Chapter 5

Nonclassicality and decoherence of photon-added squeezed coherent Schrödinger kitten states in a Kerr medium

5.1 Photon-added squeezed coherent Schrödinger cat state

Adding or subtracting photons from the field is useful in the context of engineering quantum states which would be useful in sending quantum information. Using a single photon interferometer towards realizing the coherent superpositions of two alternate sequences of photon addition and subtraction, an experimental test of the bosonic commutation relation between the annihilation and creation operators has been implemented [86,87].

The possibility of arbitrarily adding and subtracting photons to and from a light field may allow us to engineer customized fruitful quantum states. Using a conditional addition or subtraction of photons on the Gaussian entangled beams an enhancement of the entanglement has been observed [88,89].

We study the evolution of the initial ($t = 0$) multiple ($\kappa$) photon-added squeezed coherent Schrödinger cat state given by

$$|\psi^{(\kappa)}(0)\rangle = N^{(\kappa)} a^\dagger \kappa (|\xi, \alpha\rangle + c |\xi, -\alpha\rangle),$$

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad |\xi, \alpha\rangle = S(\xi)D(\alpha)|0\rangle, \quad c \in \mathbb{C}. \quad (5.1)$$

The displacement and the squeezing operators given in (5.1) read, respectively, as $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$, $S(\xi) = \exp((\xi a^2 - \xi^* a^2)/2)$, $\alpha = |\alpha| \exp(i \theta)$, $\xi = r \exp(i \vartheta)$, $\alpha, \xi \in \mathbb{C}$. The squeezing operator maintains the following unitary transformations:

$$S^\dagger(\xi) a S(\xi) = \mu a + \nu a^\dagger, \quad S^\dagger(\xi) a^\dagger S(\xi) = \mu a^\dagger + \nu^* a,$$

$$\mu = \cosh r, \quad \nu = \exp(i \vartheta) \sinh r. \quad (5.2)$$
The squeezed coherent state in the number state basis may be written \[|\xi, \alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2} - \frac{\alpha^2 \nu^*}{2\mu}\right) \sum_{n=0}^{\infty} \frac{i^n}{\sqrt{n!}\mu} \left(\frac{\nu}{2\mu}\right)^n H_n \left(-i\frac{\alpha}{\sqrt{2\mu\nu}}\right) |n\rangle, \tag{5.3}\]

where the Hermite polynomials are described by the generating function: \(\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n\). Similarly, the multiple photon-added squeezed coherent state in the number state basis reads \(a^{\dagger \kappa}|\xi, \alpha\rangle \equiv \sum_{n=\kappa}^{\infty} A_{n,\kappa}(\xi, \alpha)|n\rangle\), where the coefficients are given by

\[A_{n,\kappa}(\xi, \alpha) = \exp\left(-\frac{|\alpha|^2}{2} - \frac{\alpha^2 \nu^*}{2\mu}\right) \left(\frac{i^n}{\sqrt{n!}} \frac{\nu^{n-\kappa}}{\sqrt{\mu}} H_{n-\kappa} \left(-i\frac{\alpha}{\sqrt{2\mu\nu}}\right)\right). \tag{5.4}\]

The parity property of the Hermite polynomials \(H_n(-x) = (-1)^n H_n(x)\) relates the above coefficients as follows: \(A_{n,\kappa}(\xi, -\alpha) = (-1)^n A_{n,\kappa}(\xi, \alpha)\). The normalization constant for the initial state (5.1) reads

\[N(\kappa) = \left(\sum_{n=\kappa}^{\infty} \left|1 + |c|^2 + 2(-1)^{n-\kappa} \text{Re}(c)\right|^2 |A_{n,\kappa}(\xi, \alpha)|^2\right)^{-\frac{1}{2}}, \tag{5.5}\]

where the infinite sums relevant for the above construction may be explicitly evaluated:

\[\sum_{n=\kappa}^{\infty} |A_{n,\kappa}(\xi, \alpha)|^2 = \kappa! \sum_{p=0}^{\kappa} \sum_{\ell=0}^{p} (-1)^p \left(\frac{\kappa}{p}\right) \frac{|\nu|^{2p-\ell}}{2^p \ell! (p-\ell)!} \times H_{\ell} \left(\frac{i\alpha}{\sqrt{2\mu\nu}}\right) H_{2p-\ell} \left(\frac{i\alpha}{2\mu} + i\alpha^* \sqrt{\frac{\mu}{2\nu}}\right), \tag{5.6}\]

\[\sum_{n=\kappa}^{\infty} (-1)^{n-\kappa} |A_{n,\kappa}(\xi, \alpha)|^2 = \exp(-2|\alpha|^2) \kappa! \sum_{p=0}^{\kappa} \sum_{\ell=0}^{p} (-1)^p \left(\frac{\kappa}{p}\right) \frac{|\nu|^{2p-\ell}}{2^p \ell! (p-\ell)!} \times H_{\ell} \left(\frac{i\alpha}{\sqrt{2\mu\nu}}\right) H_{2p-\ell} \left(\frac{i\alpha}{2\mu} - i\alpha^* \sqrt{\frac{\mu}{2\nu}}\right). \tag{5.7}\]

To compute the lhs of (5.6, 5.7) we have employed the following identity of the Hermite polynomials:
\[
\sum_{n=0}^{\infty} \frac{(n + \kappa)!}{2^n(n!)^2} t^n \psi_n(x) \psi_n(y) = \frac{\kappa!}{\sqrt{1 - t^2}} \mathcal{E}_t(x, y) \sum_{p=0}^{\kappa} \sum_{\ell=0}^{p} \left( -\frac{1}{2} \right)^p \binom{\kappa}{p} \frac{1}{\ell!(p - \ell)!} \times \times \left( \frac{t}{\sqrt{1 - t^2}} \right)^{2p-\ell} H_\ell(x) H_{2p-\ell} \left( \frac{tx - y}{\sqrt{1 - t^2}} \right),
\]

(5.8)

where the exponential factor reads: \( \mathcal{E}_t(x, y) = \exp \left( -\frac{(tx)^2 - 2txy + (ty)^2}{1 - t^2} \right) \). The time evolution of the initial state is given by

\[
|\psi^{(\kappa)}(t)\rangle = \mathcal{N}^{(\kappa)} \sum_{n=\kappa}^{\infty} (1 + \epsilon)n^{-\kappa} A_{n,\kappa}(t) |n\rangle,
\]

\( A_{n,\kappa}(t) = A_{n,\kappa}(\xi, \alpha) \exp(-i((\omega - \lambda)n + \lambda n^2)t) \)

(5.9)

and the corresponding density matrix assumes the form

\[
\rho^{(\kappa)}(t) \equiv |\psi^{(\kappa)}(t)\rangle \langle \psi^{(\kappa)}(t) | = \sum_{n,m=\kappa}^{\infty} \rho_{n,m}^{(\kappa)}(t) |n\rangle \langle m|,
\]

(5.10)

where the matrix elements read

\[
\rho_{n,m}^{(\kappa)}(t) \equiv \langle n|\rho^{(\kappa)}(t)|m\rangle = (\mathcal{N}^{(\kappa)})^2 (1 + \epsilon)n^{-\kappa} (1 + \epsilon)m^{-\kappa} A_{n,\kappa}(t)A_{m,\kappa}^*(t).
\]

(5.11)

Owing to the relation (5.5) the normalization condition \( \text{Tr} \rho^{(\kappa)}(t) = 1 \) is maintained.

5.2 Phase space distributions

The density matrix (5.10) now yields the diagonal Sudarshan-Glauber \( P(\beta, \beta^*) \) representation (3.15) as a series sum

\[
P^{(\kappa)}(\beta, \beta^*, t) = \exp \left( |\beta|^2 \right) \sum_{n,m=\kappa}^{\infty} \frac{(-1)^{n+m}}{\sqrt{n!m!}} \rho_{n,m}^{(\kappa)}(t) \left( \frac{\partial}{\partial \beta} \right)^n \left( \frac{\partial}{\partial \beta^*} \right)^m \delta^{(2)}(\beta)
\]

(5.12)
that includes all derivatives of the delta function. The highly singular nature of the
distribution (5.12) provides a clear signature of the nonclassicality of the evolving state.
The $P$-representation (5.12) obeys the normalization condition (3.14).

The evolution of the Wigner distribution [38] may now be computed by substituting
the density matrix (5.10) in the series expansion (3.20):

$$W^{(\kappa)}(\beta, \beta^*; t) = \frac{2}{\pi} \exp(-2|\beta|^2) \sum_{n,m=\kappa}^\infty \frac{(2\beta^*)^n (2\beta)^m}{\sqrt{n!m!}} {}_2F_0\left(-n,-m; -\frac{1}{4}||\beta||^2\right) \rho_{n,m}^{(\kappa)}(t).$$  (5.13)

In the construction (5.13) the identity (3.26) has been used. The distribution property
(3.22) for the choice $\sigma = 1/2$ facilitates the integration in (3.21) allowing to reproduce
the distribution (5.13). This acts as a consistency check for our derivation. The expression (5.13) reveals that corresponding to the choices ($c = \pm 1$) of the even and odd
combinations of the initial state (5.1) the $W$-distribution assumes a parity symmetric
form:

$$W^{(\kappa)}(\beta, \beta^*; t) = W^{(\kappa)}(-\beta, -\beta^*; t),$$  (5.14)

This property is responsible for the generation of only even number of kitten states (Fig.
5.1 $c_1, c_2, c_3$) described in Sec. 5.2.1. On the other hand, for the choice $c = \pm i$ à la
the Yurke-Stoler type of states [90] the symmetry (5.14) is violated causing the production of
both the even and odd number of kitten states. In the study of the nonclassicality of the
photon-added squeezed kitten states the negativity $\delta_W$ (3.27) is employed extensively.

For the pure state density matrix (5.10) the definition (3.28) readily furnishes the
$Q$-function on the phase space as

$$Q^{(\kappa)}(\beta, \beta^*; t) = \frac{1}{\pi} \left(\mathcal{N}^{(\kappa)}\right)^2 \exp(-|\beta|^2) \left| \sum_{n=\kappa}^\infty \frac{\beta^{*n}}{n!} (1 + (-1)^{n-\kappa}c) A_{n,\kappa}(t) \right|^2,$$  (5.15)

which vanishes only at asymptotically large values of $|\beta|$. Parallel to the case of the
$W$-distribution the choice ($c = \pm 1$) in (5.15) imparts a parity symmetric property of the
$Q$-function as

$$Q^{(\kappa)}(\beta, \beta^*; t) = Q^{(\kappa)}(-\beta, -\beta^*; t)$$  (5.16)
that underlies the observation of only even number of kitten states in this instance. In
the general case, however, both the even and odd number of kitten states are realized.

The polar phase density (3.40) corresponding to the $Q$-function (5.15) reads

$$Q^{(\kappa)}(\tilde{\theta}) = \frac{1}{2\pi} \sum_{n,m=\kappa}^{\infty} \frac{(m+n)/2)!}{\sqrt{n!m!}} \exp\left(-i(n-m)\tilde{\theta}\right) \rho^{(\kappa)}_{n,m}(t), \quad \beta = |\beta| \exp(i\tilde{\theta}). \quad (5.17)$$

In the context of an initial coherent state subject to the Kerr Hamiltonian (4.1) the
unitary time evolution has been found [3–6] to lead to the formations of the transient
kitten states characterized by the superposition of a finite number of macroscopic (for
large values of $|\alpha|$) coherent states. The presence of the anharmonic term in (4.1) leads
to a periodicity of the Wehrl entropy $S_Q$ (3.45) that develops a series of local minima
at the rational submultiples of the said time period. The quantum superposition of the
macroscopic states are produced precisely at these local minima of the Wehrl entropy.

For our choice of the initial state (5.1) the photon-added Schrödinger squeezed kitten
configurations are also realized at the local minima of the Wehrl entropy $S_Q$ occurring
at the rational submultiples of the time period. Moreover, we observe that a similar
pattern for the photon-added squeezed kitten states also emerge for the choice of the
squeezed vacuum configuration ($\alpha = 0$) and a large squeezing parameter $r \sim 1.5$ in the
initial state (5.1). We will describe it in the next subsection.

### 5.2.1 Photon-added squeezed kitten states

In the absence of decoherence the unitary time evolution of the pure state density matrix
(5.10) leads to a periodic behavior of its Wehrl entropy $S_Q$ (Figs. 5.1 a1, a2, d1), and other
dynamical quantities such as negativity $\delta_W$ associated with the $W$-distribution (Figs.
5.1 a3, a4). One manifest characteristics in these diagrams is that a state, say, with a
larger squeezing parameter $r$ corresponds to higher values of the dynamical quantities
$S_Q$ and $\delta_W$. This may be qualitatively understood as follows: Larger value of $r$ engenders
increased occupation of higher Fourier modes of the coefficients (5.4), which, in turn,
triggers the generation of more interference terms for the $W$-distribution (5.13) as well as
the $Q$-function (5.15). This induces wider spread of the quasiprobability functions in the phase space, and, simultaneously, more numerous sign reversals of the $W$-distribution. Consequently, the Wehrl entropy and the negativity register increments with the enlargement of the squeezing parameter. Accretion of the number ($\kappa$) of photons added to the state produces the same qualitative effect.

These diagrams (Figs. 5.1 $a_1 - a_4, d_1$) display a series of local minima at rational submultiples of the time period. At these instants the evolving state of the nonlinear Kerr oscillator coincides with the superposition of a finite number of photon-added squeezed coherent states. To demonstrate the above transitory convergence of states, we construct superpositions of the photon-added squeezed kitten states and obtain their Hilbert-Schmidt distance measures [91] from the evolving Kerr oscillator states at the relevant time limits. The vanishing of the distances between the two sets of states establishes their transient indistinguishability. The Hilbert-Schmidt distance between the states characterized by the density matrices $\rho_1$ and $\rho_2$ is defined as $\left( d_{\text{HS}}(\rho_1, \rho_2) \right)^2 \equiv \text{Tr} \left( \rho_1 - \rho_2 \right)^2$, and may be expressed [91] via the corresponding $W$-distributions.

The fiducial marker state occurring as a superposition of $\kappa$-photon added $p$ squeezed kittens, and the corresponding density matrix may be listed as

$$\left| \tilde{\psi}^{(\kappa)} \right\rangle = \tilde{C}^{(\kappa)} \sum_{j=0}^{p-1} f_j a^{|\kappa|} |\xi_j, \alpha_j\rangle, \quad \tilde{\rho}^{(\kappa)} = \left| \tilde{\psi}^{(\kappa)} \right\rangle \left\langle \tilde{\psi}^{(\kappa)} \right|,$$

(5.18)

where the complex coordinates describing the ensemble of kitten states read:

$$\xi_j = \xi \exp(2i(\tilde{\vartheta} + 2\pi j/p)), \quad \alpha_j = \alpha \exp(i(\tilde{\vartheta} + 2\pi j/p)), \quad \nu_j = \nu \exp(2i(\tilde{\vartheta} + 2\pi j/p)).$$

(5.19)

The evaluation of the normalization constant

$$\tilde{C}^{(\kappa)} = \left( \sum_{j,\ell=0}^{p-1} f_j f_\ell^* \sum_{n=\kappa}^{\infty} A_{n,\kappa}(\xi_j, \alpha_j) A_{n,\kappa}^*(\xi_\ell, \alpha_\ell) \right)^{-1/2}$$

is aided by the following bilinear
Figure 5.1: For the selection of parameters $c = 1, \delta = 1$ maintained throughout these diagrams, the plots ($a_1$) and ($a_2$) refer, respectively, to the Wehrl entropy of the single photon-added case for $\alpha = 2$ with varying squeezing parameter $\xi$, and that of the multiple photon-added cases for the variables $\alpha = 2, \xi = 0.5$. Retaining the choice of the parameters as in ($a_1$)($a_2$), the diagram ($a_3$)($a_4$) studies the negativity of the Wigner distribution. Comparatively large values of each of the parameters $\xi$ and $\kappa$ enhance the population of the higher modes as well as the spread of the interference domain in the phase space. These effects, in turn, augment the observed values of $S_Q$ and $\delta_W$. The Figs. ($b_1$, $b_2$, $b_3$) and ($c_1$, $c_2$, $c_3$) correspond, respectively, to the scaled times ($\pi/8$, $\pi/12$, $\pi/16$) showing the polar density $Q(\theta)$ and the $W$-distribution for the phase space variables $\alpha = 2, \xi = 0.5$. The formation of the transient kitten structures for the evolving photon-added squeezed coherent states are evident therein. The negativity $\delta_W$ corresponding to ($c_1$, $c_2$, $c_3$) reads (1.9224, 2.6665, 2.8487) depicting a systematic increase of nonclassicality with growing complexity of the quantum states that necessitates enlarged fluctuations of the higher Fourier modes of the density matrix. Another observation is that even for the photon-added squeezed vacuum initial state ($\alpha = 0$) the evolution to the kitten structures is manifest for the dominant values of the squeezing parameter $\xi$. The initial state $|\psi_{\alpha=0}^{(\kappa=1)}(0)\rangle$ with high squeezing parameter $\xi = 1.5$ develops ($d_1$), during its evolution, a periodic structure of the Wehrl entropy, while its Wigner distribution at $t = 0$ ($\delta_W|_{t=0} = 0.4209$) is depicted in ($d_2$). The subsequent diagrams ($d_3$ – $d_5$) obviously display its evolving transitory photon-added squeezed vacuum kitten states at the scaled times ($\pi/8$, $\pi/12$, $\pi/16$). The corresponding negativity variables $\delta_W$ are given by (2.8336, 3.1297, 3.1981) reinforcing the observation that incremental complexity of the quantum states leads to higher nonclassicality.
sum obtained via the identity (5.8):

\[
\sum_{n=\kappa}^{\infty} \mathcal{A}_{n,\kappa}(\xi_j, \alpha_j) \mathcal{A}_{n,\kappa}^\ast(\xi_\ell, \alpha_\ell) = \kappa! \frac{\mu}{\sqrt{U_{j\ell}}} \exp \left( - \frac{N_{j\ell}}{U_{j\ell}} \right) \sum_{p=0}^{\kappa} \left( \frac{\kappa}{p} \right) \left( -\nu \nu^\ast \right)^p \times \\
\times \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \left( \frac{U_{j\ell}}{\nu \nu^\ast} \right)^{\frac{q}{2}} H_q \left( \frac{-i\alpha_j}{\sqrt{2\mu \nu_j}} \right) \times H_{2p-q} \left( -i(\alpha_j \nu^\ast_j + \alpha_\ell^\ast \mu) \right). 
\]

(5.20)

where the coefficients have the following structures:

\[
U_{j\ell} = \mu^2 - \nu_j \nu_j^\ast, \\
N_{j\ell} = |\alpha|^2 U_{j\ell} - \alpha_j \alpha_j^\ast + \frac{1}{2\mu} \left( U_{j\ell} (\alpha_j^2 \nu_j^\ast + \alpha_\ell^2 \nu_\ell) - \alpha_j^2 \nu_j^\ast - \alpha_\ell^2 \nu_\ell \right). 
\]

(5.21)

The Wigner distribution of the state (5.18) is given by

\[
\tilde{W}^{(\kappa)}(\beta, \beta^\ast) = \left( \tilde{c}^{(\kappa)} \right)^2 \sum_{j,\ell=0}^{p-1} f_j^\ast f_\ell^\ast \tilde{W}_{j\ell}^{(\kappa)}(\beta, \beta^\ast),
\]

(5.22)

where the \( W \)-distribution pertinent to the projection operator reads

\[
\tilde{W}_{j\ell}^{(\kappa)}(\beta, \beta^\ast) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle \beta, n|a^n|\xi_j, \alpha_j \rangle \langle \xi_\ell, \alpha_\ell|a^n|\beta, n \rangle \times \\
\frac{2}{\pi} \exp(-2|\beta|^2) \sum_{n,m=\kappa}^{\infty} \frac{(2\beta^\ast)^n(2\beta)^m}{\sqrt{n!m!}} {}_2F_0 \left( -n, -m; -; -\frac{1}{4|\beta|^2} \right) \times \\
\times \mathcal{A}_{n,\kappa}(\xi_j, \alpha_j) \mathcal{A}_{n,\kappa}^\ast(\xi_\ell, \alpha_\ell). 
\]

(5.23)

We now provide an explicit evaluation of the infinite sum in (5.23), say for the \( \kappa = 1 \) case, and determine the corresponding \( W \)-distribution:

\[
\tilde{W}_{j\ell}^{(\kappa=1)}(\beta, \beta^\ast) = \frac{2 H_{j\ell}}{\pi \langle U_{j\ell} \rangle^{\frac{3}{2}}} \exp \left( -2|\beta|^2 - F_{j\ell} \right). 
\]

(5.24)
The exponent and the amplitude factors in (5.24) are of the following form:

\[ F_{j\ell} = |\alpha|^2 + \frac{1}{2\mu}(\alpha_j^2\nu_j^* + \alpha_j^2\nu_j) - \frac{2\beta^*}{\mu}F_j - \frac{2\beta}{\mu}F_{\ell}^* \]

\[ + \frac{1}{U_{j\ell}} \left( F_{j\ell}F_{\ell}^* - \frac{\nu_j^*F_{j\ell}^2}{2\mu} - \frac{\nu_jF_{j\ell}^2}{2\mu} \right) , \]

\[ H_{j\ell} = (\mu^2 + \nu_j^*\nu_j)F_{j\ell}F_{\ell}^* - \mu\nu_j^*F_{j\ell}^2 - \mu\nu_jF_{j\ell}^2 - U_{j\ell} \left( \alpha_j\alpha_j^* - \mu^2(4|\beta|^2 - 1) \right) \]

\[ - F_{j\ell}F_{\ell}^* + 2\mu\beta^*F_j + 2\mu\beta F_{\ell}^* \] \tag{5.25}

where the phase space variables read: \( F_j = \alpha_j + \beta^*\nu_j, \) \( F_{\ell} = \alpha_j + 2\beta^*\nu_j. \) The normalization constant may be evaluated as follows:

\[ \tilde{C}(\kappa=1) = \left\{ \sum_{j,\ell=0}^{p-1} \frac{f_j f_{\ell}^*}{(U_{j\ell})^3} \exp \left( -\frac{N_{j\ell}}{U_{j\ell}} \right) \left( U_{j\ell}^2 + (\alpha_j\alpha_j^* + \nu_j\nu_j^*) \right) \right. \]

\[ + \left. \mu(\alpha_j^2\nu_j^* + \alpha_j^2\nu_j) + \nu_j\nu_j^*U_{j\ell} \right\}^{-\frac{1}{2}}. \] \tag{5.26}

The quasiprobability distribution for the benchmark state \( \tilde{W}^{(\kappa=1)}(\beta, \beta^*) \) for the single photon-added case now follows from (5.22, 5.24, 5.26). In the derivation of the expression (5.24) we have enlisted the following identities \[92\]:

\[ \sum_{n=0}^{\infty} \frac{(n+k+1)t^n}{n!} H_{n+k}(x) = (t H_{k+1}(x-t) + (k+1)H_k(x-t)) \exp(2tx - t^2), \] \tag{5.27}

\[ \sum_{n=0}^{\infty} \frac{(n+1)t^n}{2^n n!} H_n(x)H_n(y) = \frac{1 + 2t(1 + t^2)xy - t^2(1 + 2(x^2 + y^2))}{(1 - t^2)^{5/2}} \mathcal{E}_t(x, y), \] \tag{5.28}

\[ \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} H_{n+k}(x)H_n(y) = \frac{1}{(1 - t^2)^{k+1/2}} H_k \left( \frac{x - ty}{\sqrt{1 - t^2}} \right) \mathcal{E}_t(x, y), \] \tag{5.29}

\[ \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} H_{n+k}(x)H_{n+k}(y) = \frac{(-t)^{\kappa}}{(1 - t^2)^{2\kappa + 1/2}} \mathcal{E}_t(x, y) \sum_{p=0}^{\kappa} \binom{\kappa}{p} \left( \frac{\sqrt{1 - t^2}}{t} \right)^{p} \times \]

\[ \times H_p(x)H_{2\kappa-p} \left( \frac{xt - y}{\sqrt{1 - t^2}} \right). \] \tag{5.30}

The Hilbert-Schmidt distance \[91\] between the evolving state (5.13) of the Kerr osci-
lator at the specified times and the fiducial state (5.22) comprising of the superposition of the photon-added squeezed coherent states may now be expressed as follows:

$$ \left( d_{HS} \left( \rho^{(r)}(t), \tilde{\rho}^{(r)} \right) \right)^2 = \pi \int \left( W^{(r)}(\beta, \beta^*; t) - \tilde{W}^{(r)}(\beta, \beta^*) \right)^2 d^2 \beta. $$  

(5.31)

When the convergence of the two states is realized at the rational submultiples of the time period of the Wehrl entropy $S_Q$ of the Kerr oscillator (Figs. 5.1 a₁, a₂), the distance measure (5.31) goes to the precise zero limit. In specific, we now consider the case $c = 1$, where only the even number of kittens ($p = 2, 4, 6...$) are produced at times $\lambda t = \frac{\pi}{2p}$. This can be observed from the evolution of the polar density $Q(\tilde{\vartheta}$) as well as the $W$-distribution given in Figs. 5.1 (b₁, b₂, b₃) and (c₁, c₂, c₃), respectively. These transitory states are identified with the fiducial marker state (5.18) with appropriate coefficients that render the Hilbert-Schmidt distance measure (5.31) vanish at the relevant time limits. For the $c = 1$ case these coefficients are given as

$$ f_j = \exp \left( \frac{i\pi j(p + 2)}{p} \right), \quad \tilde{\vartheta} = \frac{\pi}{2p} (p - 2\kappa + 1 - \delta), \quad \delta = \frac{\omega}{\lambda}, $$  

(5.32)

where the supplementary angle of rotation $\tilde{\vartheta}$ implements complete phase space overlap between the density matrix of the reference state (5.18) and that of the transient squeezed kitten states (5.10) observed when the locally minimum Wehrl entropy configurations are realized. At the submultiples of the time period, we, therefore, find a transient production of the Yurke-Stoler type of states [90] which are a linear combination of equally phase-shifted photon-added squeezed coherent states.

The appearances of the similar kitten-like patterns in the phase space are also noticed (Figs. 5.1 d₁ – d₅) for the time evolution of the photon-added highly squeezed vacuum states $\alpha = 0, \xi \sim 1.5$. This result points towards another possibility. The well-known one mode Schwinger representation of the $su(1, 1)$ algebra [8] reads $[K_0, K_\pm] = \pm K_\pm, [K_+, K_-] = -2K_0, K_\pm \equiv \frac{1}{2} a^\dagger a^2, K_- \equiv \frac{1}{2} a^2, K_0 \equiv \frac{1}{4}(aa^\dagger + a^\dagger a)$, and, consequently, the squeezed vacuum states may be understood as particular realizations of coherent states of $su(1, 1)$ algebra [93]. The present result, therefore, indicates the possibility of
Figure 5.2: The evolution of the Hilbert-Schmidt distance $d_{\text{HS}}(\rho^{(\kappa)}(t), \tilde{\rho}^{(\kappa)})$ for $c = 1, \delta = 1, \kappa = 1$ case is studied for $p = 4$ kitten formations realized at time $\lambda t = \frac{\pi}{8}$. The diagram (a1) corresponds to the phase space variables ($\alpha = 0, \xi = 1.5$). The plots (a2) and (a3) refer to $\xi = 0.5$ while maintaining the displacement parameter to $\alpha = 2$ and $\alpha = 4$, respectively.

similar kitten-type formations existing in the evolution of other systems governed by the $su(1, 1)$ symmetry. This issue merits further investigation.

A general feature observed in the kitten-like organizations in the phase space is that higher complexity necessitates more intensified occupation of higher oscillatory Fourier modes, say, in the $W$-distribution. This, in turn, triggers an increased value of the negativity $\delta_W$ signifying the presence of more nonclassicality of the state. Maintaining the parametric choice $c = 1, \kappa = 1$, we plot in Fig. 5.2 the time variation of the Hilbert-Schmidt distance between the evolving state (5.10) and the fiducial state (5.18) for, say $p = 4$ squeezed kitten formations. We note that as the displacement parameter $\alpha$ increases, the distance collapses to the null value $d_{\text{HS}}(\rho^{(\kappa)}(t), \tilde{\rho}^{(\kappa)}) \to 0$ with growing sharpness at the appropriate submultiple of the time period: $\lambda t = \frac{\pi}{8}$. As enlarged value of $\alpha$ indicates more macroscopic nature of the state, the rising rapidity of the collapse for higher $\alpha$ points towards extremely short-lived nature of the kitten states for the macroscopic systems.

5.2.2 Optical tomogram

The optical tomogram for this model can be constructed using the expressions (5.11, 4.4)

$$\Omega^{(\kappa)}(X, \varphi; t) = \left(\mathcal{N}^{(\kappa)}\right)^2 \frac{\exp(-X^2)}{\sqrt{\pi}} \left| \sum_{n=\kappa}^{\infty} \frac{(1 + (-1)^{n-\kappa}c)}{\sqrt{2^n n!}} A_{n,\kappa}(t) \exp(-in\varphi) H_n(X) \right|^2,$$  

(5.33)
Figure 5.3: The diagrams \((a_1, a_2, a_3, a_4)\) plot the optical tomograms for the instance \(c = 1, \kappa = 1, \delta = 1\) corresponding to the scaled time \(\lambda t\) at \((0, \frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{4})\), respectively. The phase space variables read \(\alpha = 2, \xi = 0.5\).

where the normalization property (4.4) follows from (5.5, 5.9), and the orthonormality of the Hermite polynomials: \(\int dX \exp(-X^2)H_n(X)H_m(X) = \delta_{n,m}2^n\sqrt{\pi n!}\). The rhs in (5.33) establishes the parity symmetric behavior \(\Omega^{(\kappa)}(-X, \varphi; t) = \Omega^{(\kappa)}(X, \varphi; t)\) for the choice \(c = \pm 1\). The optical tomogram at \(t = 0\) may now be directly constructed as

\[
\Omega(X, \varphi; t = 0) = (\mathcal{N}^{(\kappa)})^2 \frac{\exp(-X^2)}{\sqrt{\pi \mu 2^n}} \exp \left(-|\alpha|^2 - \frac{\alpha^2 \nu^*}{2\mu} - \frac{\alpha^* \nu}{2\mu}\right) \times \\
\times |f_\varphi(\xi, \alpha) + cf_\varphi(\xi, -\alpha)|^2,
\]

(5.34)

where the structure function reads

\[
f_\varphi(\xi, \alpha) = \frac{1}{\left(1 + \frac{\nu}{\mu} \exp(-2i\varphi)\right)^{\kappa+1}} \exp \left(\frac{\nu X^2 - \alpha^2 \nu^*}{2\mu} + \sqrt{2\alpha X \exp(i\varphi})\right) \times \\
\times H_\kappa \left(\frac{X - \frac{\alpha}{\sqrt{2\mu}} \exp(-i\varphi)}{\sqrt{1 + \frac{\nu}{\mu} \exp(-2i\varphi)}}\right).
\]

(5.35)

To obtain the above expression we enlisted the identity (5.29). Each structure function in (5.34) produces a filament-like formation in the optical tomogram given in Fig. 5.3 (a_1). In general, the squared modulus of each of the structure functions at instants of time equal to the submultiples of the period of the Wehrl entropy \(S_Q\) becomes manifest in the optical tomogram \(\Omega^{(\kappa)}(X, \varphi; t)\) as the corresponding filament. The ripples observed in Fig. 5.3 (a_1) may be ascribed to the interference pattern produced by the inner product of two structure functions in (5.34).
Towards obtaining the tomogram at the scaled time $\lambda t = \pi/4$ we use the following identity:

$$
\exp \left( -i \frac{\pi}{4} n^2 \right) = \frac{1 + (-1)^n}{2} i^n + \frac{1 - (-1)^n}{2} \exp \left( -i \frac{\pi}{4} \right),
$$

(5.36)

which facilitates expressing the infinite sum (5.33) as a closed form expression involving a finite number of structure functions as follows:

$$
\Omega \left( X, \varphi; \lambda t = \frac{\pi}{4} \right) = \left( \mathcal{N}^{(\kappa)} \right)^2 \frac{\exp(-X^2)}{\sqrt{\pi \mu^2 \kappa}} \exp \left( -|\alpha|^2 - \frac{\alpha^2 \nu^*}{2\mu} - \frac{\alpha^2 \nu}{2\mu} \right) |S|^2, \quad (5.37)
$$

where the sum reads

$$
S = \frac{c + 1}{2} \left( f_\Phi(-\xi, i\alpha) + f_\Phi(-\xi, -i\alpha) \right) - \frac{c - 1}{2} \exp \left( -i \frac{\pi}{4} \right) \left( f_\Phi(\xi, \alpha) - f_\Phi(\xi, -\alpha) \right), \quad (5.38)
$$

and the phase angle associated with the structure functions is given by $\Phi = \varphi + \phi_1, \phi_1 = (\delta - 1 + 2\kappa)^2 \pi$. With the choice of the parameters $c = 1, \delta = 1$ the single photon-added case in the tomogram (5.37) becomes identical to its initial state (5.34) rendering the corresponding time period assume the value $T = \frac{\pi}{4\lambda}$. This may be observed in Fig. 5.3 (a4).

Explicit evaluation of the tomogram $\Omega(X, \varphi; t)$ at other submultiples of the time period proceeds similarly. As an example, we demonstrate this for the scaled time $\lambda t = \frac{\pi}{8}$. For this purpose we employ the following identity:

$$
\exp \left( -i \frac{\pi}{8} n^2 \right) = \frac{1 + (-1)^n}{2} \left\{ \exp \left( -i \frac{\pi}{2} \right) \frac{1 - i^n}{2} + \frac{1 + i^n}{2} \right\} + \frac{1 - (-1)^n}{2} \times
$$

$$
\times \exp \left( -i \frac{\pi}{8} \right) \left\{ \exp \left( -i \frac{\pi}{4} n \right) \frac{1 - i^{n+1}}{2} + \exp \left( i \frac{\pi}{4} n \right) \frac{1 + i^{n+1}}{2} \right\}, \quad (5.39)
$$

that allows us to establish the corresponding tomogram in the form given below:

$$
\Omega \left( X, \varphi; \lambda t = \frac{\pi}{8} \right) = \left( \mathcal{N}^{(\kappa)} \right)^2 \frac{\exp(-X^2)}{\sqrt{\pi \mu^2 \kappa}} \exp \left( -|\alpha|^2 - \frac{\alpha^2 \nu^*}{2\mu} - \frac{\alpha^2 \nu}{2\mu} \right) |\tilde{S}|^2. \quad (5.40)
$$

The closed form expression of the sum over an infinite number of modes (5.33) for the
chosen time $\lambda t = \frac{\pi}{8}$ may be furnished as a linear combination of structure functions:

$$\tilde{S} = \frac{c+1}{4} \left[ (1-i) \left( f_{\bar{g}}(\xi, \alpha) + f_{\bar{g}}(\xi, -\alpha) \right) + (1+i) \left( f_{\bar{g}}(-\xi, i\alpha) + f_{\bar{g}}(-\xi, -i\alpha) \right) \right]$$

$$+ \frac{c-1}{4} \exp \left( i\frac{\pi}{8} \right) \left[ (1-i) \left( f_{\bar{g}_+}(\xi, -\alpha) - f_{\bar{g}_+}(\xi, \alpha) \right) + (1+i) \left( f_{\bar{g}_-}(\xi, -\alpha) - f_{\bar{g}_-}(\xi, \alpha) \right) \right],$$

where the effective phase contribution reads $\tilde{\Phi} = \varphi + \phi_2, \phi_2 = (\delta - 1 + 2\kappa)\frac{\pi}{8}$. As remarked earlier, the square of the modulus of each structure function in (5.41) produces a strand in the corresponding tomogram ($c = 1, \kappa = 1, \delta = 1$ case) given in Fig. 5.3 (a3). The bilinear interference terms produced by the cross multiples of the structure functions give rise to the striations in the tomogram. These features are generic at all rational submultiples of the time period, and, in particular, may also be observed for the preceding set of parametric values in Fig. 5.3 (a2), which represents the tomogram $\Omega(X, \varphi; \lambda t = \frac{\pi}{12})$.

### 5.2.3 Nonclassical depth

To investigate the extent of nonclassicality of an arbitrary quantum state a general phase space distribution function depending on a continuous real variable $\sigma$ has been introduced [10,11] as

$$R(\beta, \beta^*; \sigma) = \frac{1}{\pi \sigma} \int P(\gamma, \gamma^*) \exp \left( -\frac{|\beta - \gamma|^2}{\sigma} \right) d^2\gamma,$$

where the distribution $R(\beta, \beta^*; \sigma)$ embodies a smoothing process induced by a Gaussian convolution with a dispersion $\sigma$. The $R$-distribution satisfies the normalizability condition for an arbitrary $\sigma$:

$$\int R(\beta, \beta^*; \sigma) d^2\beta = 1.$$

For the example considered here the distribution $R(\beta, \beta^*; \sigma)$ interpolates in the domain $\sigma \in [0, 1]$ between the highly singular $P$-representation (5.12) and the positive semidef-
inite $Q$-function (5.15). The greatest lower bound of the smoothing parameter $\sigma$ that renders the $R$-distribution of a relevant quantum state positive semidefinite may be regarded [10] as its measure of nonclassicality. Using this procedure the nonclassical depths of various states have been investigated [10,94,95]. Towards evaluating the distribution $R(\beta, \beta^*; \sigma)$ for the quantum state studied here we substitute the $P$-representation (5.12) in the definition (5.42). Utilizing the identity (3.22) we obtain

$$R^{(\kappa)}(\beta, \beta^*; \sigma; t) = \frac{1}{\pi \sigma} \sum_{n,m=\kappa}^{\infty} \frac{\sigma^{-(n+m)}}{\sqrt{n!m!}} \beta^n \beta^m \exp \left( -\frac{|\beta|^2}{\sigma} \right) \times$$

$$\times 2F_0 \left( -n, -m; -\frac{\sigma(1-\sigma)}{|\beta|^2} \right) \rho_{n,m}^{(\kappa)}(t). \quad (5.44)$$

Owing to the conservation of trace of the density matrix the construction (5.44) maintains the normalizability criterion (5.43). Employing the explicit values of the density matrix elements (5.11) we notice that at times which are rational submultiples of the period of the Wehrl entropy, the double summation in (5.44) may be reduced to a single infinite sum. To achieve this we use the following identity:

$$\sum_{n=0}^{\infty} \frac{(n+\ell)! t^n}{n! (n+\ell-\kappa)!} H_{n+\ell-\kappa}(x) = \exp(-t^2 + 2x t) \sum_{p=0}^{\kappa} \frac{t^p (\kappa - p)!}{(\kappa - p)!} \times$$

$$\times \left( \frac{\ell}{\kappa - p} \right) H_{\ell+p}(x-t). \quad (5.45)$$

The initial $t = 0$ limit of the $R$-distribution may now be readily furnished as

$$R^{(\kappa)}(\beta, \beta^*; \sigma; t = 0) = \frac{(\mathcal{N}^{(\kappa)})^2}{\pi \sigma} \exp \left( -\frac{|\beta|^2}{\sigma} \right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \frac{\sigma(\sigma-1)}{|\beta|^2} \right)^{\ell} \times$$

$$\times \left| \mathcal{G}_\ell(\xi, \alpha) + c \mathcal{G}_\ell(\xi, -\alpha) \right|^2, \quad (5.46)$$

where the mode contribution reads

$$\mathcal{G}_\ell(\xi, \alpha) = \frac{1}{\sqrt{\mu}} \exp \left( -\frac{|\alpha|^2}{2} - \frac{\alpha^2 \nu^*}{2\mu} + \frac{\beta^* \nu}{\sigma \mu} \right) \times$$

$$\times \frac{1}{\sqrt{2\mu}} \left( \nu - \frac{\sigma - \beta^*}{\sigma} \right) \left( \frac{i\beta^*}{\sigma} \right)^{\ell} \left( \frac{\nu}{2\mu} \right)^{(\kappa - \ell)} \left( \frac{\nu}{\kappa - p} \right)^{\ell+p} H_{\ell+p} \left( -\frac{i\alpha \sigma - i\beta^* \nu}{\sigma \sqrt{2\mu \nu}} \right). \quad (5.47)$$
The negativity $\delta_R$ of the $R$-distribution is plotted for the evolution of the initial state (5.1) at various times for the choice $c = 1$, $\kappa = 1$, $\delta = 1$, while maintaining the displacement and squeezing parameters $\alpha = 2$, $\xi = 0.5$, respectively.

A parallel derivation that employs the identity (5.36) provides the $R$-distribution at $\lambda t = \pi/4$:

$$R^{(n)}(\beta, \beta^*; \sigma; \lambda t = \pi/4) = \frac{(N^{(n)})^2}{\pi \sigma} \exp\left(-\frac{|\beta|^2}{\sigma}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{\sigma(\sigma - 1)}{|\beta|^2}\right)^{\ell} |G_{\ell}|^2,$$  \hspace{1cm} (5.48)

where the mode measure $G_{\ell}$ is given by

$$G_{\ell} = \frac{c + 1}{2} \left\{ G_{\ell}\left(-\xi \exp(-2i\phi_1), i\alpha \exp(-i\phi_1)\right) 
+ G_{\ell}\left(-\xi \exp(-2i\phi_1), -i\alpha \exp(-i\phi_1)\right)\right\} - \frac{c - 1}{2} \times
\exp\left(-\frac{i\pi}{4}\right) \left\{ G_{\ell}\left(\xi \exp(-2i\phi_1), \alpha \exp(-i\phi_1)\right) 
- G_{\ell}\left(\xi \exp(-2i\phi_1), -\alpha \exp(-i\phi_1)\right)\right\}. \hspace{1cm} (5.49)$$

The expressions (5.46, 5.48) indicate that for the choice $\delta = 1$, $c = 1$, $\kappa = 1$, the $R$-distribution, following the pattern observed for the Wehrl entropy and the tomogram discussed earlier, maintains a period $T = \pi/4\lambda$. Proceeding similarly, we may extract the $R$-distribution at other rational submultiples of the period. For instance, aided by the
identity (5.39), we compute the $R$-distribution at $\lambda t = \frac{\pi}{8}$:

$$
R^{(n)}(\beta; \sigma; \lambda t = \frac{\pi}{8}) = \frac{(N^{(n)})^2}{\pi \sigma} \exp\left(-\frac{|\beta|^2}{\sigma}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{\sigma(\sigma - 1)}{|\beta|^2}\right)^\ell |\tilde{G}_\ell|^2, \tag{5.50}
$$

where the entry $\tilde{G}_\ell$ is listed as follows

$$
\tilde{G}_\ell = \frac{c + 1}{4} \left\{ (1 - i) \left( G_\ell\left(\xi \exp(-2i\phi_2), \alpha \exp(-i\phi_2)\right) 
+ G_\ell\left(-\xi \exp(-2i\phi_2), -\alpha \exp(-i\phi_2)\right) \right) 
+ (1 + i) \left( G_\ell\left(-\xi \exp(-2i\phi_2), i\alpha \exp(-i\phi_2)\right) 
+ G_\ell\left(-\xi \exp(-2i\phi_2), -i\alpha \exp(-i\phi_2)\right) \right) \right\} 
- \frac{c - 1}{4} \exp\left(i\frac{\pi}{8}\right) (1 - i) \times 
\times \left( G_\ell\left(\xi \exp(-2i\left(\phi_2 + \frac{\pi}{4}\right)), \alpha \exp(-i\left(\phi_2 + \frac{\pi}{4}\right))\right) 
- G_\ell\left(\xi \exp(-2i\left(\phi_2 + \frac{\pi}{4}\right)), -\alpha \exp(-i\left(\phi_2 + \frac{\pi}{4}\right))\right) 
+ G_\ell\left(\xi \exp(-2i\left(\phi_2 - \frac{\pi}{4}\right)), \alpha \exp(-i\left(\phi_2 - \frac{\pi}{4}\right))\right) 
- G_\ell\left(\xi \exp(-2i\left(\phi_2 - \frac{\pi}{4}\right)), -\alpha \exp(-i\left(\phi_2 - \frac{\pi}{4}\right))\right) \right). \tag{5.51}
$$

As the expressions of the $R$-distribution obtained above contain one or more infinite summations, it becomes difficult to procure an analytical solution of the nonnegativity constraint $R^{(n)}(\beta, \beta^*; \sigma; t) \geq 0$ satisfied by the least value of $\sigma$. To proceed, we mimic the well-known description of negativity of the $W$-distribution advanced in [46]. One way of numerically estimating the extent of nonclassicality of the quantum state (5.10) is to introduce a measure of the negative volume of the $R$-distribution for the whole range $[0, 1]$ of the variable smoothing parameter $\sigma$:

$$
\delta_R(\sigma) = \int |R(\beta, \beta^*; \sigma)| d^2\beta - 1. \tag{5.52}
$$
The least value of the parameter $\sigma_L$ yielding a nonnegative phase space volume of the $R$-distribution $\delta_R(\sigma \geq \sigma_L) = 0$ may now be regarded as a measure of nonclassicality of the state. In Fig. 5.4 we observe that $\delta_R(\sigma) \to 0$ is realized only as $\sigma \to 1$ i.e. when the $R$-distribution coincides with the $Q$-function. A general feature emerging in Fig. 5.4 is that in the range $0 < \sigma < 1$ the states with a greater number of kitten-like structures (say, at $\lambda t = \frac{\pi}{8}$ in comparison with $\lambda t = \frac{\pi}{4}$) also possess a higher degree of negativity $\delta_R(\sigma)$. The expressions (5.48, 5.50) indicate that for the latter the existence of a larger number of interference terms between various mode contributions $G_\ell$ gives rise to more oscillatory nature of the corresponding distribution $R^{(\kappa)}(\beta; \sigma; \lambda t = \frac{\pi}{8})$, which concomitantly now engenders a higher negativity at $\lambda t = \frac{\pi}{8}$. Another characteristic becomes evident in Fig. 5.4. At times such as $\lambda t = \frac{\pi}{4\sqrt{2}}$ which are irrational multiples of the period, the kitten-like formations are not realized in the phase space. These states, however, portray consistently large negativity measure $\delta_R(\sigma)$ as they are characterized by rapidly oscillatory $R$-distribution.

5.3 Decoherence models

5.3.1 Amplitude decay model

In the Born-Markov approximation at zero temperature the master equation (4.5) relevant for the amplitude decay process for the Kerr Hamiltonian (4.1) is given by the choice $\mathcal{X} = a$. Following [85] we introduce the superoperators acting on an arbitrary density matrix $\rho$ associated with the system interacting with the reservoir degrees of freedom. In the Heisenberg picture the action of the superoperators may be enlisted as

$$
S \rho \equiv -i[H, \rho], \quad J \rho \equiv 2\gamma a \rho a^\dagger, \quad L \rho \equiv -\gamma(a^\dagger a \rho + \rho a^\dagger a), \\
R \rho \equiv a^\dagger a \rho - \rho a^\dagger a.
$$

The algebraic structure satisfied by the superoperators reads

$$
[S, J] = 2i\lambda R J, \quad [S, L] = 0, \quad [S, R] = 0, \quad [L, J] = 2\gamma J, \quad [L, R] = 0, \quad [R, J] = 0.
$$
The above commutation relations admit [85] a factorized form of the evolution of the density matrix of the dissipative system. In particular, the amplitude decay process now restricts the initial state (5.1) to evolve to the mixed state density matrix given below:

$$\rho^{(\kappa)}(t) = \exp(S t) \exp(L t) \exp \left\{ \frac{1 - \exp \left( -2(\gamma + i\lambda\mathcal{R})t \right)}{2(\gamma + i\lambda\mathcal{R})} J \right\} \rho^{(\kappa)}(0). \tag{5.55}$$

The above evolution equation in conjunction with the operator structure (5.53) now readily furnishes the amplitude damped elements of the density matrix (5.10) in the number state basis:

$$\rho^{(\kappa)}_{n,m}(t) = \exp \left( -i\omega(n-m)t \right) \exp \left( -i\lambda(n-m)(n+m-1)t \right) \times \exp \left( -\gamma(n+m)t \right) \sum_{\ell=0}^{\infty} \frac{(n+\ell)! (m+\ell)!}{n!m!} \frac{1}{\ell!} \times \left( \frac{\gamma(1 - \exp(-2(\gamma + i\lambda(n-m))t))}{\gamma + i\lambda(n-m)} \right)^{\ell} \rho^{(\kappa)}_{n+\ell,m+\ell}(0). \tag{5.56}$$

The boundary values of the matrix elements $\rho^{(\kappa)}_{n+\ell,m+\ell}(0)$ may be easily read off from the construction of the initial state (5.1). In (5.56) the higher mode components of the density matrix decay more rapidly, and, consequently, in the long time limit only the ground state remains populated: $\rho^{(\kappa)}_{n,m}(t \to \infty) \to \delta_{n,0} \delta_{m,0}$. The construction (5.56) of the density matrix elements immediately furnishes the Husimi $Q$-function (3.28) for the system undergoing the amplitude decay:

$$Q(\beta, \beta^*, t) = \frac{1}{\pi} \exp(-|\beta|^2) \sum_{n,m=0}^{\infty} \frac{\beta^n \beta^m}{\sqrt{n!m!}} \rho^{(\kappa)}_{n,m}(t) \to \frac{1}{\pi} \exp(-|\beta|^2). \tag{5.57}$$

Consequently, the asymptotic value of the corresponding Wehrl entropy (3.45) in the presence of amplitude dissipation terms in the Lindblad equation reads $S_Q(t \to \infty) \to 1 + \log \pi$, which is the universal lower bound of $S_Q$ [96]. This is observed in Figs. 5.5 (a1, a2, a8). The decoherence phenomenon and the loss of nonclassicality in the presence of amplitude dissipation (5.56) are evident from the asymptotic behavior of the negativity $\delta_W$ of the $W$-distribution (5.13): $\delta_W(t \to \infty) \to 0$ (Figs. 5.5, a3, a4, a9).
Figure 5.5: For the parametric values $c = 1, \delta = 1, \alpha = 2$ the amplitude decay model is studied. Single photon-added case ($\kappa = 1$) is investigated in plots ($a_1$ – $a_7$). The illustrations ($a_1$, $a_2$) study the evolution of the Wehrl entropy; first, for various values of the squeezing parameter $\xi$ at damping coefficient $\gamma = 0.5$ ($a_1$), and subsequently for a fixed squeezing $\xi = 0.5$ and varying damping strengths ($a_2$). Similarly the depictions ($a_3$, $a_4$) refer to the evolution of negativity $\delta W$ under the identical set of parameters described in ($a_1$) and ($a_2$), respectively. The diagrams ($a_5$ – $a_7$) sketch the optical tomogram for the variables $\xi = 0.5$ and $\gamma = 0.5$ at times $\lambda t = 0.06, \lambda t = 0.3, \lambda t = 6$, respectively. Lastly, the graphs ($a_8$, $a_9$) produce the Wehrl entropy and negativity $\delta W$ for the multiple photon-added cases ($\kappa = 0 - 3$), where the squeezing parameter and the damping constant read $\xi = 0.5, \gamma = 0.5$.

For a more squeezed state with a larger parameter $r$ the higher harmonics in the density matrix (5.10) are activated causing an increment in the corresponding $\delta W$. This effect becomes more pronounced as higher harmonics induce more rapid oscillations on phase space. An increase in $\delta W$ for an enhanced squeezing parameter $r$ is observed at times $t \ll \gamma^{-1}$. However, since the higher harmonics experience increased damping, the negativity $\delta W$ for a larger $r$ also dissipates faster (Fig. 5.5 $a_3$) at $t \gtrsim \gamma^{-1}$. A parallel reason also induces similar behavior (Fig. 5.5 $a_9$) for an increased number of photon added ($\kappa$) state. We also enlist the optical tomogram (4.4) following from the evolution (5.56) of the density matrix subject to amplitude damping:

$$\Omega(X, \varphi; t) = \frac{\exp(-X^2)}{\sqrt{\pi}} \sum_{n,m=0}^{\infty} \frac{\exp(-i\varphi(n - m))}{\sqrt{2^{n+m}n!m!}} H_n(X) H_m(X) \rho_{n,m}^{(\kappa)}(t).$$  \hspace{1cm} (5.58)

The Figs. 5.5 ($a_5$ – $a_7$) illustrating the tomogram (5.58) at various times establish that
the higher harmonics are progressively erased due to dissipation, and in the \( t \to \infty \) limit the tomogram (Fig. 5.5 a) reduces to the trivial example of the occupation of the ground state alone: \( \Omega(X, \varphi; t \to \infty) \to \frac{1}{\sqrt{\pi}} \exp(-X^2) \).

### 5.3.2 Phase damping model

For the phase damping model the interaction between the system and the reservoir degrees of freedom is represented by the choice \( X = a^\dagger a \) in the evolution (4.5) of the mixed state density matrix. As the superoperator algebra is fully commutative in the present case the elements of the density matrix in the number state basis (5.10) assume the following localized form in their Fourier indices:

\[
\rho^{(\kappa)}_{n,m}(t) = \exp\left(-i\omega(n-m)t\right) \exp\left(-i\lambda(n-m)(n+m-1)t\right) \times \\
\times \exp\left(-\gamma(n-m)^2t\right) \rho^{(\kappa)}_{n,m}(0), \tag{5.59}
\]

where the initial values of the elements \( \rho^{(\kappa)}_{n,m}(0) \) may be procured via (5.1). It is evident from the phase damping process (5.59) that while the magnitudes of the diagonal components \( \rho^{(\kappa)}_{n,n}(t) \) remain invariant, the off diagonal elements undergo attenuation with time. Moreover, the damping rapidly increases as we progressively move away from the diagonal elements. In the long time \( t \gg \gamma^{-1} \) limit the damped density matrix elements assume a diagonal form, which via (5.11) reads:

\[
\rho^{(\kappa)}_{n,m}(t \to \infty) \to \delta_{n,m} \rho^{(\kappa)}_{n,n}(0) = \delta_{n,m} \left(\mathcal{N}^{(\kappa)}\right)^2 \left(1 + (-1)^{n-\kappa}c\right) \mathcal{A}_{n,\kappa}(\xi,\alpha) \right)^2. \tag{5.60}
\]

The diagonal asymptotically steady state density matrix (5.60) now immediately provides the \( t \gg \gamma^{-1} \) limit of the quasiprobability function (3.28):

\[
Q^{(\kappa)}(\beta, \beta^*; t \to \infty) = \frac{\left(\mathcal{N}^{(\kappa)}\right)^2}{\pi\mu} \exp\left(-|\alpha|^2 - |\beta|^2 - \frac{\alpha^2 \nu^*}{2\mu} - \frac{\alpha^* \nu}{2\mu}\right) |\beta|^{2\kappa} \times \\
\times \sum_{n=0}^{\infty} (1 + |c|^2 + 2(-1)^n \text{Re}(c)) \frac{1}{(n!)^2} \left(\frac{|\nu\beta|^2}{2\mu}\right)^n \left|H_n\left(-\frac{i\alpha}{\sqrt{2\mu\nu}}\right)\right|^2. \tag{5.61}
\]
The sum involving the Hermite polynomials in (5.61) may be recast [97] in terms of the modified Bessel functions in the following form that is particularly amenable for asymptotic evaluations:

\[
\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} H_n(x) H_n(y) = I_0(2t) I_0(4\sqrt{xy}) + \sum_{\ell=1}^{\infty} (-1)^\ell \left( \frac{x^{2\ell} + y^{2\ell}}{x^\ell y^\ell} \right) I_\ell(2t) I_{2\ell}(4\sqrt{xy}), \tag{5.62}
\]

where \(I_n(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell! \Gamma(n+\ell+1)} \left( \frac{x}{2} \right)^{n+2\ell} \). The construction (5.61, 5.62) along with the phase space integral (3.45) allows us to compute the asymptotic limit of the Wehrl entropy \(S_Q\). The Figs. 5.6 (a1, a2, a8) describe the evolution of \(S_Q\) in a phase damped system. As the initial diagonal elements of the density matrix are preserved in the dissipation process (5.60), the asymptotic limit of \(S_Q\) achieves a steady state value. Moreover, a larger value of the squeezing parameter \(r\) reflecting more extensive occupation in the phase space results in a comparatively enhanced asymptotic limit of \(S_Q\). For various choices of \(r\) the corresponding long time limits of \(S_Q\) are given in Table 5.1. Analogously, a higher value of \(\kappa\) for the multiple photon-added states is linked with an increased phase space occupation of the \(Q\)-function, which, in turn, produce a concomitant increment in asymptotic values of \(S_Q\) (Table 5.2).

Contrasting the amplitude damping process, the conserved diagonal density matrix elements (5.60) in the phase damping model manifest nonclassicality even in the long time limit: \(t \gg \gamma^{-1}\). A quantitative indicator of this phenomenon is revealed by the asymptotic limit of the Wigner distribution

\[
W^{(\kappa)}(\beta, \beta^*; t \to \infty) = \frac{2}{\pi \mu} \left( N^{(\kappa)} \right)^2 \exp \left( -|\alpha|^2 - 2|\beta|^2 - \frac{\alpha^2 \nu^*}{2\mu} - \frac{\alpha^*^2 \nu}{2\mu} \right) \times \frac{2|\beta|^{2\kappa}}{(n!)^2} \frac{1}{(\mu^2)^n} \times \left( \frac{2|\nu|^{2\kappa}}{\mu^2} \right)^n \times \left| H_n \left( -\frac{i\alpha}{\sqrt{2\mu \nu}} \right) \right|^2 \frac{1}{\nu} \left( \begin{array}{c} (n + \kappa), -(n + \kappa); -\frac{1}{4|\beta|^2} \end{array} \right) \tag{5.63}
\]

obtained via (5.13, 5.60). The limiting value (5.63) of the \(W\)-distribution now facilitates
the computation, \textit{à la} (3.27), of the negativity $\delta W$ represented in Figs. 5.6 (a$_3$, a$_4$, a$_9$). Attenuation of the off-diagonal elements $\rho_{n,m}^{(a)}(t) \forall n \neq m$ with time initially triggers the decline in $\delta W$ in the scale $t \lesssim \gamma^{-1}$, but subsequently an asymptotically ($t \gg \gamma^{-1}$) stable value of $\delta W$ is realized due to the undamped diagonal density matrix elements.

The asymptotic value of the $W$-distribution (5.63) reveals that even though its spread in the phase space increases with increasing $r$, the domains with negative values of $W^{(\kappa)}(\beta, \beta^*; t \to \infty)$ contracts with the increment of the squeezing parameter, as the latter diminishes the arguments of the Hermite polynomials. Consequently, a larger value of $r$ is found to yield a reduced asymptotic value of the negativity $\delta W$ (Fig. 5.6 a$_3$, Table 5.1). To validate above arguments, we have plotted the asymptotic distribution (5.63) for $\gamma = 0.5$, while maintaining the squeezing parameters $r = 0.05$ (Fig. 5.7 a$_1$) and $r = 0.7$ (Fig. 5.7 a$_2$), respectively. On the other hand, an increased photon-added ($\kappa > 1$) state induces a small augmentation in the sign reversals in the $W^{(\kappa)}(\beta, \beta^*; t \to \infty)$ distribution due to the presence of the increasingly higher order Laguerre polynomial $\, _2F_0(-(n+\kappa),-(n+\kappa);\cdots-|\beta|^2)^{-1}$ in the infinite sum. This causes a modest increment in the asymptotic value of $\delta W$ with increasing $\kappa$ (Fig. 5.6 a$_9$, Table 5.2).

Finally, we enlist the asymptotic behavior of the tomogram (5.58) subject to the evolution (5.60) characterizing the phase damping model:

$$
\Omega^{(\kappa)}(X, \varphi; t \to \infty) = \left( \frac{N^{(\kappa)}}{\sqrt{\pi} \mu 2^\kappa} \right) \exp \left( -|\alpha|^2 - \frac{\alpha^2 \mu^*}{2\mu} - \frac{\alpha^*^2 \nu}{2\mu} - X^2 \right) \times 
\times \sum_{n=0}^{\infty} (1 + |c|^2 + 2(-1)^n \text{Re}(c)) \frac{1}{(n!)^2} \left( \frac{|\nu|}{4\mu} \right)^n \times 
\times (H_{n+\kappa}(X))^2 \left| H_n \left( -\frac{i\alpha}{\sqrt{2\mu\nu}} \right) \right|^2.
$$

(5.64)

The tomogram $\Omega^{(\kappa)}(X, \varphi; t)$ reproduced in Figs. 5.6 (a$_5$–a$_7$) at incremental times exhibit gradual disappearance of phase relationships triggered by the dissipation process. The limiting value (5.64) arrived at $t \gg \gamma^{-1}$ is independent of the phase $\varphi$ of the quadrature variable. This is confirmed in Fig. 5.6 a$_7$. 

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Figure 5.6: Choosing the initial state with parameters $c = 1, \alpha = 2$ the phase damping model is studied for the single photon-added case $\kappa = 1$ in diagrams (a$_1$ - a$_7$). The plots (a$_1$) and (a$_2$) indicate the evolution of the Wehrl entropy. In (a$_1$) the squeezing parameter $r$ is varied at damping coefficient $\gamma = 0.5$, whereas (a$_2$) probes varying damping strength $\gamma$ for a fixed $r = 0.5$. The evolution of negativity is investigated in (a$_3$) and (a$_1$) while maintaining the parameters identical to those described in (a$_1$) and (a$_2$), respectively. The asymptotic values of the $S_Q$ and $\delta_W$, observed sequentially in Figs. (a$_1$) and (a$_3$), are given in Table 5.1 for various choices of the squeezing parameter $r$. The optical tomogram of the phase damped system is explored in (a$_5$ - a$_7$) at increasing times (i) $\lambda t = 0.15$, (ii) $\lambda t = 1.1$, (iii) $\lambda t = 5$ while the parameters $r = 0.5, \gamma = 0.5$ remain constants. Diagrams (a$_8$, a$_9$) examine the variations in the Wehrl entropy and negativity for the multiple photon-added cases ($\kappa > 1$) for fixed values of the parameters $r = 0.5, \gamma = 0.5$. The asymptotic values of the variables $S_Q$ and $\delta_W$ obtained in plots a$_8$ and a$_9$, respectively, are registered in Table 5.2.

Figure 5.7: The plots depict, for varying squeezing parameter $r$, the asymptotic limits of the $W$-distribution for the phase damping model given in (5.63). For the choice $\xi = 0.05$ the diagram (a$_1$) is associated with the negativity $\delta_W(t \to \infty) = 0.270258$, while a larger squeezing $\xi = 0.7$ produces (a$_2$) a significantly lower negativity $\delta_W(t \to \infty) = 0.067378$, even though the phase space occupation of the latter example is much higher. The other parameters are chosen as $c = 1, \kappa = 1, \delta = 1, \gamma = 0.5, \alpha = 2$. 
\[ \kappa = 1, \gamma = 0.5 \]

<table>
<thead>
<tr>
<th>( r = 0.1 )</th>
<th>( r = 0.3 )</th>
<th>( r = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.832996</td>
<td>4.133899</td>
<td>4.461696</td>
</tr>
<tr>
<td>0.243214</td>
<td>0.156939</td>
<td>0.102281</td>
</tr>
</tbody>
</table>

Table 5.1: Asymptotic \((t \gg \gamma^{-1})\) values of the Wehrl entropy \((S_Q)\) and the negativity \((\delta_W)\) for increasing squeezing parameter \(r\) are listed following Figs. 5.6 \((a_1)\) and \((a_3)\), respectively. The parameters and the phase space variable are fixed at \(c = 1, \kappa = 1, \delta = 1, \gamma = 0.5, \alpha = 2\).

<table>
<thead>
<tr>
<th>( r = 0.5, \gamma = 0.5 )</th>
<th>( \kappa = 0 )</th>
<th>( \kappa = 1 )</th>
<th>( \kappa = 2 )</th>
<th>( \kappa = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_Q(t \rightarrow \infty) )</td>
<td>4.367690</td>
<td>4.461696</td>
<td>4.533932</td>
<td>4.593092</td>
</tr>
<tr>
<td>( \delta_W(t \rightarrow \infty) )</td>
<td>0.093517</td>
<td>0.102281</td>
<td>0.105443</td>
<td>0.107013</td>
</tr>
</tbody>
</table>

Table 5.2: Asymptotic \((t \gg \gamma^{-1})\) limits of the dynamical variables \((S_Q, \delta_W)\) for multiple photon-added states for a fixed squeezing parameter are entered following the observations in Figs. 5.6 \((a_8)\) and \((a_9)\), respectively. The parameters and the phase space variable are fixed at \(c = 1, \delta = 1, r = 0.5, \gamma = 0.5, \alpha = 2\).

5.4 Conclusion

We have studied the evolution of a superposition of an arbitrary number of photon-added squeezed coherent cat-type states in a nonlinear Kerr medium. Owing to the nonlinearity of the medium the dynamical quantities such as the Wehrl entropy \(S_Q\) and the negativity \(\delta_W\) of the \(W\)-distribution show a periodic structure, and these quantities exhibit a series of local minima at the rational submultiples of the said period. By using the Hilbert-Schmidt distance between the quantum states we demonstrate that our evolving state transitorily coincides with, in general, the Yurke-Stoler type of photon-added squeezed kitten states, which maintain a uniform rotation of the phase space variables on the complex plane. For the choice of the phase space variables with macroscopically large magnitudes the kitten formations of the quantum states show extremely short-lived behavior. These transient kitten-like states allow closed form construction of the corresponding tomograms which provide the alternate description of the quantum states to the one provided by the quasiprobability distributions. With the increase in number of lobes in the kitten formations, the number of interference terms increases triggering more quantumness of the corresponding states. More complex quantum states embody the dominance of higher Fourier modes of the density matrix which, via rapid oscillations in the phase space, correspondingly produce more nonclassicality in the general
$R$-distribution. The nonclassical depths of the states studied here are observed to attain maximum possible value. The amplitude and the phase dissipation models are studied via their related Lindblad equations. The phase damping model has the property that in the long time limit the dynamical quantities assume nontrivial asymptotic values, and the nonclassicality of the quantum states are partially retained. Even though the higher squeezing parameter is responsible for wider spreading of the quasiprobability functions in the phase space, it, in the long time limit, also produces less oscillatory behavior. The asymptotic limit of the negativity, therefore, decreases with increased squeezing.