Chapter 3

Existence of solutions and approximate controllability of fractional nonlocal neutral impulsive stochastic differential equations of order $1 < q < 2$ with infinite delay and Poisson jumps

3.1 Introduction

The most important advantages of using fractional differential equations in these and other applications is their nonlocal property. The study of impulsive differential systems have become an important object of an investigation in recent years, because their applications in many areas such as mechanics, electrical engineering, drug administration, threshold theory in biology, ecology, etc. For instance we refer [18, 27, 55, 56, 81, 95, 125]. Neutral differential systems with impulses aries in many areas of applied mathematics and these systems have been studied during the last decades. Cui et al. [36] studied the existence
result for fractional neutral stochastic integro-differential equations with infinite delay. The concept of controllability plays an important role in the analysis and design of control systems. Yan et al. [140] established some sufficient conditions for the approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay. Sakthivel [114] established the approximate controllability of second-order stochastic differential equations with impulsive effects. Sakthivel et al. [118] studied the exponential stability of second-order stochastic evolution equations with Poisson jumps. Toufik Guendouzi et al. [49] established a set of sufficient conditions for the existence and controllability of fractional neutral stochastic integro-differential equations with infinite delay.

Controllability of nonlinear integer order differential systems was studied by many authors. Chang et al. [31] studied the controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces. Vijayakumar et al. [134] studied the controllability for a class of fractional neutral integro-differential equations with unbounded delay. Arthi et al. [5] studied the controllability of second order impulsive differential equations with state-dependent delay. Further it is important to note that event driven dynamics become very useful in most fields of application and lead to stochastic differential equations with jumps [35, 110, 111, 117, 118, 132]. Controllability of the stochastic dynamical system in infinite dimensional spaces are derived using different kinds of approaches [97, 98, 100, 115]. The existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations have been well developed [119]. Shu et al. [124] discussed the existence and uniqueness of mild solutions for fractional differential systems with nonlocal conditions of order $1 < q < 2$. Sakthivel et al. [116] studied a class of fractional control systems of nonlinear fractional nonlocal dynamical systems of order $1 < q < 2$, they established a new set of sufficient conditions for the controllability of nonlinear fractional systems by using fixed point analysis approach and extended the result to study the approximate controllability of the corresponding systems with nonlocal conditions.
However the controllability of fractional nonlocal neutral impulsive stochastic differential equations of order $1 < q < 2$ with infinite delay and Poisson jumps have not yet been considered in the literature and this fact is the motivation of this work. Motivated by the above discussion, our current consideration in this chapter is the existence of mild solutions and approximate controllability of fractional nonlocal neutral impulsive stochastic differential equations of order $1 < q < 2$ with infinite delay and Poisson jumps in Hilbert space described by

$$c D^q_t[x(t) + f_1(t, x_t)] = [Ax(t) + Bu(t)]dt + \int_0^t f_2(t - s)x(s)ds + \int_{-\infty}^t g(\tau, x_\tau)d\omega(\tau)$$

$$+ \int_Z h(t, x_t, \eta)\widetilde{N}(dt, d\eta), \quad t \in J := [0, b] \setminus \{t_1, \ldots, t_n\}$$

$$x_0(t) = \phi(t) + m_1(x_{t_1}, \ldots, x_{t_m})(t), \quad t \in (-\infty, 0],$$

$$x'(0) = \xi,$$

$$\Delta x(t_k) = I_k(x_{t_k}),$$

$$\Delta x'(t_k) = \tilde{I}_k(x_{t_k}), \quad k = 1, 2, \ldots, n.$$  \hspace{1cm} (3.1)

Here, the state variable $x(\cdot)$ takes values in a real separable Hilbert space $H$ with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|_H$. The fractional derivative $c D^q_t$, $1 < q < 2$ is understood in the Caputo sense. $A$, $(f_2(t))_{t \geq 0}$ are closed linear operators defined on a common domain which is dense in Hilbert space $H$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < b$ be the given time points. The control function $u(\cdot)$ is given in $L_2(J, U)$ of admissible control functions with $U$ as a Hilbert space. $B$ is a bounded linear operator from $U$ into $H$. Also $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on $H$. Let $\{\omega(t)\}_{t \geq 0}$ be a given $K$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Let $\tilde{q} = \{\tilde{q}(t) : t \in D_{\tilde{q}}\}$ be a stationary $\tilde{F}_t$ Poisson point process with characteristic measure $\lambda$. Let $N(dt, d\eta)$ be the Poisson counting measure associated with $\tilde{q}$. Then $N(t, Z) = \sum_{s \in D_{\tilde{q}}, s \leq t} I_Z(\tilde{q}(s))$ with measurable set $Z \in \tilde{B}(K - \{0\})$, which denotes the Borel $\sigma$-field of $K - \{0\}$. Let $\widetilde{N}(dt, d\eta) = N(dt, d\eta) - dt\lambda(d\eta)$ be the compensated martingale that is independent of $\omega(t)$. Let $P_2([0, b] \times Z; H)$ be the space all mapping $\chi : J \times Z \to H$ for which $\int_0^b \int_Z \mathbb{E}\|\chi(t, \eta)\|^2_H dt d\lambda(d\eta) < \infty$. We can define the H-valued...
stochastic integral \( \int_0^h \int_Z \chi(t, \eta) \tilde{N}(dt, d\eta) \), which is a centred square integrable martingale.

The histories \( x_t \) represents the function defined by \( x_t : (-\infty, 0] \to H, \ x_t(\theta) = x(t + \theta) \), for \( t \geq 0 \) belong to some phase space \( C_h \) described axiomatically. Further \( f_1 : J \times C_h \to H, \ g : J \times C_h \to L_{Q}(K, H) \) and \( h : J \times C_h \times Z \to H \) are nonlinear functions. Moreover \( m_1 : C_h \to C_h \) is a continuous function. \( I_k \) and \( \tilde{I}_k : H \to H \) are appropriate functions.

The symbol \( \Delta \zeta(t) \) represents the jump of the function \( \zeta \) at \( t \) which is defined by \( \Delta \zeta(t) = \zeta(t^+) - \zeta(t^-) \). The initial data \( \phi = \{ \phi(t) : t \in (-\infty, 0] \} \) is an \( \mathcal{F}_0 \)-measurable \( C_h \)-valued stochastic process independent of Brownian motion \( \{ \omega(t) \} \) and Poisson point process \( \tilde{\mathcal{Q}}(\cdot) \) with finite second moment. Further \( \xi(t) \) is an \( \mathcal{F}_1 \)-measurable \( H \)-valued random variable independent of \( \omega(t) \) and Poisson point process \( \tilde{q} \) with finite second moment.

### 3.2 Preliminaries

The complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and its associated norm definitions are similar to the section 2.2.

Consider the abstract fractional integro-differential equations as in the form

\[
D_q^tx(t) = Ax(t) + \int_0^t f_2(t - s)x(s)ds,
\]

\[
x(0) = x_0 \in H, \quad x'(0) = 0,
\]

which has an associated \( q \)-resolvent operator of bounded linear operators \( (R_q(t))_{t \geq 0} \) on \( H \).

**Definition 3.2.1.** A one-parameter family of bounded linear operators \( (R_q(t))_{t \geq 0} \) on \( H \) is called an \( q \)-resolvent operator of (3.2) if the following conditions are verified.

(a) The function \( R_q(\cdot) : [0, \infty) \to \mathcal{L}(H) \) is strongly continuous and \( R_q(0)x = x \) for all \( x \in H \) and \( q \in (1, 2) \).

(b) For \( x \in D(A), R_q(\cdot)x \in C([0, \infty), [D(A)]) \cap C^1((0, \infty), H) \), and

\[
D_q^tR_q(t)x = AR_q(t)x + \int_0^t f_2(t - s)R_q(s)xds
\]
\[ D_q^t R_q(t)x = R_q(t)Ax + \int_0^t R_q(t-s)f_2(s)xds \]

for every \( t \geq 0 \).

In order to see the existence of \( q \)-resolvent operator for problem (3.2), we have considered the following conditions:

\((P_1)\) The operator \( A : D(A) \subseteq H \to H \) is a closed linear operator with \([D(A)]\) dense in \( H \). Let \( 1 < q < 2 \). For some \( \tilde{\varphi}_0 \in (0, \frac{\pi}{2}) \), for each \( \tilde{\varphi} < \tilde{\varphi}_0 \), there is a positive constant \( C_0 = C_0(\tilde{\varphi}) \) such that \( \lambda \in p(A) \) for each \( \lambda \in \Sigma_{0,q}\beta = \{ \lambda \in \mathbb{C} : \lambda \neq 0, \arg(\lambda) < q\beta \} \), where \( \beta = \tilde{\varphi} + \frac{\pi}{2} \) and \( \|R(\lambda,A)\| \leq \frac{C_0}{|\lambda|} \) for all \( \lambda \in \Sigma_{0,q\beta} \).

\((P_2)\) For all \( t \geq 0 \), \( f_2(t) : D(f_2(t)) \subseteq H \to H \) is a closed linear operator, \( D(A) \subseteq D(f_2(t)) \) and \( f_2(t)x \) is strongly measurable on \((0, \infty)\) for each \( x \in D(A) \). There exists \( b_1(\cdot) \in \mathcal{L}_{loc}^1(\mathbb{R}^+) \) such that \( \hat{b}_1(\lambda) \) exists for \( Re(\lambda) > 0 \) and \( \|f_2(t)x\|_H \leq b_1(t)\|x\|_1 \), for all \( t > 0 \) and \( x \in D(A) \). Moreover, the operator valued function \( \hat{f}_2 : \Sigma_{0,\tilde{\varphi}} \to \mathcal{L}([D(A)],H) \) has an analytical extension (still denoted by \( \hat{f}_2 \)) to \( \Sigma_{0,\beta} \) such that \( \|\hat{f}_2(\lambda)x\|_H \leq \|\hat{f}_2(\lambda)\|_H \|x\|_1 \) for all \( x \in D(A) \), and \( \|\hat{f}_2(\lambda)\| = O\left(\frac{1}{|\lambda|}\right) \) as \( |\lambda| \to \infty \).

\((P_3)\) There exists a subspace \( D \subseteq D(A) \) dense in \([D(A)]\) and a positive constant \( M_0 \) such that \( A(D) \subseteq D(A), \hat{f}_2(\lambda)(D) \subseteq D(A) \), and \( \|A\hat{f}_2(\lambda)x\|_H \leq M_0\|x\|_H \) for every \( x \in D \) and for all \( \lambda \in \Sigma_{0,\beta} \).

In the sequel for \( r_1 > 0 \) and \( \theta \in (\frac{\pi}{2}, \beta) \), \( \Sigma_{r_1,\theta} = \{ \lambda \in \mathbb{C} : \lambda \neq 0, |\lambda| > r_1, |\arg(\lambda)| < \theta \} \) for \( \Gamma_{r_1,\theta} \). \( \Gamma_{r_1,\theta}^i, \ i = 1, 2, 3 \) are the paths \( \Gamma_{r_1,\theta}^1 = \{ te^{i\theta} : t \geq r_1 \} \), \( \Gamma_{r_1,\theta}^2 = \{ te^{i\theta} : t \geq r_1 \} \), and \( \Gamma_{r_1,\theta} = \cup_{i=1}^3 \Gamma_{r_1,\theta}^i \) oriented counterclockwise. In addition \( \rho_q(F_q) \) are the sets
\[
\rho_q(F_q) = \{ \lambda \in \mathbb{C} : F_q(\lambda) = \lambda^{q-1}(\lambda^qI - A - \hat{f}_2(\lambda))^{-1} \in \mathcal{L}(H) \}.
\]

Define the operator family \( (\mathcal{R}_q(t))_{t \geq 0} \) by
\[
\mathcal{R}_q(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r_1,\theta}} e^{\lambda t} F_q(\lambda) d\lambda, & t > 0 \\ I, & t = 0 \end{cases}
\]  
(3.3)
Theorem 3.2.2. Assume that conditions \((P_1) - (P_3)\) are fulfilled. Then there exists a unique \(q\)-resolvent operator for problem (3.2).

Theorem 3.2.3. [4] The function \(R_q : [0, \infty) \to \mathcal{L}(H)\) is strongly continuous and \(R_q : (0, \infty) \to \mathcal{L}(H)\) is uniformly continuous.

Now we consider the non-homogeneous problem
\[
D^q_t x(t) = Ax(t) + \int_0^t f_2(t - s)x(s)ds + f(t), \quad t \in [0, b],
\]
x(0) = x_0, \quad x'(0) = 0, \quad (3.4)
where \(q \in (1, 2)\) and \(f \in \mathcal{L}^1([0, b], H)\). In the sequel, \(R_q(\cdot)\) is the operator function defined by (3.3).

\(^cD^q_t \tilde{h}(t)\) represent the Caputo derivative of order \(q > 0\) of \(\tilde{h}\) is defined by
\[
^cD^q_t \tilde{h}(t) = \int_0^t g_{n-q}(t - s)\frac{d^n}{ds^n} \tilde{h}(s)ds,
\]
where \(n\) is the smallest integer greater than or equal to \(q\) and \(g_T(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \ t > 0, \ T \geq 0\).

Definition 3.2.4. [134] A function \(x : J \to H, 0 < b\) is called a classical solution of (3.4) on \([0, b]\) if \(x \in C([0, b], \mathcal{D}(A)) \cap C([0, b], H), g_{n-q} \ast x \in C^1([0, b], H), \ n = 1, 2, \) the conditions \(x(0) = x_0, \ x'(0) = 0\) holds and equation (3.4) is verified on \(J\).

Definition 3.2.5. [134] Let \(q \in (1, 2)\), we define the family \((T_q(t))_{t \geq 0}\) by
\[
T_q(t)x = \int_0^t g_{q-1}(t - s)R_q(s)xds,
\]
for each \(t \geq 0\).

Lemma 3.2.6. [4] The function \(R_q(\cdot)\) is exponentially bounded in \(\mathcal{L}(H)\).

Lemma 3.2.7. [4] The function \(R_q(\cdot)\) is exponentially bounded in \(\mathcal{L}(H)\), then \(T_q(\cdot)\) is exponentially bounded in \(\mathcal{L}(H)\).

Lemma 3.2.8. [4] The function \(R_q(\cdot)\) is exponentially bounded in \(\mathcal{L}([D(A)])\), then \(T_q(\cdot)\) is exponentially bounded in \(\mathcal{L}([D(A)])\).
In the next result we denote by \((-A)^\beta\) the power of the operator \(-A\), there exists a constant \(M_1\) such that \(\|(−A)^\beta\| \leq M_1\) for \(0 \leq \beta \leq 1\).

**Lemma 3.2.9.** [4] Suppose that the conditions \((P_1) - (P_3)\) are satisfied. Let \(q \in (1, 2)\) and \(\beta \in (0, 1)\) such that \(q\beta \in (0, 1)\), then there exists a positive number \(M_1\) such that

\[
\|(-A)^\beta R_q(t)\| \leq M_1 e^{rtq^\beta},
\]

\[
\|(-A)^\beta T_q(t)\| \leq M_1 e^{rtq(1-\beta)^{-1}}
\]

for all \(t > 0\).

**Remark 3.2.10.** [6] If \(\hat{f}_2(\lambda)(-A)^{-\beta}y = (-A)^{-\beta}\hat{f}_2(\lambda)y\) for \(y \in [D(A)]\). Also we see that for \(\beta \in (0, 1)\) and \(x \in [D(-A)^{\beta}]\), we have
\[
(-A)^\beta R_q(t)x = R_q(t)(-A)^\beta x \quad \text{and} \quad (-A)^\beta T_q(t)x = T_q(t)(-A)^\beta x.
\]

Let \(f \in \mathcal{L}_1([0, b], H)\). Let \(x_0 \in D(A)\). A function \(x \in \mathcal{L}_1([0, b], H)\) is called a mild solution of (3.4) if
\[
x(t) = R_q(t)x_0 + \int_0^t T_q(t-s)f(s)ds, \quad t \in [0, b].
\]

It is clear from the preceding definition that \(R_q(\cdot)x_0\) is a solution of the problem (3.2) on \((0, \infty)\) for \(x_0 \in D(A)\).

### 3.3 The main results

**Definition 3.3.1.** [49] [116] [134] An \(H\)-valued stochastic process \(\{x(t) : t \in (-\infty, b]\}\) is said to be a mild solution of the system (3.1) if

(i) \(x(t)\) is \(\mathcal{F}_t\)-adapted and measurable for \(t \geq 0\),

(ii) \(x(t)\) is continuous on \([0, b]\) almost surely and for each \(s \in [0, t)\), the function \(AT_q(t-s)f_1(s, x_s)\) is integrable such that the following stochastic integral equation is satisfied:
\[
x(t) = R_q(t)\left(\phi(0) + f_1(0, \phi) + m_1(x_{i_1}, \ldots, x_{i_m})(0)\right) - f_1(t, x_t)
\]
For each \( \phi \in \mathcal{C} \),

\[
\mathbb{E}\|f_1(t, x_1, \ldots, x_m)\|_H^2 \leq \int_{\mathbb{R}^m} \mathbb{E}\|f_1(\xi_1, \ldots, \xi_m)\|_H^2 \leq M \quad \text{for } m = 1, \ldots, M_{f_1, (-A)^\beta}.
\]

\[\mathbb{E}\|(-A)^\beta f_1(t, x_1, \ldots, x_m)\|_H^2 \leq M \quad \text{for } m = 1, \ldots, M_{f_1, (-A)^\beta}.
\]

\[\mathbb{E}\|(-A)^\beta f_1(t, x_1) - (-A)^\beta f_1(t, x_2)\|_H^2 \leq M \quad \text{for } m = 1, \ldots, M_{f_1, (-A)^\beta}.
\]

\[\mathbb{E}\|K_1(t)\|_H^2 \leq M_k.
\]
The functions \( g : J \times C_h \to \mathcal{L}(K, H) \) and \( h : J \times C_h \times Z \to H \) satisfies the following properties:

(i) The function \( g : J \times C_h \to \mathcal{L}(K, H) \) is a continuous and measurable function.

(ii) The non-linear function \( h \) is a Borel measurable function which satisfy the Lipschitz condition and there exists positive constants \( M_h > 0 \) and \( L_h > 0 \) such that for every \( x, x_1, x_2 \in C_h \) and \( t \in J \),

\[
\begin{align*}
\int_Z \| h(t, x, \eta) \|^2_H \lambda(d\eta) &\leq M_h(1 + \|x\|^2_{C_h}), \\
\int_Z \| h(t, x_1, \eta) - h(t, x_2, \eta) \|^2_H \lambda(d\eta) &\leq M_h \|x_1 - x_2\|^2_{C_h}, \\
\int_Z \| h(t, x, \eta) \|^{4}_H \lambda(d\eta) &\leq L_h(1 + \|x\|^4_{C_h}), \\
\int_Z \| h(t, x_1, \eta) - h(t, x_2, \eta) \|^{2}_H \lambda(d\eta) &\leq L_h \|x_1 - x_2\|^2_{C_h}.
\end{align*}
\]

(iii) There are positive integrable functions \( m, n \in \mathcal{L}^1([0,b]) \) and continuous non-decreasing functions \( A_g, A_h : [0,\infty) \to (0,\infty) \) such that for every \( (t, x) \in J \times C_h \), we have

\[
\int_0^t \mathbb{E}\|g(t, x)\|^2_H ds \leq m(t) A_g \|x\|^2_{C_h}, \quad \liminf_{\tau \to \infty} \frac{A_g(\tau)}{\tau} = v_2 < \infty,
\]

\[
\int_0^t \mathbb{E}\|h(t, x, \eta)\|^2_H ds \leq n(t) A_h(1 + \|x\|^2_{C_h}), \quad \liminf_{\tau \to \infty} \frac{A_h(\tau)}{\tau} = v_3 < \infty.
\]

The functions \( I_k \) and \( \bar{T}_k : H \to H \) are continuous and there exists positive constants \( M_{I_k}, M_{\bar{T}_k}, k = 1, 2, \ldots, n \) such that

\[
\begin{align*}
\mathbb{E}\| I_k(x) - I_k(y) \|^2_H &\leq M_{I_k} \|x - y\|^2_{C_h}, \quad x, y \in C_h, \quad k = 1, 2, \ldots, n, \\
\mathbb{E}\| \bar{T}_k(x) - \bar{T}_k(y) \|^2_H &\leq M_{\bar{T}_k} \|x - y\|^2_{C_h}, \quad x, y \in C_h, \quad k = 1, 2, \ldots, n.
\end{align*}
\]

The functions \( I_k \) and \( \bar{T}_k : H \to H \) are continuous and there exists positive non-decreasing functions \( \Theta_k, \overline{\Theta}_k : [0,\infty) \to (0,\infty), k = 1, 2, \ldots, n \) such that for every \( x \in C_h \) and \( k = 1, 2, \ldots, n \), we have

\[
\mathbb{E}\| I_k(x) \|^2_H \leq \Theta_k A_{I_k} \|x\|^2_{C_h}, \quad \liminf_{\tau \to \infty} \frac{A_{I_k}(\tau)}{\tau} = v_4 < \infty.
\]
\[ \mathbb{E}\|T_k(x)\|_H^2 \leq \Theta_k A T_k \|x\|_{L^2}^2, \quad \liminf_{\tau \to \infty} \frac{A T_k(\tau)}{\tau} = \nu_5 < \infty. \]

In order to study the approximate controllability for the fractional control system (3.1), we introduce the approximate controllability of its linear part

\[ D_q^b x(t) = Ax(t) + Bu(t), \]
\[ x(0) = x_0, \quad x'(0) = 0. \]

(3.5)

For this purpose, we need to introduce the relevant operator

\[ \psi_0^b = \int_0^b T_q(b-s)B B^* T_q^*(b-s)ds, \]
\[ R(\epsilon, \psi_0^b) = (\epsilon I + \psi_0^b)^{-1} \text{ for } \epsilon > 0, \]

where \( B^* \) denotes the adjoint of \( B \) and \( T_q^*(t) \) is the adjoint of \( T_q(t) \). It is clear that the operator \( \psi_0^b \) is a linear bounded operator.

\((H_{16})\) \( \epsilon R(\epsilon, \psi_0^b) \to 0 \) as \( \epsilon \to 0^+ \) in the strong operator topology.

Note that the assumption \((H_{16})\) is equivalent to the fact that the linear fractional control system (3.5) is approximately controllable on \( J \).

**Definition 3.3.2.** [49] Let \( x_b(\phi, \xi; u) \) be the state value of (3.1) at the terminal time \( b \) corresponding to the control \( u \) and the initial value \( \phi \). Introduce the set

\[ R(b, \phi, \xi) = \{ x_b(\phi, \xi; u)(0) : u(\cdot) \in L^2(J, U) \} \]

which is called the reachable set of (3.1) at the terminal time \( b \) and its closure in \( H \) is denoted by \( \overline{R(b, \phi, \xi)} \). The system (3.1) is said to be approximately controllable on the interval \( J \) if \( \overline{R(b, \phi, \xi)} = H \).

**Lemma 3.3.3.** [95] For any \( \hat{x} \in L^2(\Omega, H) \) there exists \( \hat{\phi} \in L^2_{\phi}(\Omega, L^2(0, b : L^2)) \) such that

\[ \hat{x}_b = \mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)d\omega(s). \]

For any \( \epsilon > 0 \), \( k = 1, 2, \ldots, n \) and \( \hat{x} \in L^2(\Omega, H) \), we define the control function as,

\[ u^\epsilon(t) = B^* T_q^*(b-t)(\epsilon I + \psi_0^b)^{-1} \left[ \mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)d\omega(s) \right]. \]
\[ -\mathcal{R}_q(b) \left( \phi(0) + f_1(0, \phi) + m_1(x_{t_1}, \ldots, x_{t_m})(0) \right) + f_1(t, x_t) \]
\[ + B^* T_q^*(b - t) \int_0^t (\epsilon I + \psi_0^b)^{-1} AT_q(b - s) f_1(s, x_s) ds \]
\[ + B^* T_q^*(b - t) \int_0^t \int_s^t (\epsilon I + \psi_0^b)^{-1} T_q(b - s) f_2(t - s) f_1(\nu, x_{\nu}) d\nu ds \]
\[ - B^* T_q^*(b - t) \int_0^t (\epsilon I + \psi_0^b)^{-1} T_q(b - s) Bu^\epsilon(s) ds \]
\[ - B^* T_q^*(b - t) \int_0^t (\epsilon I + \psi_0^b)^{-1} T_q(b - s) \sum_{0 < t_k < t} T_q(b - t_k) I_k(x_{t_k}) \]
\[ - B^* T_q^*(b - t) (\epsilon I + \psi_0^b)^{-1} \sum_{0 < t_k < t} T_q(b - t_k) I_k^\epsilon(x_{t_k}). \]  

(3.6)

**Theorem 3.3.4.** Assume that the assumptions \((H_3)\) - \((H_{15})\) hold. Then for each \(\epsilon > 0\) and \(k = 1, 2, \ldots, n\), the fractional control system (3.1) has a mild solution on \(J\) provided that

\[
\left[ 4 \left( M_{f_1} + M_a^2 \frac{b_{2q} b^3}{\epsilon^2 \beta^2} M_{f_1,(-A)^{\beta}} \right) \right]^2 + 4 t^2 M_a^2 \frac{b_{2q} b^3}{\epsilon^2 \beta^2} M_{f_1,(-A)^{\beta}} \int_0^t \mu(s) ds
\]
\[ + 8 M^2 b^2 t^2 \left( Tr(Q) A_q \sup_{s \in J} m(s) + (M_b + \sqrt{T_b}) A_h \sup_{s \in J} n(s) \right) \]
\[ + 4 M^2 b^2 \left( \sum_{k=1}^n A_{I_k} \sup_{s \in J} \Theta_k(s) + \sum_{k=1}^n A_{T_k} \sup_{s \in J} \Theta_k^\epsilon(s) \right) \right] \left[ 9 + \frac{99}{c_2} (M M_B)^4 b^2 \right] < 1
\]

**Proof.** Let \(C(((-\infty, b], H)\) be the space of all continuous \(H\)-valued stochastic process \(\{\zeta(t) : t \in (-\infty, b]\) and \(C_b = \{ x : x \in C(((-\infty, b], H), x_0 \in C_h \}\). Let \(\| \cdot \|_b\) be the semi-norm defined by \(\| x \|_b = \| x_0 \|_{C_h} + \sup_{0 \leq s \leq b} (\mathbb{E}\| x(s) \|^2)\)\(^\frac{1}{2}\), \(x \in C_b\). For any \(\epsilon > 0\) and \(k = 1, 2, \ldots, n\), define the operator \(\Phi^\epsilon : C_b \to C_b\) by

\[
(\Phi^\epsilon x)(t) = \begin{cases} 
\phi(t), & t \in (-\infty, 0] \\
\mathcal{R}_q(t) \left( \phi(0) + f_1(0, \phi) + m_1(x_{t_1}, \ldots, x_{t_m})(0) \right) \\
- f_1(t, x_t) - \int_0^t A_T(t - s) f_1(s, x_s) ds \\
- \int_0^t \int_s^t T_q(t - s) f_2(t - s) f_1(\nu, x_{\nu}) d\nu ds + \int_0^t T_q(t - s) Bu^\epsilon(s) ds \\
+ \int_0^t T_q(t - s) \int_s^t g(\tau, x_\tau) d\omega(\tau) ds + \int_0^t T_q(t - s) \mathcal{N}(ds, d\eta) \\
+ \sum_{0 < t_k < t} T_q(t - t_k) I_k(x_{t_k}) + \sum_{0 < t_k < t} T_q(t - t_k) I_k^\epsilon(x_{t_k}), & t \in J
\end{cases}
\]
Using Holder’s inequality and assumption \((H_{11})\), we get
\[
\mathbb{E}\left\| \int_0^t AT_q(t-s)f_1(s, x_s)ds \right\|_H^2 \leq M_a^2 \left( \int_0^t (t-s)^{q\beta-1}ds \right) \times \left( \int_0^t (t-s)^{q\beta-1} \mathbb{E}\left\| (-A)^{\beta} f_1(s, x_s) \right\|_H^2 ds \right) \\
\leq M_a^2 M_{f_1(-A)^{\beta}} \frac{b\beta}{q\beta} \int_0^t (t-s)^{q\beta-1}(1 + \|x_s\|_{E_{\mathbb{C}^n}}^2)ds,
\]
where \(M_a = M_1 e^{\epsilon(t-s)}\). By Bochner’s theorem and Lemma \(2.2.1\), \(AT_q(t-s)f_1(s, x_s)\) is integrable on \(J\). Similarly \(T_q(t-s)f_2(t-s)f_1(\nu, x_\nu)d\nu\) and \(T_q(t-s)Bu^\epsilon(s)\) are integrable on \(J\). Next we shall prove that \(\Phi^\epsilon\) has a fixed point, by Sadovskii’s theorem, which is then a mild solution of the fractional control system \(3.1\). Define
\[
\tilde{\phi} = \begin{cases} 
\phi(t) & t \in (-\infty, 0] \\
\mathcal{R}_q(t)(\phi(0) + m_1(x_{t_1}, ..., x_{t_m})(0)), & t \in J.
\end{cases}
\]
It is clear that \(\tilde{\phi} \in C_b\). Let \(x(t) = \tilde{\phi}(t) + z(t), t \in (-\infty, b]\). Then \(x\) satisfies \(3.1\) if and only if \(z_0 = 0\) and
\[
z(t) = \mathcal{R}_q(t)f_1(0, \tilde{\phi}) - f_1(t, \tilde{\phi}_t + z_t) - \int_0^t AT_q(t-s)f_1(s, \tilde{\phi}_s + z_s)ds \\
- \int_0^t \int_0^s T_q(t-s)f_2(t-s)f_1(\nu, \tilde{\phi}_\nu + z_\nu)d\nu ds + \int_0^t T_q(t-s)Bu^\epsilon(s)ds \\
+ \int_0^t T_q(t-s)\left[ \int_{-\infty}^s g(\tau, \tilde{\phi}_\tau + z_\tau) \omega(\tau) ds \right] d\tau + \int_0^t \int Z T_q(t-s)\mathfrak{h}(s, \tilde{\phi}_s + z_s, \eta)\tilde{\mathcal{N}}(ds, d\eta) \\
+ \sum_{0 < t_k < t} T_q(t-t_k)I_k(\tilde{\phi}_{tk} + z_{tk}) + \sum_{0 < t_k < t} T_q(t-t_k)I_k(\tilde{\phi}_{tk} + z_{tk}), \quad k = 1, 2, ..., n,
\]
where \(u^\epsilon(t)\) is defined by \(3.6\). Let \(C_b^0 = \{z \in C_b : z_0 = 0 \in C_b\}\). For any \(z \in C_b^0\), we can have \(\|z\|_b = \|z_0\|_{\mathbb{C}^n} + \sup_{0 \leq s \leq b} (\mathbb{E}\|z(s)\|^2)^{\frac{1}{2}} = \sup_{0 \leq s \leq b} (\mathbb{E}\|z(s)\|^2)^{\frac{1}{2}}\). Hence \((C_b^0, \| \cdot \|_b)\) is a Banach space. Set \(C_r = \{y \in C_b^0 : \|y\|_b \leq r\}\) for each positive number \(r\). Then for each \(r\), \(C_r\) is a bounded closed convex set in \(C_b^0\). Then by Lemma \(2.2.1\) for \(z \in C_r\), we have
\[
\|\tilde{\phi}_t + z_t\|^2_{E_{\mathbb{C}^n}} \leq 2(\|z_t\|^2_{\mathbb{C}^n} + \|\tilde{\phi}_t\|^2_{\mathbb{C}^n}) \\
\leq 4(t^2 \sup_{0 \leq s \leq t} \mathbb{E}\|z(s)\|^2 + \|z_0\|^2_{\mathbb{C}^n} + t^2 \sup_{0 \leq s \leq t} \mathbb{E}\|\tilde{\phi}(s)\|^2 + \|\tilde{\phi}_0\|^2_{\mathbb{C}^n}) \\
\leq 4(t^2 + t^2 M^2\mathbb{E}\|\phi(0)\|^2_H + \|\phi\|^2_{E_{\mathbb{C}^n}})
\]
(3.7)
Define the map $\Phi$ on $C_b^0$ as

$$
(\Phi z)(t) = \begin{cases} 
0, & t \in (-\infty, 0] \\
R_q(t) f_1(0, \phi) - f_1(t, \bar{\phi} + z_t) - \int_0^t AT_q(t-s) f_1(s, \bar{\phi} + z_s) ds \\
- \int_0^t \int_0^s T_q(t-s) f_2(t-s) f_1(\nu, \bar{\phi} + z_s) d\nu d\nu + \int_0^t T_q(t-s) Bu^\epsilon(s) ds \\
+ \int_0^t T_q(t-s) \left[ \int_{-\infty}^{\infty} g(\tau, \bar{\phi} + z_s) d\omega(\tau) \right] ds \\
+ \sum_{0 < t_k < t} T_q(t-t_k) I_k(\bar{\phi}_t + z_{t_k}), & k = 1, 2, \ldots n, t \in J.
\end{cases}
$$

Then $\Phi$ is well-defined on $C_r$ for each $r > 0$. Also the operator $\Phi^\epsilon$ has a fixed point if and only if $\Phi$ has a fixed point. Let $\Phi = \Phi_1 + \Phi_2$. Then the operators $\Phi_1$ and $\Phi_2$ are defined on $C_r$ by

$$(\Phi_1 z)(t) = R_q(t) f_1(0, \phi) - f_1(t, \bar{\phi} + z_t) - \int_0^t AT_q(t-s) f_1(s, \bar{\phi} + z_s) ds \\
+ \sum_{0 < t_k < t} T_q(t-t_k) I_k(\bar{\phi}_t + z_{t_k}), & k = 1, 2, \ldots n,
$$

$$(\Phi_2 z)(t) = -\int_0^t \int_0^s T_q(t-s) f_2(t-s) f_1(\nu, \bar{\phi} + z_s) d\nu d\nu + \int_0^t T_q(t-s) Bu^\epsilon(s) ds \\
+ \int_0^t T_q(t-s) \left[ \int_{-\infty}^{\infty} g(\tau, \bar{\phi} + z_s) d\omega(\tau) \right] ds \\
+ \sum_{0 < t_k < t} T_q(t-t_k) I_k(\bar{\phi}_t + z_{t_k}), & k = 1, 2, \ldots n.
$$

In order to prove this theorem, we shall prove the next theorem.

**Theorem 3.3.5.** Assume that the above assumptions $(H_9) - (H_{15})$ hold. Then $\Phi_1$ is a contraction mapping and $\Phi_2$ is compact.

**Proof.** The proof of this theorem needs the following Lemmas.

**Lemma 3.3.6.** Consider the above assumptions $(H_9) - (H_{15})$. For each $\epsilon > 0$, there exists a positive number $r$ such that $\Phi(C_r) \subset C_r$.

**Proof.** Suppose let us assume that $\Phi(C_r) \not\subset C_r$. Then for each positive number $r$, there exists a function $z^\epsilon(\cdot) \in C_r$, but $\Phi(z^\epsilon) \not\in C_r$, that is $E\| (\Phi z^\epsilon)(t) \|_H^2 > r$ for some $t = t(r) \in J$. Then for each $\epsilon > 0$ and $k = 1, 2, \ldots, n$, we have

$$
r \leq E\| (\Phi z^\epsilon) \|_H^2
$$
\[
\begin{align*}
&\leq 9\mathbb{E}\left\| \mathcal{R}_q(t)f_1(0, \phi) \right\|^2_H + 9\mathbb{E}\left\| f_1(t, \tilde{\phi}_s + z^r_t) \right\|^2_H \quad + 9\mathbb{E}\left\| \int_0^t A T_q(t-s) f_1(s, \tilde{\phi}_s + z^r_s) ds \right\|^2_H \\
&\quad + 9\mathbb{E}\left\| \int_0^t \int_0^s T_q(t-s) f_2(t-s)f_1(\nu, \tilde{\phi}_\nu + z^r_\nu) d\nu ds \right\|^2_H \quad + 9\mathbb{E}\left\| \int_0^t T_q(t-s) Bu'(s) ds \right\|^2_H \\
&\quad + 9\mathbb{E}\left\| \int_0^t T_q(t-s) \left[ \int_{-\infty}^s g(\tau, \tilde{\phi}_\tau + z^r_\tau) d\omega(\tau) \right] ds \right\|^2_H \\
&\quad + 9\mathbb{E}\left\| \int_0^t \int_Z T_q(t-s) h(s, \tilde{\phi}_s + z^r_s, \eta) \tilde{N}(ds, d\eta) \right\|^2_H \\
&\quad + 9 \sum_{0 < t_k < t} \mathbb{E}\left\| T_q(t-t_k) I_k (\tilde{\phi}_{t_k} + z^r_{t_k}) \right\|^2 + 9 \sum_{0 < t_k < t} \mathbb{E}\left\| T_q(t-t_k) T_k (\tilde{\phi}_{t_k} + z^r_{t_k}) \right\|^2.
\end{align*}
\]

If the right side terms are represented by \( I_i, \ i = 1, 2, \ldots, 9 \), then \( r \leq 9 \sum_{i=1}^9 I_i \). Evaluating each term separately, we get

\[
I_1 \leq M^2 \mathbb{E}\left\| f_1(0, \phi) \right\|^2_H \leq M^2 M f_1 (1 + \| \phi \|_{C_0}^2) \quad (3.8)
\]

\[
I_2 \leq \mathbb{E}\left\| f_1(t, \tilde{\phi}_s + z^r_s) \right\|^2_H \leq M f_1 (1 + \| \tilde{\phi}_s + z^r_s \|^2_H) \leq M f_1 (1 + 4(t^2 r + t^2 M^2 \mathbb{E}\| \phi(0) \|^2_H + \| \phi \|_{C_0}^2)) \quad (3.9)
\]

\[
I_3 \leq \mathbb{E}\left\| \int_0^t A T_q(t-s) f_1(s, \tilde{\phi}_s + z^r_s) ds \right\|^2_H \\
&\leq M^2 a \left( \int_0^t (t-s)^{q-1} ds \right) \left( \int_0^t (t-s)^{q-1} \mathbb{E}\| (\mathcal{A})^2 f_1(s, \tilde{\phi}_s + z^r_s) \|^2_H ds \right) \\
&\leq M^2 \frac{b q^2 a}{q \beta} \left( \int_0^t (t-s)^{q-1} \mathbb{E}\| (\mathcal{A})^2 f_1(s, \tilde{\phi}_s + z^r_s) \|^2_H ds \right) \\
&\leq M^2 \frac{b q^2 a}{q \beta} \int_0^t (t-s)^{q-1} M f_{1,-(\mathcal{A})^2}(1 + \| \tilde{\phi}_s + z^r_s \|_{C_0}^2) ds \\
&\leq M^2 M f_{1,-(\mathcal{A})^2} \frac{b q^2 a}{q \beta^2} (1 + 4(t^2 r + t^2 M^2 \mathbb{E}\| \phi(0) \|^2_H + \| \phi \|_{C_0}^2)) \quad (3.10)
\]

\[
I_4 \leq \mathbb{E}\left\| \int_0^t \int_0^s T_q(t-s) f_2(t-s)f_1(\nu, \tilde{\phi}_\nu + z^r_\nu) d\nu ds \right\|^2_H \\
&\leq M^2 \int_0^t \int_0^s (t-s)^{q-1} \mu(t-s) \mathbb{E}\| (\mathcal{A})^2 f_1(\nu, \tilde{\phi}_\nu + z^r_\nu) \|^2_H d\nu ds.
\]
\[
\begin{align*}
&\leq M_a \frac{b^{2q\beta}}{q^2}\|M_{f_1,-(\cdot)^p}\|_2 (1 + \|\phi\|_{\mathcal{H}}^2) \int_0^b \mu(s) ds \\
&\leq M_a M_{f_1,-(\cdot)^p} \frac{b^{2q\beta}}{q^2} (1 + 4(t^2 r + t^2 M^2 \mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2 + \|\phi\|^2_{\mathcal{H}})) \int_0^b \mu(s) ds \\
&I_5 \leq \mathbb{E}\left(\| \int_0^t T_q(t-s)Bu^*(s) \| ds \right)^2 \\
&\leq M^2 M_B^2 \int_0^t \mathbb{E}\|u^*(s)\|^2 ds, \text{ where } \|B\| = M_B \\
&\leq M^2 M_B^2 b^2 \mathbb{E}\|u^*(s)\|^2 \\
&\mathbb{E}\|u^*(s)\|^2 \leq \left(\frac{11}{\varepsilon^2} M^2 M_B^2 \right) \left(\frac{11}{\varepsilon^2} M^2 M_B^2 \right) M_R,
\end{align*}
\]

where \( \mathcal{Z} = \|\phi\|_{\mathcal{H}}^2 + \|\mathcal{A}(x_{t_1}, \ldots, x_{t_m})(t)\|_{\mathcal{H}}^2 + \|\xi\|^2_{\mathcal{H}} \)

Hence, we have

\[
I_5 \leq \left(\frac{11}{\varepsilon^2} M^2 M_B^2 \right) M_R,
\]

where

\[
M_R = 2 \|\mathbb{E}\tilde{x}_b\|^2 + 2 \int_0^b \mathbb{E}\|\tilde{\phi}(s)\|^2 ds + M^2(\|\phi\|^2_{\mathcal{H}} + \mathcal{Z}) + M^2 M_{f_1}(1 + \|\phi\|^2_{\mathcal{H}}) \\
+ \left( M_{f_1} + M^2 M_{f_1,-(\cdot)^p} \frac{b^{2q\beta}}{q^2} (1 + \int_0^b \mu(s) ds) \right) (1 + 4(t^2 r + t^2 M^2 \mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2 + \|\phi\|^2_{\mathcal{H}})) \\
+ M^2 b^2 \left( 2 M_K + 2 Tr(Q) A_\beta (4(t^2 r + t^2 M^2 \mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2 + \|\phi\|^2_{\mathcal{H}})) \right) \sup \limits_{s \in J} m(s) \\
+ M^2 b^2 (2 M_b + 2 \sqrt{T_B}) A_\beta (1 + 4(t^2 r + t^2 M^2 \mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2 + \|\phi\|^2_{\mathcal{H}})) \sup \limits_{s \in J} n(s)
\]
Combining these equations (3.8) - (3.18)

In the same way, we get

\[
\begin{align*}
I_6 & \leq E \left\| \int_0^t T_\eta(t - s) \left[ \int_{-\infty}^s g(\tau, \bar{\eta} + z_\tau^r) d\omega(\tau) \right] ds \right\|_{H^2}^2 \\
& \leq M^2 b^2 \left( 2M_K + 2T_r(Q) \int_0^t E \left\| g(\tau, \bar{\eta} + z_\tau^r) \right\|_{B}^2 d\tau \right) \\
& \leq M^2 b^2 \left( 2M_K + 2T_r(Q) A_g(4(t^2 r + \ell^2 M^2 E \|\phi(0)\|^2_{H^2} + \|\phi\|^2_{C_{bh}})) sup_{s \in J} m(s) \right)
\end{align*}
\]  

(3.15)

In the same way, we get

\[
\begin{align*}
I_7 & \leq E \left\| \int_0^t \int_{J} T_\eta(t - s) h(s, \bar{\phi}_s + z_s^r, \eta) \tilde{N}(ds, d\eta) \right\|_{H^2}^2 \\
& \leq 2M^2 \int_0^t \int_{J} E \left\| h(s, \bar{\phi}_s + z_s^r, \eta) \right\|_{H^2}^2 \lambda(\eta) ds \\
& \quad + 2M^2 \left( \int_0^t \int_{J} E \left\| h(s, \bar{\phi}_s + z_s^r, \eta) \right\|_{H^2}^4 \lambda(\eta) ds \right)^{1/2} \\
& \leq M^2(2M_h + 2\sqrt{L_b}) \int_0^t \left( 1 + \left\| \frac{\bar{\phi}_s + z_s^r}{C_{bh}} \right\|_{H^2}^2 \right) ds \\
& \leq M^2 b^2 \left( 2M_h + 2\sqrt{L_b} \right) A_b(1 + 4(t^2 r + \ell^2 M^2 E \|\phi(0)\|^2_{H^2} + \|\phi\|^2_{C_{bh}})) sup_{s \in J} n(s)
\end{align*}
\]  

(3.16)

\[
I_8 \leq E \left\| \sum_{0 < t_k < t} T_\eta(t - t_k) I_k(x_{t_k}) \right\|_{H^2}^2, k = 1, 2, \ldots, n \\
\leq M^2 \sum_{k=1}^n \theta_k(t) A_{t_k}(\left\| \bar{\phi}_t + z_t^r \right\|_{C_{bh}}^2) \\
\leq M^2 \sum_{k=1}^n A_{t_k}(4(t^2 r + \ell^2 M^2 E \|\phi(0)\|^2_{H^2} + \|\phi\|^2_{C_{bh}})) sup_{t \in J} \theta_k(t)
\]

(3.17)

In the same way, we get

\[
I_9 \leq M^2 \sum_{k=1}^n A_{t_k}(4(t^2 r + \ell^2 M^2 E \|\phi(0)\|^2_{H^2} + \|\phi\|^2_{C_{bh}})) sup_{s \in J} \bar{\sigma_k}(s), k = 1, 2, \ldots, n.
\]

(3.18)

Combining these equations (3.8) - (3.18), we get

\[
\begin{align*}
r & \leq L_0 + 36 \left( M_{f_1} + M_{1,q} \int_{0}^{b} \mu(s) ds \right) t^2 r + 99 \left( M_{2} M_B^2 b^2 \right) \\
& \times \left( \frac{1}{\ell^2} M_B M^2 \right) M_R + 18 M^2 b^2 T_r(Q) A_g(4(t^2 r + \ell^2 M^2 E \|\phi(0)\|^2_{H^2} + \|\phi\|^2_{C_{bh}})) sup_{s \in J} m(s)
\end{align*}
\]

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Lemma 3.3.7. Let assumptions \((H_9) - (H_{15})\) hold. Then \(\Phi_1\) is a contraction mapping.

Proof. Let \(v_1, v_2 \in C_r\). Then for \(k = 1, 2, ..., n\), we have

\[
\mathbb{E}\|\Phi(v_1)(t) - \Phi(v_2)(t)\|_H^2 \\
\leq 4M^2 \mathbb{E}\|M^{1.5}_t + M^{1.5}_t\|_H^2 + 4M^2 \mathbb{E}\|v_1(s) - v_2(s)\|_H^2 \\
+ 4M^2 \mathbb{E}\|v_1(s) - v_2(s)\|_C^2 \\
\leq 4M^2 \mathbb{E}\|M^{1.5}_t + M^{1.5}_t\|_H^2 + 4M^2 \mathbb{E}\|v_1(s) - v_2(s)\|_C^2.
\]

Hence \(\Phi_1\) is a contraction mapping, by assumptions \((H_{11})\), \((H_{14})\) and the assumption in Theorem 33.35.
Assume that \( r > 0 \) and \( \Phi_2(C_r) \subset C_r \).

**Lemma 3.3.8.** Let assume that assumptions \((H_9)\) - \((H_{15})\) hold. Then \( \Phi_2 \) maps bounded sets into bounded sets.

**Proof.** For each \( t \in J, z \in C_r \) and \( \epsilon > 0 \) from \((3.7)\), we have

\[
\|\tilde{\phi}_t + z_t\|_{C_h}^2 \leq 4(l^2 + l^2 M^2 \mathbb{E}\|\phi(0)\|_H^2 + \|\phi\|_{C_h}^2) = q'
\]

\[
\|\Phi_2z(t)\|_H^2 \leq 4\mathbb{E}\int_0^t \int_0^s T_q(t-s)f_2(t-s) + \tilde{\phi}_r + \phi(0) + \phi(0)\|H^2 + \|\phi\|_{C_h}^2) \int_0^t \mu(s)ds + 4\left(\frac{11}{e^2} (MM_B)^4 b^2 M_R \right)
\]

\[
+ 4M^2 b^2 \left(2M_K + 2T R(Q) A_g (4(l^2 + l^2 M^2 \mathbb{E}\|\phi(0)\|_H^2 + \|\phi\|_{C_h}^2) \sup_{s \in J} m(s) \right)
\]

\[
+ 4M^2 b^2 (2M_b + 2\sqrt{L_b}) A_b (4(l^2 + l^2 M^2 \mathbb{E}\|\phi(0)\|_H^2 + \|\phi\|_{C_h}^2) \sup_{s \in J} n(s) \right)
\]

\[
\leq 4M^2 b^2 \left(1 + q' \right) \int_0^b \mu(s)ds + 4\left(\frac{11}{e^2} (MM_B)^4 b^2 M_R \right)
\]

\[
+ 4M^2 b^2 \left(2M_K + 2T R(Q) A_g q' \sup_{s \in J} m(s) \right) + 4M^2 b^2 (2M_b + 2\sqrt{L_b}) A_b q' \sup_{s \in J} n(s) = \tilde{C}.
\]

Thus for each \( z \in C_r \), \( \|\Phi_2z\|_H^2 \leq \tilde{C} \). \( \square \)

**Lemma 3.3.9.** Let assumptions \((H_9)\) - \((H_{15})\) holds. Then the set \( \{\Phi_2z : z \in C_r\} \) is an equicontinuous family of functions on \( J \).

**Proof.** Let \( 0 < \epsilon_1 < t < b \) and \( \delta_1 > 0 \) such that \( \|T_q(s_1) - T_q(s_2)\| < \epsilon_1 \) with \( |s_1 - s_2| < \delta_1 \) for \( s_1, s_2 \in J \). For \( z \in C_r \), \( 0 < |h_1| < \delta_1 \), \( t + h_1 \in J \), we have

\[
\mathbb{E}\|\Phi_2z(t + h_1) - \Phi_2z(t)\|_H^2
\]
Applying the assumptions \((H_9)-(H_{15})\) and Holder’s inequality, we get

\[
\mathbb{E}\|\Phi_2 z(t + h_1) - \Phi_2 z(t)\|^2_H \\
\leq 8M^2 \mathbb{E}\left\| \int_t^{t+h_1} f_2(t-s) f_1(\nu, \tilde{\phi}, z_\nu) dv ds \right\|^2_H \\
+ 8\mathbb{E}\left\| \int_t^{t+h_1} B u'(s) ds \right\|^2_H \\
+ 8M^2 \mathbb{E}\left\| \int_t^{t+h_1} g(\tau, \tilde{\phi}, z_\tau) d\omega(\tau) \right\|^2_H \\
+ 8\mathbb{E}\left\| \int_t^{t+h_1} \left[ \int_{-\infty}^{s} g(\tau, \tilde{\phi}, z_\tau) d\omega(\tau) \right] ds \right\|^2_H \\
+ 8\mathbb{E}\left\| \int_t^{t+h_1} \left[ \int_{-\infty}^{s} g(\tau, \tilde{\phi}, z_\tau) d\omega(\tau) \right] ds \right\|^2_H \\
+ 8\mathbb{E}\left\| \int_t^{t+h_1} \left[ \int_Z h(s, \tilde{\phi}, z_\phi, \eta, \eta) \tilde{N}(ds, d\eta) \right] \right\|^2_H \\
+ 8\mathbb{E}\left\| \int_t^{t+h_1} \left[ \int_Z h(s, \tilde{\phi}, z_\phi, \eta, \eta) \tilde{N}(ds, d\eta) \right] \right\|^2_H \\
\]

Hence for \(\epsilon_1\) sufficiently small, the right side of the above inequality tends to zero as \(\epsilon, h_1 \to 0\). Also the compactness of \(T_q(t), \ t > 0\) implies the continuity in the uniform operator topology. Hence the set \(\{\Phi_2 z : z \in C_r\}\) is equicontinuous.

**Lemma 3.3.10.** Let assumptions \((H_9)-(H_{15})\) holds. Then \(\Phi_2\) maps \(C_r\) into a precompact set in \(C_r\).

**Proof.** Let \(0 < t \leq b\) be fixed and \(\epsilon_1\) be a real number satisfying \(0 < \epsilon_1 < t\). Define the
operator $\Phi_2$ on $C_r$ by

$$\Phi_2(t) = -\int_0^{t-\epsilon_1} T_q(t-s)f_2(t-s)\left[\int_0^s f_1(\nu, \tilde{\phi}_\nu + z_\nu) d\nu\right] ds$$

$$+ \int_0^{t-\epsilon_1} T_q(t-s) Bu^*(s) ds$$

$$+ \int_0^{t-\epsilon_1} T_q(t-s)\left[\int_{-\infty}^s g(\tau, \tilde{\phi}_\tau + z_\tau) d\omega(\tau)\right] ds$$

$$+ \int_0^{t-\epsilon_1} T_q(t-s)\left[\int_{Z} h(s, \tilde{\phi}_s + z_s, \eta) \tilde{N}(ds,d\eta)\right]$$

But for $t > 0$, $T_q(t)$ is a compact operator. Hence the set $\{(\Phi_2 z)(t) : z \in C_r\}$ is precompact in $H$ for every $\epsilon_1 \in (0,t)$. Also for each $z \in C_r$, we have

$$\mathbb{E}\|\Phi_2z(t) - (\Phi_2 z)(t)\|^2_H$$

$$\leq 4\epsilon_1^2 \mathbb{E}\left\|\int_0^{t-\epsilon_1} T_q(t-s)f_2(t-s)\left[\int_0^s f_1(\nu, \tilde{\phi}_\nu + z_\nu) d\nu\right] ds\right\|^2_H$$

$$+ 4\epsilon_1^2 \mathbb{E}\left\|\int_0^{t-\epsilon_1} T_q(t-s) Bu^*(s) ds\right\|^2_H$$

$$+ 4\epsilon_1^2 \mathbb{E}\left\|\int_0^{t-\epsilon_1} T_q(t-s)\left[\int_{-\infty}^s g(\tau, \tilde{\phi}_\tau + z_\tau) d\omega(\tau)\right] ds\right\|^2_H$$

$$+ 4\epsilon_1^2 \mathbb{E}\left\|\int_0^{t-\epsilon_1} T_q(t-s)\left[\int_{Z} h(s, \tilde{\phi}_s + z_s, \eta) \tilde{N}(ds,d\eta)\right]\right\|^2_H$$

$$\mathbb{E}\|\Phi_2z(t) - (\Phi_2^{\epsilon_1, \delta_1} z)(t)\|^2_H$$

$$\leq 4\epsilon_1^2 M^2_a \int_{t-\epsilon_1}^{t} \left[\frac{b q^a}{q^a} M_1 \frac{t}{s} (t-s)^{q^a-1} (1 + q^a) \int_0^b \mu(s) ds + M_2 \frac{11}{q^a} M_2 M_B M_R\right.$$}

$$+ 2 M_K + 2 T r(Q) m(s) A_g q^r + (2 M_B + 2 \sqrt{M_B}) m(s) A_g q^r\] ds \to 0,$$

as $\epsilon_1, \delta_1 \to 0$.

Hence there are relatively compact sets arbitrarily close to the set $\{\Phi_2 z(t) : z \in C_r\}$ and hence the set $\{(\Phi_2 z)(t) : z \in C_r\}$ is also precompact in $C_r$. Hence from the Lemmas 3.3.6-
under the Theorem 3.5 and Arzela-Ascoli theorem satisfies all the conditions of Sadovskii’s fixed point theorem (Theorem 1.5.24), the fractional stochastic system (3.1) has a mild solution on $J$. \hfill \square

## 3.4 Approximate controllability

**Theorem 3.4.1.** Assume that the assumptions of Theorem 3.4 and $(H_0) - (H_{16})$ hold. Also the functions $f_1$, $g$ and $h$ are uniformly bounded on their respective domains. Further, if $R_q(t)$ and $T_q(t)$ are compact, then the fractional stochastic control system (3.1) is approximately controllable on $J$.

**Proof.** Let $x^*$ be a fixed point of the operator $\Phi^x$. By using the stochastic Fubini theorem, we get

$$x^*(b) = \hat{x}_b - \epsilon (\epsilon I + \psi_0^b)^{-1} \left[ \mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)d\omega(s) \right]$$

$$+ f_1(b, x_b) - \epsilon \int_0^b (\epsilon I + \psi_0^b)^{-1} AT_q(b - s) f_1(s, x_s^*) ds$$

$$+ \epsilon \int_0^b \int_0^s (\epsilon I + \psi_0^b)^{-1} T_q(b - s) f_2(b - s) f_1(s, x_s^*) ds$$

$$+ \epsilon \int_0^b (\epsilon I + \psi_0^b)^{-1} T_q(b - s) \left[ \int_{-\infty}^{s} g(\tau, x_s^*) d\omega(\tau) \right] ds$$

$$+ \epsilon \int_0^b \int_{\mathbb{Z}} (\epsilon I + \psi_0^b)^{-1} T_q(b - s) h(s, x_s^*, \eta) \tilde{N}(ds, d\eta)$$

$$+ \epsilon (\epsilon I + \psi_0^b)^{-1} \sum_{0 < t_k < t} T_q(b - t_k) I_k(x_{t_k}^*) + \epsilon (\epsilon I + \psi_0^b)^{-1} \sum_{0 < t_k < t} T_q(b - t_k) \mathcal{T}_k(x_{t_k}^*)$$

The properties of $f_1$, $g$ and $h$ implies that $\|g(\tau, x_s^*)\|^2 + \|h(s, x_s^*, \eta)\|^2 \leq K_1$ and $\|f_1(s, x_s^*)\|^2 \leq K_2$. Also the properties of $I_k$ and $\mathcal{T}_k$ implies that $\|I_k(x_{t_k}^*)\|^2 \leq K_3$ and $\|\mathcal{T}_k(x_{t_k}^*)\|^2 \leq K_4$. Then a subsequence $\{f_1(s, x_s^*), g(\tau, x_s^*), h(s, x_s^*, \eta), I_k(x_{t_k}^*), \mathcal{T}_k(x_{t_k}^*)\}$ converges weakly to, say, $\{f_1(s), g(s, \tau), h(s, \eta), I_k(x), \mathcal{T}_k(x)\}$. From the above equation,
we have

\[ \mathbb{E}\|x^*(b) - \hat{x}_b\|^2 \]
\[ \leq 15 \left\| \epsilon (I + \psi_h^b)^{-1} \left[ \mathbb{E}\tilde{x}_b - \mathcal{R}_q(b) \left( \phi(0) + f_1(0, \phi) + m_1(x_{i_1}, \ldots, x_{i_m})(0) \right) \right] \right\|^2 \]
\[ + 15\mathbb{E} \left\| \epsilon (I + \psi_h^b)^{-1} f_1(b, x_b) \right\|^2 \]
\[ + 15\mathbb{E} \left( \int_0^b \left\| \epsilon (I + \psi_h^b)^{-1} \phi(s) \right\|^2 ds \right) \]
\[ + 15\mathbb{E} \left( \int_0^b \left\| \epsilon (I + \psi_h^b)^{-1} AT_q(b-s) \right\|^2 ds \right) \]
\[ \times [f_1(s, x_s^\epsilon) - f_1(s)] \left\| ds \right\|^2 + 15\mathbb{E} \left( \int_0^b \left\| \epsilon (I + \psi_h^b)^{-1} AT_q(b-s) f_1(s) \right\|^2 ds \right) \]
\[ + 15\mathbb{E} \left( \int_0^b \int_0^s \left\| \epsilon (I + \psi_h^b)^{-1} T_q(b-s) f_2(b-s) [f_1(s, x_s^\epsilon) - f_1(s)] ds \right\|^2 \]
\[ + 15\mathbb{E} \left( \int_0^b \left\| \epsilon (I + \psi_h^b)^{-1} T_q(b-s) \left[ \int_{-\infty}^s [g(\tau, x_\tau^\epsilon) - g(s, \tau)] ds \right] \right\|^2 \]
\[ + 15\mathbb{E} \left( \int_0^b \left\| \epsilon (I + \psi_h^b)^{-1} T_q(b-s) \left[ \int_{-\infty}^s g(s, \tau) ds \right] \right\|^2 \]
\[ + 15\mathbb{E} \left( \int_0^b \int_Z \left\| \epsilon (I + \psi_h^b)^{-1} T_q(b-s) \left[ h(s, x_s^\epsilon, \eta) - h(s, \eta) \right] \tilde{N}(ds, d\eta) \right\|^2 \]
\[ + 15\mathbb{E} \left\| \sum_{0 \lt k \lt t} \epsilon (I + \psi_h^b)^{-1} T_q(b-t_k) [I_k(x_{t_k}) - I_k(x)] \right\|^2 \]
\[ + 15\mathbb{E} \left\| \sum_{0 \lt k \lt t} \epsilon (I + \psi_h^b)^{-1} T_q(b-t_k) I_k(x) \right\|^2 \]
\[ + 15\mathbb{E} \left\| \sum_{0 \lt k \lt t} \epsilon (I + \psi_h^b)^{-1} T_q(b-t_k) \bar{I}_k(x) \right\|^2 \]
\[ + 15\mathbb{E} \left\| \sum_{0 \lt k \lt t} \epsilon (I + \psi_h^b)^{-1} T_q(b-t_k) \bar{T}_k(x) \right\|^2 \]

Hence for all \( 0 \leq s \leq b \), the operator \( \epsilon (I + \psi_h^b)^{-1} \rightarrow 0 \) strongly as \( \epsilon \rightarrow 0 \) and moreover \( \| \epsilon (I + \psi_h^b)^{-1} \| \leq 1 \). Thus by the Lebesque dominated convergence theorem and the compactness of \( \mathcal{R}_q(t) \) and \( T_q(t) \) implies that \( \mathbb{E}\|x^*(b) - \hat{x}_b\|^2 \rightarrow 0 \). This gives the approximate controllability of the control system (3.1). \( \square \)

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3.5 Example

Consider the following fractional order partial stochastic differential system in the form

\[ cD_t^q [z(t, x) + \int_{-\infty}^{t} \int_{0}^{\pi} b_0(t - s, \eta, x) z(s, \eta) d\eta ds] = \frac{\partial^2}{\partial x^2} z(t, x) + \int_{0}^{t} (t - s)^\delta e^{-\gamma(t-s)} \frac{\partial^2}{\partial x^2} z(s, x) ds + \mu(t, x) + \int_{-\infty}^{t} a_0(s - t) z(s, x) d\omega(s) + \int_{Z} \eta \left( \int_{-\infty}^{t} a_1(s - t) z(s, x) ds \right) \tilde{N}(dt, d\eta), (t, x) \in J \times [0, \pi] \]

where \( cD_t^q \) is the Caputo fractional partial derivative of order \( 1 < q < 2 \), \( \mu : J \times [0, \pi] \rightarrow [0, \pi] \) is continuous. \( c_k, d_k \) are continuous for \( k = 1, 2, \ldots, n \) and \( C_i, i = 1, 2, \ldots, m \) are fixed numbers. Define the operator \( m_1 : C^m_k \rightarrow C_h \) by \( m_1(z_{t_1}, z_{t_2}, \ldots, z_{t_m})(x) = \sum_{i=0}^{m} C_i z(t_i + x) \) and \( \| m_1(\cdot) \|_H \leq M_{m_1} \). \( I_k \) and \( \tilde{I}_k : H \rightarrow H \) are appropriate functions. Let \( 0 = t_0 < t_1 < t_2 < \ldots < t_n < b \) be the given time points and the symbol \( \Delta \zeta(t) \) represents the jump of the function \( \zeta \) at \( t \) defined by \( \Delta \zeta(t) = \zeta(t^+) - \zeta(t^-) \). Also \( \omega(t) \) denotes a standard one-dimensional Wiener process defined on a stochastic basis \( (\Omega, \mathcal{F}_t, \mathbb{F}) \).

To write the above system \((3.20)\) into the abstract form \((3.1)\), we can choose the space \( H = U = L^2([0, \pi]) \). \( C_h \) is the phase space. Define \( A : D(A) \subseteq H \rightarrow H \) by \( Ax = x'' \), \( (Az)(x) = \frac{\partial^2 z(x)}{\partial x^2} \) with the domain \( D(A) = \{ x \in H : x, x' \text{ are absolutely continuous}, x'' \in H \text{ and } x(0) = x(\pi) = 0 \} \). The operator \( A \) is the infinitesimal generator of an analytic semigroup \( \rho(A) = C\{ -n^2 : n \in \mathbb{N} \} \) and for \( \vartheta \in (0, 1) \) and \( q \vartheta \in (\frac{1}{2}, \pi) \) there exists \( M_{q\vartheta} > 0 \) such that \( \| R(\lambda, A) \| \leq M_{q\vartheta} |\lambda|^{-1} \) for all \( \lambda \in \Sigma_{q\vartheta} \) and the fractional power \((-A)^\gamma : D((-A)^\gamma) \subset H \rightarrow H\) of \( A \) is given by \((-A)^\gamma x = \sum_{n=1}^{\infty} n^{2\gamma} \langle x, w_n \rangle w_n\),

\[ 0 \leq t \leq b, \quad 0 \leq x \leq \pi, \]

\[ z(\theta, x) = \phi(\theta, x) + \sum_{i=0}^{m} C_i z(t_i + x), \quad \theta \leq 0, \quad x \in [a, \pi], \quad t \in (-\infty, 0], \]

\[ [z(t_k^+) - z(t_k^-)]x = I_k(z(t_k))x = \int_{-\infty}^{t} c_k(t_k - s) z(s, x) ds, \quad k = 1, 2, \ldots, n, \]

\[ [z'(t_k^+) - z'(t_k^-)]x = \tilde{I}_k(z(t_k))x = \int_{-\infty}^{t} d_k(t_k - s) z(s, x) ds, \quad k = 1, 2, \ldots, n, \quad (3.20) \]
where \( D((-A)\gamma) = \{ x \in H : (-A)^\gamma x \in H \} \). Hence, \( A \) is sectorial of type and the properties \((P_1)\) hold. We also consider the operator \( f_2(t) : D(A) \subseteq H \to H, t \geq 0, f_2(t)x = t^\delta e^{-\tau t}Ax \) for \( x \in D(A) \). Moreover it is easy to verify that conditions \((P_2)\) and \((P_3)\) are satisfied with \( b_0(t) = t^\delta e^{-\tau t} \) and \( D = C_0^\infty([0, \pi]) \) is the space of infinitely differentiable functions that vanish at \( x = 0 \) and \( x = \pi \). Therefore \( (3.2) \) has an associated \( q \)-resolvent operators \( (R_q(t))_{t \geq 0} \) on \( H \). Define the operators \( f_1 : J \times C_h \to D((-A)^\gamma), g : J \times C_h \times \Omega \to H \), \( \eta : J \times C_h \times Z \to H \) and \( I_k, T_k : H \to H \) by

\[
\begin{align*}
  f_1(\psi)(x) &= \int_{-\infty}^0 \int_0^\pi b_0(s, \overline{\nu}, x)\psi(s, \overline{\nu})d\overline{\nu}ds, \\
  g(\psi)(x) &= \int_{-\infty}^0 a_0(s)\psi(s, x)d\omega(s), \\
  \eta(\psi)(x) &= \int_{-\infty}^t a_1(s)\psi(s, x)ds, \\
  I_k(\psi)(x) &= \int_{-\infty}^t c_k(s)\psi(s, x)ds, \quad k = 1, 2, \ldots, n, \\
  T_k(\psi)(x) &= \int_{-\infty}^t d_k(s)\psi(s, x)ds, \quad k = 1, 2, \ldots, n.
\end{align*}
\]

Then the functions \( f_1, g \) and \( \eta \) are continuous. The bounded linear operator \( B : U \to H \) is defined by \( Bu(t)(x) = \mu(t, x) \), \( 0 \leq x \leq \pi \). Moreover it is clear that \( f_1(J \times C_h) \subset D((-A)^{\frac{3}{2}}) \) and \( \|(-A)^{\frac{3}{2}}f_1\|_{L(C_h, H)} \leq L_{f_1} \). By Lemma \((3.2, 8)\), there exists \( M_\eta > 0 \) such that \( \|b_0(t)AT_q(t)\|_{L(D((-A)^{\frac{3}{2}}), H)} \leq \frac{M_\eta}{\gamma t^{1/2-\eta}} \) for \( \bar{\eta} \in (0, 1) \) and \( t > 0 \). Now we present a phase space \( C_h \). Let \( h(s) = e^{2s}, s < 0 \), then \( l = \int_{-\infty}^0 h(s)ds = \frac{1}{2} \). Let \( \|\phi\|_{C_h} = \int_{-\infty}^0 \|h(s)\sup_{-\infty}^0 \mathbb{E}\|\phi(0)\|^2 \frac{1}{2}ds \). Then \( (C_h, \|\cdot\|_{C_h}) \) is a Banach space. For \( (t, \varphi) \in J \times C_h \), we have \( \varphi(\theta)(\cdot) = \phi(\theta, \cdot) \) and \( \varphi'(\theta)(\cdot) = \xi(\theta, \cdot), (\varphi, \cdot) \in (-\infty, 0] \times [0, \pi] \). Let \( z(t)(\cdot) = z(t, \cdot) \). Let \( (\Omega, \mathfrak{F}, \mathbb{P}) \) be a complete probability space and \( \{ K(t) : t \in J \} \) is a Poisson point process taking values in the space \( K = [0, \infty) \) with a \( \sigma \)-finite intensity measure \( \lambda(dy) \). The Poisson counting measure \( \tilde{N}(dt, dy) \) is induced by \( K(\cdot) \) and the compensating martingale measure is denoted by \( \tilde{N}(dt, dy) := N(dt, dy) - dt\lambda(dy) \). Hence with the above choices, the system \((3.20)\) can be rewritten to the abstract form \((3.1)\) and all the conditions of Theorem \((3.3, 4)\) are satisfied. Thus there exists a mild solution for the system \((3.20)\). Moreover all
the conditions of Theorem 3.4.1 are satisfied and hence the system (3.20) is approximately controllable on $J$. 