\phi_0^2} \right] \tag{24}
\]

so that the vortex velocity becomes,

\[
v = \frac{J \phi_0 - 2J\pi^2 A^2 B^2 \phi_0^{-1}}{c[\eta - i\omega M + (ik/\omega)]} \tag{25}
\]

where flux is quantized within the area \( A \) such that \( BA = \phi_2 \). This velocity produces the electric field of

\[
E_\phi = \frac{-J \phi_0 B + J(2\pi^2 A^2 \phi_0^{-1})B^3}{c^2[\eta - i\omega M + (ik/\omega)]} \tag{26}
\]

from which the effective penetration depth is found to become,

\[
-\frac{4\pi i\omega}{c^2} \lambda_{\text{eff}}^2 = -\frac{4\pi i\omega}{c^2} \lambda_2^2 + \frac{\phi_0 B}{c^2[\eta - i\omega M + (ik/\omega)]} - \frac{(2\pi^2 A^2 \phi_0^{-1})B^3}{c^2[\eta - i\omega M + (ik/\omega)]} \tag{27}
\]

Upon comparing with (14) we find that the field is

\[
B_{\text{eff}} = B \left[ 1 - \frac{2\pi^2 A^2 B^2}{\phi_0^2} \right] \tag{28}
\]
so that in the approximation which leads to the square root of the field as in (18), the penetration depth varies as

$$\lambda_{eff} \propto B^{1/2}[1 - aB^2]^{1/2}$$  \hspace{1cm} (29)

where $a = 2\pi^2 A^2 / \phi_0^2$ and in the approximation which gives penetration depth linear in field,

$$\lambda_{eff} \propto B[1 - aB^2]$$  \hspace{1cm} (30)

in which the bending in $\lambda_{eff}(B)$ varies as $B^3$. Retaining the full cosine term in the resistivity, it is seen that

$$\rho = \frac{4\pi i \omega \lambda_0^2}{c^2} + \frac{\phi_0 B}{c^2 \eta} \cos[2\pi \phi_x / \phi_0]$$  \hspace{1cm} (31)

which oscillates as a function of magnetic field. We observe that $\rho$ is proportional to the square of the penetration depth. Thus oscillations occur in the square of the penetration depth as a function of magnetic field. In the eq.(31), $\phi_0$ may be replaced by $n\phi_0$ where $n$ is an integer. Then it is seen that resistivity is quantized,

$$\rho \approx nB$$  \hspace{1cm} (32)

in units of $\phi_0 / c^2 \eta$ leaving out the phase factor as equal to unity and the first term which does not depend on the field. Similarly, the square of the effective London penetration depth, $\lambda_{eff}^2 \propto nB$, gets quantized.

Case IV. The voltage as a function of field can be determined by our theory. The electric field generated by the motion of fluxoids in the magnetic field is determined by

$$E = -\frac{J\phi_0 B(\eta - im)}{c^2(\eta^2 + m^2)}$$  \hspace{1cm} (33)

The voltage is applied across a distance $r$ so that, $V = Er$, is the voltage. Differentiating this voltage with respect to the magnetic field $B$, we find the effective vortex velcoity,
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\( v = \frac{\partial V}{\partial B} \) as,

\[
v = \frac{-J \phi_o r (\eta - im)}{c^2(\eta^2 + m^2)} = -v_1 + iv_2 \tag{34}
\]

with

\[
v_1 = \frac{J \phi_o \eta r}{c^2(\eta^2 + m^2)} \tag{35}
\]

\[
v_2 = \frac{J \phi_o m r}{c^2(\eta^2 + m^2)} \tag{36}
\]

The real part \( v_1 \) is nonzero even if \( m = 0 \). This is the velocity which can be greater than the velocity of light in the medium for

\[
\frac{J \phi_o \eta r}{\eta^2 + m^2} > c^3 \tag{37}
\]

For small \( m \), \((J \phi_o \eta)^{1/3} > c \) in a medium is allowed and \( v_2 \) may also be larger than \( c \). The ratio of the two velocities \( v_1/v_2 = \eta/m \) indicates that one of the two velocities may be greater than \( c \) depending on the ratio \( \eta/m \). The system is thus in a dynamic equilibrium and can switch from one velocity to another.

**Comparison with experimental results**

(a) Eskildsen et al [12] have obtained the change in the penetration depth in single crystal of ErNi\(_2\)B\(_2\)C as a function of magnetic field by r.f. kinetic inductance measurements. We have found that except at small fields, the measured values are in accord with the predicted dependence of the penetration depth on the square root of magnetic field as shown by the expression (18), \( \lambda_L \propto B^{1/2} \), for large viscosity. In Fig.1 we show the measured values of the penetration depth as a function of the square root of the magnetic field which agree with the theory.

(b) In the case of small viscosity, the equations (21) and (23) predict that the penetration depth is linearly proportional to the magnetic field. Sonier et al [13] have
measured the magnetic field dependence of the penetration depth $\lambda_{ab}(H)$ in the vortex state of NbSe$_2$ at $0.6 \ T_c$ and also at $0.33 \ T_c$. In both the cases the penetration depth is seen to linearly increase with increasing magnetic field. In particular the data agrees with eq. (23) for

$$\lambda_{ab}(o) = 1323 \ \text{Å}, \ \beta = 1.61 \ \text{at} \ T = 0.33T_c \ \text{and}$$

$$\lambda_{ab}(o) = 1436 \ \text{Å}, \ \beta = 1.56 \ \text{at} \ T = 0.60T_c.$$ 

The data for $T = 0.33T_c$ are displayed in Fig.2 showing the amount of agreement between the theory and the experimental measurements.
Fig. 2 The measured values of the penetration depth in NbSe$_2$ are seen to be linearly proportional to the magnetic field for $T = 0.33T_c$. The data are taken from Sonier et al [13] and the calculation is given by eq.(23).

Sonier et al [14] have also measured the $\lambda_{ab}(H)$ in YBa$_2$Cu$_3$O$_{6.6}$ in both the twinned as well as the detwinned samples. For twinned samples, $H_{c2} = 70T$, $\lambda_{ab}(0,0) = 1586\,\text{Å}$ and $\beta = 6.6$ and for the detwinned samples $\lambda_{ab}(0,0) = 1699\,\text{Å}$ and $\beta = 5.0$. In both the cases eq.(23) is well obeyed.

(c) The magnetic field penetration depth $\lambda_L$ in CeRu$_2$ has been obtained using transverse field $\mu SR$ by Yamashita et al [15]. At 2K for a field of 0.9846 T the penetration depth is found to be $1983\pm15\,\text{Å}$. As the field is increased to 1.9621 T the penetration depth
increases to $2101 \pm 11 \text{Å}$ but upon further increase of field to $2.9312 \text{T}$ the penetration depth reduces but increases to $2937 \pm 24 \text{Å}$ upon further increase of field to $3.9003 \text{T}$. This type of oscillatory behaviour is qualitatively in accord with that predicted by (20) and is caused by the phase factor dependent term in the tunneling current. Considering the phase factor (20) becomes,

$$\lambda_{eff} = \lambda(o) \left[ 1 + \frac{\phi_o B}{8\pi \omega \lambda_L^2 m} \cos \frac{2\pi \phi_x}{\phi_o} \right]$$

(38)

Between two fields $B_1$ and $B_2$ the cosine term shows one half oscillation when

$$\left( \frac{2\pi \phi_{x1}}{\phi_o} - \frac{\pi}{2} \right) - \left( \frac{2\pi \phi_{x2}}{\phi_o} - \frac{\pi}{2} \right) = \pi$$

(39)

Since the field is quantized in an area $A$, we write $\phi_{x1} = AB_1$ and $\phi_{x2} = AB_2$ so that one half oscillation is obtained when

$$B_1 - B_2 = \frac{\phi_o}{\pi A}$$

(40)

In Fig.3 we have shown the experimental data points [15] of penetration depth as a function of field in CeRu$_2$ along with a qualitative curve drawn as a guide to the eyes. It is found that the data are in qualitative agreement with the interpretation given above. The period of oscillations in Fig.3 is about $2 \times 10^4 \text{Gauss}$. Substituting this value for $B_1 - B_2$ and the value of $\phi_o$ we find that $A^{1/2} \simeq (10/\pi)^{1/2} 10^{-6} \text{cm}$ which is quite reasonable for the size of the current loops.
Oscillation of period of about 2T is found in CeRu$_2$. The experimental data is taken from Yamashita et al [15] and the theory is given by eq. (38).

(d) For Josephson junctions the voltage frequency relation is $2eV = h\nu$ from which the voltage is found to be

$$V = \frac{hc\nu}{2e} = \phi_0 \frac{\nu}{c}$$

where $V$ is the applied voltage and $\nu$ is the microwave frequency. The voltage can be written in terms of the unit flux, $\phi_0 = hc/2e$. For a stack of $N$ intrinsic Josephson junctions

$$V = N\phi_0 \frac{\nu}{c}.$$
The voltage is being applied across the thickness $t$ of the junction so that

$$2eV = 2eBvt$$

where $\bar{v}$ is the standard Swihart velocity [16-18]

$$V = B\bar{v}t$$

so that for $N$ junctions

$$V = N\bar{v}Bt.$$  \hfill (45)

The plot of voltage as a function of field in a stack of $N$ Josephson junctions is thus predicted to be linear with only one slope. It has been shown in (34) that the velocity becomes complex so that the system can jump from one state with velocity $v_1$ to another state of velocity $v_2$. The $c$-axis in $Bi_2Sr_2CaCu_2O_{8+y}$ is analogous to a stack of $N$ Josephson junctions. The tunneling voltage along this direction as a function of field measured by Hechtfsicher et al [19] shows two slopes as expected for a system which switches from $v_1$ to $v_2$ at a field of 17 kOe. Therefore, the measurements are in accord with the theory. It may be noted that due to the flux lattice and its melting [20], the ordinary expression for the London penetration depth requires Stephen-Bardeen corrections owing to the viscosity and oscillations in the vortex lattice. Portis et al [21] have also pointed out the importance of the term containing mass of the vortex to determine the surface resistivity which measures the penetration depth.

**Conclusions**

The penetration depth in type-II superconductors is determined by viscous oscillatory forces. For large viscosity the change in penetration depth as a function of magnetic field depends on the square root of the magnetic field. For small viscosity, the penetration
depth depends linearly on the magnetic field. For intermediate cases due to quantized phase factor we expect oscillations in the penetration depth as a function of magnetic field. The vortex velocity becomes complex when viscous forces are comparable with those of oscillatory motion of the vortex. In such a case a transition from one vortex velocity to another vortex velocity is predicted. It may be noted that the current depends \[22\] on the symmetry of the gap of the superconductor. Therefore the square of the penetration depth depends on the symmetry of the superconducting state which requires detailed investigation. All of the theoretical results are found to be in agreement with various measurements in ErNi$_2$B$_2$C, NbSe$_2$ and YBa$_2$Cu$_3$O$_7$, CeRu$_2$ and c-axis voltage in Bi$_2$Sr$_2$CaCu$_2$O$_8$. 


References


REFERENCES


