CHAPTER I
PRELIMINARIES

1.1 Introduction

A graph G consists of a finite nonempty set $V = V(G)$ of p points together with a prescribed set X of q unordered pairs of distinct points of V. The graph G is denoted by $G = (V(G), E(G))$. Each pair $x = \{u, v\}$ of points in X is a line of G, and x is said to join u and v. The line $x = \{u, v\}$ is denoted by uv. Then the graph G is said to be of order p and size q, where $p = |V(G)|$ and $q = |E(G)|$. If the graph G is fixed then for simplicity it is sometimes written V for $V(G)$ and E for $E(G)$. A graph G that has order p and size q is called a (p, q) graph **[12]**.

A graph is said to be finite if the order p is finite. A graph is called simple if there is no loop (an edge that has both endpoints the same) or multiple lines (more than one line between two points). From now on, every graph mentioned in this thesis is a simple graph. There are many ways to represent a graph. However, traditionally graph is represented by a diagram. A dot represents a vertex and a curve, usually a line segment to represents an edge.

Graph theory is one of the most interesting and more application oriented topics in Mathematics which mainly evolved with the rise of computer age. Graph theory has application in diversified fields like computer technology, communication network, electrical network and social science. Graphs have been proved a powerful mathematical tool to explain structure of molecules and to explain flow of control with the help of graph structures.

Labeling of a graph G is an assignment of integers either to vertices or edges or both, subject to certain conditions. A dynamic survey on graph labeling is regularly updated by Gallian **[11]** and is published in Electronic Journal of Combinatorics.

Most graph labelings trace their origins to the one presented in 1967 by Rosa **[23].** One of these labelings, widely known as the graceful labeling, originated as a means of attacking the conjecture of Ringel **[22]**, which states that the complete graph of order $2m$+1 can be decomposed into m copies of a given tree of size m. Many types of labeling have been studied by several authors, some of them being, α-labeling, β-labeling, graceful labeling, harmonious labeling, sequential labeling, sigma labeling, sum labeling, sum square labeling, integral sum labeling, magic labeling, prime labeling, arithmetic labeling, geometric labeling, harmonic labeling, mean labeling, mean square labeling, cordial labeling, and so on **[11, 23]**.

Harary **[13, 14]** introduced the concepts of sum and integral sum graphs. Chen, Harary, Mary Florida, Somasundaram, Nicholas, and Vilfred **[7-9, 13, 14, 21, 32-39]** studied general properties of sum and integral sum graphs. We study some properties of sum and integral sum graphs.

In this chapter, we present the basic definitions and theorems on graphs, sum graphs G_n and integral sum graphs $G_{m,n}$ and some of earlier results on sum and integral sum graphs which are needed in the subsequent chapters.

For standard terminologies and notations in graph theory used in this thesis, we follow Harary **[12]**, Chartrand and Lesniak **[5]**, Bondy and Murty **[4]**, W. D. Wallis **[43]**, D.B. West **[44]**, and Gallian **[11]**. For number theoretic terminology we refer to Amird Aczel **[2].**

1.2 Basic definitions

Definition 1.2.1 [5] A *graph* G is a finite set of objects called *vertices* (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G *is* denoted by $V(G)$ while the *edge set* of G is denoted by $E(G)$.

The edge $e = \{u, v\}$ is said to join the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are adjacent vertices, while u and e are *incident as are* v and e. If two vertices are *not joined* by an edge then we say that they are *non-adjacent* vertices. If two distinct edges are incident with a common vertex, then they are said to be *adjacent edges*. It is convenient to denote an edge by uv or vu rather than by $\{u, v\}$.

Definition 1.2.2 [5] The cardinality of the vertex set of a graph G is called the order of G and is denoted by $p(G)$, or simply by p: that is $p = |V(G)|$. The cardinality of the edge set of G is called the size of G and is denoted by $q(G)$ or q: *that is*, $q = |E(G)|$.

A graph of *order* p and *size* q is called a (p, q) *graph*. A graph with exactly one vertex is called a *trivial graph* implying that the order of a *nontrivial graph* is *at least* 2. A $(1,0)$ *graph* is called a *trivial graph*. A graph with no edge is called *an empty graph*.

It is customary to define or describe a graph by means of a diagram in which each vertex is represented by a point and each edge $e = uv$ is represented by a line segment or curve joining the points corresponding to u and v.

Definition 1.2.3 [5] A *loop* is an edge that joins a vertex to itself. If more than one edge join two vertices then these edges are called parallel edges **[5]** or multiple edges **[12]**.

A *simple graph* is a graph having no loops or parallel edges (multiple edges).

Unless mentioned otherwise, graphs treated in this thesis are simple and finite.

Definition 1.2.4 A graph H is a s*ubgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$: in that case G is called a *supergraph* of H.

If H is subgraph of G then we say that H is contained in G as a subgraph and we write $H \subseteq G$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G.

Definition 1.2.5 If a subgraph H of a graph G has the same vertex set as G, then H is called a *spanning subgraph* of G.

Definition 1.2.6 A subgraph F of a graph G is called *vertex-induced subgraph* or simply *induced subgraph* of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well.

If S is a nonempty set of vertices of a graph G, then the subgraph of G induced by S is the *induced subgraph* with vertex set S. This induced subgraph is denoted by $\langle S \rangle$ or $G[S]$.

For any subset S of vertices of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S.

For a nonempty set X of $E(G)$, the subgraph $\langle X \rangle$ induced by X is the graph whose vertex set consists of those vertices of G incident with at least one edge of X and whose edge set is X. This subgraph is called an *edge-induced subgraph* of G. The induced subgraph $\langle X \rangle$ is also denoted by $G[X]$.

Definition 1.2.7 [12] The *removal of a vertex* v_i from a graph G results in that subgraph $G - v_i$ of G consisting of all vertices of G except v_i and all edges not incident with v_i. Thus $G - v_i$ is the maximal subgraph of G not containing v_i. *The removal of an edge* x_i from G yields the spanning subgraph $G - x_i$ containing all edges of G except x_i. Thus $G - x_i$ is the maximal subgraph of G not containing x_i . The *removal of a set of vertices or edges* from G is defined by the removal of single elements in succession. More generally, if X is a set of edges of G, then $G - X$ is the spanning subgraph of G with $E(G - X) = E(G) - X$.

The induced subgraph $\langle V(G) - \{v\} \rangle$ is denoted by $G - v$: it is a subgraph of G obtained by the removal of v and all edges incident with v.

If $e \in E(G)$, the spanning subgraph with edge set $E(G) - \{e\}$ is denoted $G - e$: it is the subgraph of G obtained by the removal of e. For an edge e of a graph G, the subgraph $G - e$ is a spanning subgraph consisting of all edges of G except e.

Definition 1.2.8 If v_i and v_j are not adjacent in G, the addition of edge $v_i v_j$ results in the smallest supergraph of G containing the edge $v_i v_j$ denoted by $G + v_i v_j$.

Definition 1.2.9 A *walk* of a graph G is an alternating sequence of vertices and edges v_0, e_1, v_1, e_2, . . . , v_{n-1}, e_n, v_n, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it.

This walk joins v_0 and v_n and may also be denoted by $v_0 v_1 . . . v_{n-1} v_n$: is sometimes called a $v_0 - v_n$ walk. It is *closed* if $v_0 = v_n$ and is *open* otherwise. It is a *trail* if all the lines are distinct, and a *path* if all the points (and thus necessarily all the lines) are distinct. If the walk is closed, then it is a *cycle* provided its n points are distinct and $n \geq 3$. The *length of a walk* $v_0 v_1 . . . v_{n-1} v_n$ is n, the number of occurrences of edges in it. The number of edges in a cycle is said to be the *length of the cycle*. The graph consisting of a cycle with n vertices is denoted by C_n and hence C_n is of length n – 1. C_3 is called a *triangle.* The graph consisting of a *path* with n vertices is denoted by P_n. P_n has n – 1 lines and hence is said to be of length n – 1.

Definition 1.2.10 A graph is *connected* if every pair of vertices is joined by a path. A *maximal connected subgraph* of G is called a *connected component* or simply a *component* of G.

A graph is called *disconnected* if it is not connected. Thus a disconnected graph has at least two components.

Definition 1.2.11 The *degree* of a vertex v in a graph G is defined to be the number of edges of G incident with v and is denoted by $deg_G\ v$ or deg (v) or simply $d(v)$.

In a (p, q) graph, $0 \leq$ deg $(v) \leq p - 1$, for each vertex v. The minimum degree among the vertices of G is denoted by *min deg* G or $\delta\,(G)$ while $\Delta(G)$ = *max deg* G is the largest such number.

If $\delta(G) = \Delta(G) = r$, then all vertices have the same degree. A graph G is *regular of degree* r if every vertex of G has degree r. Such a graph is called a *r-regular graph* and we write $deg\ G = r$. A regular graph of degree 0 has no edge at all. If G is regular of degree 1, then every component contains exactly one edge and if it is regular of degree 2, then every component is a cycle, and conversely. A 3-regular graph is called a *cubic graph*.

Definition 1.2.12 A vertex of degree 0 in G is called an *isolated vertex* and a vertex of degree 1 is called a *pendant vertex* or a *leaf* or an *end vertex* of G.

Definition 1.2.13 A vertex adjacent to a pendant vertex in a graph G is called a *support*. An edge $e = uv$ is said to be an *isolated edge* if $\deg(u) = \deg(v) = 1$.

Definition 1.2.14 An edge incident with a pendant vertex is called a *pendant edge*.

Definition 1.2.15 The *neighborhood* $N(u)$ of a vertex u in a graph G is the set of all vertices v of G that are adjacent to u.

Definition 1.2.16 A graph G is *complete* if every two distinct vertices of G are adjacent.

A complete graph of order n is a regular graph of degree $n-1$ and is denoted by K_n. Therefore, K_n has the maximum possible size for a graph with n vertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices in K_n is $\binom{n}{2}$ and so the size of K_n is $\frac{n(n-1)}{2}$. K_3 is called the *triangle*. Note that $K_3 = C_3$.

Definition 1.2.17 A *clique* in a graph is a set of pair-wise *adjacent* vertices and an *independent set or a stable set* in a graph is a set of pair-wise non-adjacent vertices.

Definition 1.2.18 [44] A graph G is a *split graph* if its vertices can be partitioned into a clique and a stable set or independent set.

Definition 1.2.19 The complement $\overline{G}$ or G^c of a graph G is that graph whose vertex set is $V(G)$ and such that for each pair u, v of vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.

It is clear that $E(G) + E(G^c) = \binom{n}{2}$. The complement $\overline{K_n}$ or K_n^c of the complete graph K_n has n vertices and no edges and is referred to as the *empty graph* of order n.

Definition 1.2.20 [5] A graph G_1 is isomorphic to a graph G_2 if there exists a one-one mapping ϕ, called an isomorphism, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency, that is, $uv \in V(G_1)$ if and only of $\phi(u)\phi(v) \in V(G_2)$. If G_1 is isomorphic to a graph G_2, then we say G_1 and G_2 are *isomorphic* or *equal* and write $G_1 \cong G_2$ or $G_1 = G_2$.

Definition 1.2.21 [5] An automorphism of a graph G is an isomorphism between G and itself.

Definition 1.2.22 A graph is *self-complementary* if $G = \overline{G}$. If G is a self-complementary graph of order n then its size is $\frac{n(n-1)}{4}$. Since only one of n or n–1 is even, either $4 \mid n$ or $4 \mid (n-1)$, then either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Definition 1.2.23 A *bigraph* (or a *bipartite graph*) is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. (V_1, V_2) is called a bipartition of G. If further G contains every edge joining a vertex of V_1 to a vertex of V_2 then G is called a *complete bigraph*.

A complete bipartite graph with partite sets V_1 and V_2, where $|V(G_1)| = r$ and $|V(G_2)| = s$, is denoted by $K_{r,s}$. Clearly, $K_{r,s}$ is of order $r + s$ and size rs.

Definition 1.2.24 A star is a complete bigraph $K_{1,n}$.

Definition 1.2.25 For $n \geq 4$, the wheel W_n is defined to be the graph $K_1 + C_{n-1}$.

Definition 1.2.26 Two graphs G_1 and G_2 are called *vertex disjoint graphs* if $V(G_1) \cap V(G_2) = \varphi$.

Definition 1.2.27 [5] Let G_1 and G_2 be two vertex disjoint graphs. A *union* of G_1 and G_2 denoted by $G = G_1 \cup G_2$, is the graph that consists of $V(G) = V(G_1) \cup V(G_2)$, and $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = k \,.\, H$ or simply kH.

The *intersection,* $G_1 \cap G_2$, of G_1 and G_2, is defined analogously.

Definition 1.2.28 The *join* of two graphs G_1 and G_2 denoted by $G = G_1 + G_2$ is the graph that has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. Using the *join* operation, we see that $K_{r,s} = \overline{K}_r + \overline{K}_s$.

Definition 1.2.29 *One point union* of any number of connected graphs is obtained by identifying one vertex from each graph.

Definition 1.2.30 [11] When k copies of C_n share a common edge, it will form an *n-gon book of k pages* and is denoted by $B(n, k)$. The common edge is called the *spine* or *base of the book.*

Definition 1.2.31 [43] A *triangular book* $B(3, n)$ or $B_{3,n}$ consists of n triangles with a common edge and it can be described as $ST(n) + K_1 = P_2 + (n.K_1)$ $(or\ K_2 + (n.K_1))$ where $ST(n)$ denotes the star with *n* leaves.

Let us denote the triangular book $B(3, n)$ with the spine (u, v) by $TB_n(u, v) = P_2$ $(u, v) + n.\ K_1$. Note that $TB_0 = K_2$ represent *a book without pages* or a *trivial book.*

Definition 1.2.32 A *fan graph* $\boldsymbol{F_{n-1}}$ is the graph obtained by taking $\boldsymbol{n-3}$ concurrent chords at a vertex in a cycle $\boldsymbol{C_n}$, $n \geq 3$. The vertex at which all the $\boldsymbol{n-3}$ chords are concurrent is called the *apex vertex*.

The fan graph $\boldsymbol{F_n}$ can be described as $\boldsymbol{F_n} = \boldsymbol{P_n} + \boldsymbol{K_1}$ where $\boldsymbol{P_n}$ is a path on $\boldsymbol{n}$ vertices, $\boldsymbol{n} \geq \boldsymbol{2}$.

Definition 1.2.33 [17] Let G_1 and G_2 be subgraphs of G. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$ (a vertex of G) then we say that G is the *vertex-amalgamation* of G_1 and G_2 at vertex *v*, denoted by $G = G_1 \vee^1_{\{v\}} G_2$. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{u, v\}$ (two distinct vertices of G), then we say that G is the 2-vertex-amalgamation of G_1 and G_2 at vertices *u* and *v*, denoted by $G = G_1 \vee^1_{\{u,v\}} G_2$.

Definition 1.2.34 [17] Given a collection of graphs G_1, G_2, …,G_k and for each graph G_i a fixed vertex $v_i \in V(G_i)$, the amalgamation of G_1, G_2, …, G_k, denoted by Amal $\{G_i^{(k)}, v_i\}$, is the graph obtained by taking the union of all G_i's and identifying $v_1, v_2, \ldots, v_k$.

If $G_1, G_2, \ldots, G_k$ are subgraphs of G and $G = G_1 \cup G_2 \cup \ldots \cup G_k$ and $G_1 \cap G_2 \cap \ldots \cap G_k = \{v\}$ (a vertex of G) then we say that G is the vertex-amalgamation or simply an amalgamation of $G_1, G_2, \ldots, G_k$ at the vertex v, denoted by G = Amal$\{G_i^{(k)}, v\}$,. Alternatively we may call such graphs as *one point union* of G_i's.

Definition 1.2.35 [17] If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{\text{e}\}$ (an edge of G), then we say that G is the *edge-amalgamation* of G_1 and G_2 on edge e, denoted by $G = G_1 \vee_{\{e\}}^2 G_2$.

Definition 1.2.36 [17] Given a collection of graphs $G_1, G_2, \ldots, G_k$ and for each graph G_i a fixed edge $e_i \in V(G_i)$, the graph obtained by the identification of e_i's is called an edge-amalgamation of $G_1, G_2, \ldots, G_k$.

If $G_1, G_2, \ldots, G_k$ are subgraphs of G and $G = G_1 \cup G_2 \cup \ldots \cup G_k$ and $G_1 \cap G_2 \cap \ldots \cap G_k = \{e\}$(an edge of G), then we say G is an edge-amalgamation of $G_1, G_2, \ldots, G_k$ on the edge e, denoted by Amal $\{G_i^{(k)}, e\}$.

Definition 1.2.37 A graph G is said to be *decomposable* into the subgraphs, $H_1, H_2, \ldots, H_k$, if each subgraph H_i, $1 \le i \le k$, has no isolated vertices and $\{E(H_1), E(H_2), \ldots, E(H_k)\}$ is a partition of $E(G)$.

Note that these subgraphs H_i are not required to be spanning subgraphs of G. If each subgraph H_i is a spanning subgraph of G, then the decomposition is a *factorization* of G. If each $H_i \cong H$ for some graph H, then the graph G is *H-decomposable* and the decomposition is an *H-decomposition*.

Definition 1.2.38 If a graph G has a *factorization* into subgraphs $H_1, H_2, \ldots, H_k$, then, by definition, each subgraph (factor) H_i, $1 \le i \le k$, has no isolated vertices and is required to be a spanning subgraph of G.

A *factor* of a graph is a spanning subgraph of G. A *k-factor* is a spanning k-regular subgraph. Thus, a 1-factor is a spanning 1-regular subgraph of G.

Alavi [1] introduced the concept of *Ascending Subgraph Decomposition* (*ASD*).

Definition 1.2.39 [1] *Ascending Subgraph Decomposition* (*ASD*) is defined as the decomposition of graph G with size $\binom{n+1}{2}$ into *n* subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = i$ for $1 \leq i \leq n$.

Nagarajan **[20]** generalized *ASD* into (*a*, *d*) - *Ascending Subgraph Decomposition* ((*a*, *d*)-*ASD*).

Definition 1.2.40 [20] (*a*, *d*) - *Ascending Subgraph Decomposition* ((*a*, *d*) - *ASD*) of *G*, with size $(2a + (n-1)d)\,n/2$ is defined as the decomposition of G into *n* subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices in such a way that each G_i is isomorphic to a proper subgraph of G_{i+1}, $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$ for $1 \leq i \leq n$.

A graph labeling is an injection from the vertex set (or edge set or both) of the given graph into a set of numbers (real numbers).

The concept of *sum graph* was discovered by Harary **[13]** at the Nineteenth Southeastern conference on Combinatorics at Baton Rouge in 1988. Since then a variety of research has been done on these family of graphs. For a detailed survey on sum graphs and integral sum graphs, please refer to the Dynamic Survey on Graph Labeling by Gallian **[11]**.

Definition 1.2.41 A graph G is a *sum graph or* $\sum N$*-graph* if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices *u* and *v* is also a label on some vertex in G.

If G is a properly labeled sum graph, then the vertex with the highest label in G cannot be adjacent to any other vertex. Thus every sum graph must contain isolated vertex or vertices. A connected graph cannot be a sum graph.

Definition 1.2.42 The *sum number* of a graph G, denoted by $\sigma(G)$, is defined as the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph.

If G is not a sum graph, adding a finite number of *isolated vertices* to it always yields a sum graph and the *sum number*, $\sigma(G)$ of G is the smallest number of isolated vertices so added. If G is a *sum graph* with respect to a *label set* S, then G can be denoted as $G = G^+(S)$ and in that case $\sigma(G) = 0$.

Definition 1.2.43 A labeling that makes G together with $\sigma(G)$ isolated vertices a sum graph is called an *optimal sum graph labeling*. A labeling that realizes $G + \sigma(G) \cdot K_1$ as a sum graph is said to be an *optimal sum graph labeling*.

Since the time of introduction of sum labeling of graphs by Harary **[13]**, the problem of finding an optimal labeling for a family of graphs has been shown to be difficult, even for fairly simple graphs.

Definition 1.2.44 In the sum labeling of a graph, vertices whose labels correspond to an edge *uv* are said to be *working vertices*. It has been realized that certain graphs can only be labeled in such a way that all the working vertices are also isolates. Such graphs are called *exclusive* sum graphs otherwise it is called *inclusive*.

In 1994, Harary **[14]** introduced integral sum graph. An *integral sum graph* is also defined just as sum graph, the difference being that the label set is *a set of integers* instead of natural numbers.

Definition 1.2.45 [14] A graph G is an *integral sum graph or ∑Z-graph* if the vertices of G can be labeled with distinct integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label on some vertex in G.

If G is not an integral sum graph, adding a finite number of *isolated vertices* to it always yields an integral sum graph.

Definition 1.2.46 The *integral sum number* $\zeta(G)$ is the smallest non-negative integer s such that $G \cup s.K_1$ is an integral sum graph.

By definition it is clear that $\zeta(G) \leq \sigma(G)$ for all graph G, and G is an *∑Z-graph if and only if* $\zeta(G) = 0$.

Harary **[13]** introduced a family of sum graphs G_n over a set of positive integers $\{1, 2, \ldots, n\}$.

Graphs G_1, G_2, G_{3}, G_4 and G_9 are illustrated in Figure 1.1 and in Figure 1.2.

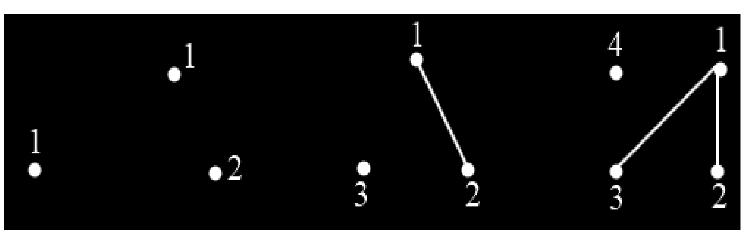

Fig. 1.1. $\boldsymbol{G_1}$ $\boldsymbol{G_2}$ $\boldsymbol{G_3}$ $\boldsymbol{G_4}$

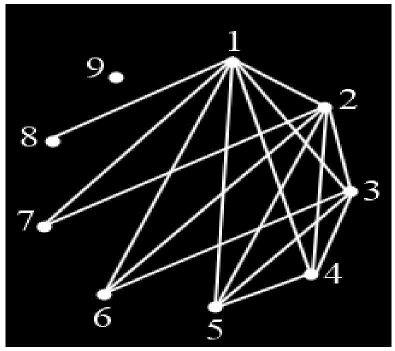

Fig. 1.2. G_9.

Harary **[14]** also introduced a family of integral sum graphs $G_{n,n}$ over a set of integers $[-n, n]$ where $[-n, n]$ denotes the set of integers $S = \{-n, -n+1, \dots, -2, -1, 0, 1, 2, \dots, n\}$. See Figure 1.3.

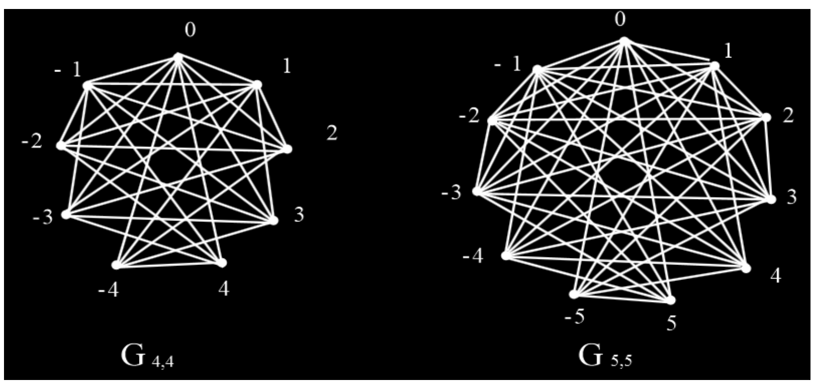

Fig. 1.3.

Harary **[14]** specified the structure of the graph $G_{n,n}$ in terms of G_n as $G_{n,n} = K_1 + (G_n + G_n)$. In this equation, the K_1-term is realized by the integer 0, which obviously is adjacent to all other elements of S. The two G_n-terms are $G^+(\{1, 2, 3, \dots, n\})$ and $G^+(\{-1, -2, -3, \dots, -n\})$. Vilfred **[34]** generalized $G_{n,n}$ to integral sum graph, $G_{m,n}$, defined maximal integral sum graph and obtained different properties, $m, n \in N \cup \{0\}$.

Definition 1.2.47 [34] The integral sum graph $G_{m,n}$ is defined as $G_{m,n}= G^{+}(S)$ where $S = [-m, n]$, $m, n \in N \cup \{0\}$.

For m, $n \in N$, $G_{m,n}$ is Hamiltonian and $G_{0,n}$ is non-Hamiltonian **[34]**. Clearly $G_{m,n} = K_1 + (G_m + G_n) = G_{n,m}$ and $G_{m,n}$ is an integral sum graph of order $m+n+1$. Also it is clear that the integral sum graphs, $G_{m,n}$ and $G_{n,m}$ without vertex labels, are the same.

Definition 1.2.48 [37] Let $\boldsymbol{G}$ be a connected graph with n vertices. Define $\boldsymbol{V}_{\Delta}(\boldsymbol{G}) = \{x \in \boldsymbol{V}(\boldsymbol{G}) \,/\, \boldsymbol{deg}(\boldsymbol{x}) = |\boldsymbol{V}(\boldsymbol{G})| - 1\}$.

Definition 1.2.49 [35] A graph G is an *anti-sum graph* or *anti-ΣN-graph* if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is not a vertex label in G.

An *anti-integral sum graph* or *anti-ΣZ-graph* is also defined just as anti-sum graph, the difference being that the labels may be any distinct integers.

Clearly, f is an anti-integral sum labeling of G if and only if f is an integral sum labeling of G^c. Thus, we call anti-sum labeling of G as *complementary sum labeling* of G and anti-integral sum labeling as *complementary integral sum labeling.*

Definition 1.2.50 [34] Two graphs G and H are said to be *comparable* if and only if either G is a subgraph of H or H is a subgraph of G. Otherwise, they are called *non-comparable graphs.*

Comparison of sum graphs or *integral sum graphs of same order* means comparison of graphs of same order that are constructed using the definition of sum labeling or integral sum labeling but after the construction the graphs are considered without the vertex labels for comparison until otherwise it is mentioned.

Whenever, we say integral sum graph $G = G^+(S)$ without the vertex labels, it means after construction of graph G using the definition of integral sum labeling on the label set S, the graph G is considered without the vertex labels.

Definition 1.2.51 [34] An integral sum graph (sum graph) G is said to be a *maximal integral sum graph (maximal sum graph)* if G is not a spanning sub-graph of any other integral sum graph (sum graph), without labels.

The *maximal integral sum graph(s)* of a given order, say n, is a maximal integral sum graph of order n with the maximum number of edges. Clearly, G_n is the maximal sum graph of order n, $n \in N$.

Chen, Mary Florida, Nicholas, Somasundaram, and Vilfred [7-9, 21, 32-39] studied several properties of $G_{m,n}$. Chen **[7]** has given some properties of integral sum labelings of graphs G with $\Delta(G) < |V(G)| - 1$ whereas Nicholas, Somasundaram and Vilfred **[21]** provided some general properties of connected integral sum graphs G with $\Delta(G) = |V(G)| - 1$. Vilfred and Florida **[34]** defined and investigated properties of maximal integral sum graphs. Nicholas, Somasundaram and Vilfred **[21]** have shown that connected integral sum graphs G other than K_3 with the property that G has exactly two vertices of maximum degree are unique and that a connected integral sum graph G other than K_3 can have at most two vertices of degree $|V(G)| - 1$. Vilfred and Nicholas **[38, 39]** have shown that the following graphs are integral sum graphs; banana trees, the union of any number of stars, fans $P_n + K_1$ ($n \geq 2$). Dutch windmills $K_3(m)$, and the graph obtained by starting with any finite number of integral sum graphs $G_1, G_2, \ldots, G_n$ and any collections of n vertices with $v_i \in G_i, 1 \leq i \leq n$ and creating a graph by identifying $v_1, v_2, \ldots, v_n$. The same authors also proved that $G+v$ where G is a union of stars is an integral sum graph.

We list out here some of the results already proved on sum and integral sum graphs.

Theorem 1.2.52 [7] Let f be an integral sum labeling of a non-trivial graph G of order n. Then, $f(x) \neq 0$ for every vertex x of G if and only if the maximum degree $\Delta(G) < n - 1$.

Theorem 1.2.53 [37] Let f be an integral sum labeling of a connected graph G of order n with at least two vertices of degree $n - 1$ each, $n \geq 3$. If $y \in V(G)$ such that $d(y) = n - 1$ and $f(y) \neq 0$, then for every vertex $v \in V(G), f(v) = k.f(y)$, where $k \in \{0, 1, -1, -2, \ldots, -n-2\}$.

Theorem 1.2.54 [37] Every integral sum graph G of order n, except K_3, has at the most two vertices of degree $n - 1$.

Theorem 1.2.55 [37] For every $n \geq 4$, there is an integral sum graph of order n with exactly two vertices of degree $n - 1$. This graph is unique up to isomorphism and is denoted by $G_{\Delta n}$.

Theorem 1.2.56 For any graph G_n, $|E(G_n)| = \frac{1}{2}(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor)$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Without loss of generality, let $m \leq n$ for the integral sum graph $G_{m,n}$.

Theorem 1.2.57 [37] For $n \in \mathrm{N}$, $G_{1,n} \cong G_{\Delta(n+2)}$ and for $2 \leq n$, $G_{1,n}$ has exactly two vertices of degree $n+1$. For $2 \leq m \leq n$, $G_{0,n}$ and $G_{m,n}$ contain exactly one vertex of degree n and $m+n$, respectively.

Theorem 1.2.58 [34] For $3 \leq m + n$, $|E(G_{m,n})| = \frac{1}{4}(m^2+n^2+3(m+n)+4mn) - \frac{1}{2}(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$ where $\lfloor x \rfloor$ denotes the floor of $x, m, n \in N \cup \{0\}$. In particular, $|E(G_{0,n})| = \frac{n(n+3)}{4} - \frac{1}{2}(\lfloor \frac{n}{2} \rfloor)$, $|E(G_{n,n})| = \frac{3n(n+1)}{2} - \lfloor \frac{n}{2} \rfloor$ and $|E(G_{n-1,n})| = \frac{n(3n-1)}{2}$, $n \in \mathrm{N}$.

Theorem 1.2.59 [37] Let G be any connected graph and $x, u, w \in V(G)$. If f is an *∑Z-labeling* with $f(x) = -f(u)$, then for each $v \in N(u)$ such that $f(u) + f(v) = f(w), w = x$ or $xw \in E(G)$.

Theorem 1.2.60 [37] Let f be an integral sum labeling of a graph G of order n, $4 \leq n$. If G has atleast two vertices of degree $n-1$ each, then, (1) there exists a vertex x of degree $n-1$ such that $f(x) = 0$ and (2) for every vertex $y \neq x$, and of degree $n-1$, there exists a vertex y' with degree less than $n-1$ such that $f(y) + f(y') = 0$.

Theorem 1.2.61 [37] For $n \geq 3$, the graph $G_{\Delta n}$ is Hamiltonian.

Theorem 1.2.62 [37] $|E(G_{\Delta n})| = \begin{cases} \frac{1}{4}(n^2+2n-4) \text{ when n is even} \\ \frac{1}{4}(n^2+2n-3) \text{ when } n \text{ odd} \end{cases}$

Theorem 1.2.63 [34] For $m, n \in N$, $G_{m,n}$ is Hamiltonian and $G_{0,n}$ is non-Hamiltonian.

Theorem 1.2.64 [34] (i) $|E(G_{1,3})| = |E(G_{2,2})|$; for $2 \leq m$, $|E(G_{1,4m-1})| < |E(G_{2m,2m})| = |E(G_{2m-1,2m+1})|$; for $1 \leq m$, $|E(G_{1,4m})| < |E(G_{2m,2m+1})|$, $|E(G_{1,4m+1})| < |E(G_{2m,2m+2})| <$

|E($G_{2m+1,2m+1}$)| and (ii) for $1 \leq m$, $G_{2m,2m} \neq G_{1,4m-1}$, $G_{2m,2m} \neq G_{2m-1,2m+1}$, $G_{2m+1,2m+1} \neq G_{1,4m+1}$ and $G_{2m+1,2m+1} \neq G_{2m,2m+2}$.

Lemma 1.2.65 [34] Let $k \leq n-k, r \leq n-r, k \neq r$ and $k, n, r \in N$. Then, $K_{k,n-k}$ and $K_{r,n-r}$ are non-comparable graphs.

For example, graphs K_m and K_n are comparable graphs for $m, n \in N$. But graphs $K_{2,5}$ and $K_{3,4}$ are non-comparable graphs.

Theorem 1.2.66 [35] For 1 ≤ r < n, $G_r \cup G^+([r+1, n])$ is a spanning subgraph of G_n which is a spanning subgraph of G_1+$G^+([2, n])$.

Theorem 1.2.67 [35] For 4 ≤ n and 2 ≤ r < n, *labeled graphs* G_n and G_r+$G^+([r+1, n])$ are non-comparable.

Theorem 1.2.68 [35] For $n \geq 5$, G_n and G_2+$G^+([3, n])$, *without the vertex labels,* are non-comparable and for $n \geq 4$, $|E(G_2 + G^+([3, n]))| = |E(G^+([3, n]))| + 2(n-2) = |E(G_n)| + 2$.

Theorem 1.2.69 [34] Let $k \leq n-k, r \leq n-r$ and $k, r, n \in N$. Then, $G_{k,n-k}$ and $G_{r,n-r}$, without the vertex labels, are comparable if and only if $k = r$.

Corollary 1.2.70 [34] Let $k \leq n-k, r \leq n-r$ and $k, n, r \in N$. Then, for $k \neq r$, $G_{k,n-k}$ and $G_{r,n-r}$, without the vertex labels, are non-comparable graphs.

Theorem 1.2.71 [34] $G_{0,n}$ is a spanning subgraph of $G_{1,n-1}$, without the vertex labels, n∈ N.

Clearly, if integral sum graph G^{+}(S) is such that 0∈S, then the vertex with label 0 has the maximum degree $|V(G^{+}(\mathrm{S}))|-1$ in G^{+}(S). Hence, *the maximal integral sum graph contains* 0 *as a vertex label.*

Theorem 1.2.72 [34] Let S $= [-m, m], 2 \le m, x, y \in Z$ *and* $x, y \notin S$. Then, $|\mathrm{E}(G^{+}(S \cup \{x, y\}))|$ is maximum when $\{x, y\} = \{m+1, -m-1\}$.

Theorem 1.2.73 [34] For $m \in N$, $G_{m,m}$ is *the maximal* integral sum graph of order $2m+1$ and for $2 \le m$, $G_{m-1,m}$ is *the maximal* integral sum graph of order $2m$.

Corollary 1.2.74 [34] For $m \in N$, *the maximal* integral sum graph of order n are (i) $G_{2m,2m}$ and $G_{2m-1,2m+1}$ when $n = 4m+1$; (ii) $G_{2m,2m+1}$ when $n = 4m+2$; (iii) $G_{2m+1,2m+1}$ when $n = 4m+3$; and (iv) $G_{2m-1,2m}$ when $n = 4m$.

Theorem 1.2.75 [34] For $3 \le n$, $G_{0,n}$ is not a maximal integral sum graph of order $n+1$.

Theorem 1.2.76 [34] For $r \le n-r$ and $r, n-r \in N$, $G_{r,n-r}$ is *a maximal* integral sum graph of order $n+1$.

Theorem 1.2.77 [35] For every sum graph G of order n, there exists a unique edge-disjoint anti-sum graph H of the same order such that $G \cup H \cong K_n$, $n \ge 2$.

Theorem 1.2.78 [35] For every integral sum graph G of order n, there exists a unique edge-disjoint anti-integral sum graph H of the same order such that $G \cup H \cong K_n$, $n \ge 2$.

Theorem 1.2.79 [35] For $1 \le r < n$, *labeled graph* G_n^c is a spanning subgraph of G_r^c $+(G^{+}([\mathrm{r}+1, \mathrm{n}]))^c$.

Theorem 1.2.80 [35] For n ≥ 4 and n > r ≥ 2, *labeled graphs* G_n^c and $G_r^c \cup (G^+([r+1,n]))^c$ are non-comparable.

Theorem 1.2.81 [35] For $n \geq 5$, G_n^c and $G_2^c \cup (G^+([3,n]))^c$, *without the vertex labels,* are non-comparable.

Theorem 1.2.82 [35] For $n \geq 2$,

(i) $K_{2n} \cong (G_{2n-1} \cup G_{2n-1}^c) \cup K_1 (2n) \cup (\cup_{i=1}^{n-1}((n-i,2n) \cup (n+i,2n))) \cup (n,2n)$;

(ii) $K_{2n} \cong G^+([1,n-1] \cup [n+1,2n]) \cup (G^+([1,n-1] \cup [n+1,2n]))^c \cup K_1 (n) \cup (\cup_{i=1}^{n-1}((n,n-i) \cup (n,n+i))) \cup (n,2n)$;

(iii) $K_{2n+1} \cong G_{2n} \cup G_{2n}^c \cup K_1$ (2n+1) $\cup (\cup_{i=1}^{n}((2n+1,n+1-i) \cup (2n+1,n+i)))$

(iv) $K_{2n+1} \cong G^+([1,n] \cup [n+2,2n+1]) \cup (G^+([1,n] \cup [n+2,2n+1]))^c \cup K_1$ (n+1) $\cup (\cup_{i=1}^{n}((n+1,n+1-i) \cup (n+1,n+1+i)))$.

Theorem 1.2.83 [35] For n ≥ 2,

(i) $K_{2n} \cong G^+([1,n-1] \cup [n+2,2n]) \cup (G^+([1,n-1] \cup [n+2,2n]))^c \cup P_2$ (n,n+1) $\cup (\cup_{i=1}^{n-1}((n,n-i) \cup (n,n+1+i) \cup (n+1,n-i) \cup (n+1,n+1+i)))$ and

(ii) $K_{2n+1} \cong G^+([1,n-1] \cup [n+2,2n+1]) \cup (G^+([1,n-1] \cup [n+2,2n+1]))^c \cup P_2$ (n,n+1) $\cup$ (n,n+2) $\cup$ (n+1,n+2) $\cup (\cup_{i=1}^{n-1}((n,i) \cup (n,2n+2-i) \cup (n+1,i) \cup (n+1,2n+2-i)))$.

Theorem 1.2.84 [35] For n ≥ 2,

(i) $K_{2n} \cong P_2(n, n+1) \cup (\cup_{j=2}^{n}(P_2(n-j+1, n+j) \cup (\cup_{i=1}^{2j-2}((n-j+1, n-j+i+1) \cup (n+j, n-j+i+1)))))$ and

(ii) $K_{2n+1} \cong K_1(2n+1) + (\cup_{j=1}^{n}(P_2(n+1-j, n+j) \cup (\cup_{i=1}^{2j-1}(n+j, n-j+i) \cup (n-j, n-j+i))))$.