Chapter 4

Mathematical formulation
4.1- Rabi Oscillations:

The regime of very high $Q$, where $\frac{\Omega_{ef}}{\Delta \omega_c} > 1$, manifests totally new behaviour [75]. The radiation remains in the cavity, so long that there is a high Probability it will be reabsorbed by the atom dissipates. Spontaneous emission becomes reversible, as the atom and the field exchange excitation of the rate $\Omega_{ef}$. Such behaviour is a well known feature of the interaction of an atom with a classical monochromatic field, these so called "Rabi-Oscillations" [76].

In cavity QED, however the atom couples to its own one – photon field without an external applied field. Thus this effect is called "Vacuum Rabi Oscillation" [77].

The vacuum Rabi oscillation is increased i.e. -

$$\Omega_{R}(N) = \Omega_{ef} \sqrt{N} \quad (4.1)$$

we know that Rabi frequency -

$$\Omega_{ef} = \frac{D_{ef}(\pi D)^{1/2}}{\left(\hbar \varepsilon_r VT\right)^{1/2}} \quad (4.2)$$

from equation (4.1) and (4.2) we get -

$$\Omega_{R}(N) = \frac{D_{ef}(\pi D)^{1/2}}{\left(\hbar \varepsilon_r VT\right)^{1/2}} \sqrt{N} \quad (4.3)$$

On the other hand for n-Photons in the cavity the vacuum Rabi oscillation and Rabi frequency-
\[ \Omega_R(n) = \Omega_{ef} \sqrt{n + 1} \quad (4.4) \]

\[ \Omega_R(n) = \frac{\Omega_{ef}(\pi D)^{1/2}}{(\hbar \varepsilon r VT)^{1/2}} \sqrt{n + 1} \quad (4.5) \]

Rabi oscillation induced by a small thermal field in a cavity. The probability \( P_e(t) \) the state at time \( t \) is

\[ P_e(t) = \sum_n P(n) \cos^2 \left( \frac{1}{2} \Omega_{ef} \sqrt{n + 1} \cdot t \right) \quad (4.6) \]

from Equation (4.2) and Equation (4.6) we get

\[ P_e(t) = \sum_n P(n) \cos^2 \left( \frac{1}{2} \frac{D_{ef}(\pi D)^{1/2}}{(\hbar \varepsilon r VT)^{1/2}} \sqrt{n + 1} \cdot t \right) \quad (4.7) \]

where \( P \) is the probability of photon in cavity.

The electric field strengths \( \vec{E}_1(t) = E_1(t) \vec{e}_z \)

Putting in above Equation \( j = 1, 2 \) for two parts of microwave

\[ \vec{E}_1(t) = E_1(t) \vec{e}_z \quad (4.8) \]

\[ \vec{E}_2(t) = E_2(t) \vec{e}_z \quad (4.9) \]

a pair of coupled damped harmonic oscillator

\[ \frac{\partial^2 E_1(t)}{\partial t^2} + \gamma_1 \frac{\partial}{\partial t} E_1(t) + \omega_1^2 E_1(t) = F_1(t) - \sigma E_1(t) \quad (4.10) \]

for a first part of the microwave.

Similarly,

\[ \frac{\partial^2 E_2(t)}{\partial t^2} + \gamma_2 \frac{\partial E_2(t)}{\partial t} + \omega_2^2 E_2(t) = F_2(t) - \sigma E_2(t) \quad (4.11) \]
for a second part of the microwave.

we may write equation (4.10) and (4.11), we get -

\[
\left( \frac{\partial^2}{\partial t^2} + \gamma_1 \frac{\partial}{\partial t} + \omega_1^2 \right) E_1(t) = F_1(t) - \sigma E_1(t) \tag{4.12}
\]

i.e. \[
\left( \frac{\partial^2}{\partial t^2} + \gamma_2 \frac{\partial}{\partial t} + \omega_2^2 \right) E_2(t) = F_2(t) - \sigma E_2(t) \tag{4.13}
\]

Where \( \omega_1 \) and \( \omega_2 \) are the angular frequencies of first and second part of microwave, \( \gamma_1 \) and \( \gamma_2 \) is the decay rate and a coupling strength \( F_j(t) \) in the coupling of the antenna where \( j=1,2 \).

The coupled differential equation (4.12) and (4.13) can be written in Green function method.

\[
G(\omega) = \frac{1}{4\pi} \frac{e^{i\omega|r-r'|}}{|r-r'|} \tag{4.14}
\]

The inverse of Green function is given by -

\[
G^{-1}(\omega) = \omega_1^2 - H(\omega) \tag{4.15}
\]

The non-Hermitian two-level Hamiltonian function written in the matrix form-

\[
H(\omega) = \begin{pmatrix}
\omega_1^2 - i\omega\gamma_1 & \sigma \\
\sigma & \omega_2^2 - i\omega\gamma_2
\end{pmatrix} \tag{4.16}
\]

equation (4.16) may be written in determinant form-

\[
H(\omega) = \begin{vmatrix}
\omega_1^2 - i\omega\gamma_1 & \sigma \\
\sigma & \omega_2^2 - i\omega\gamma_2
\end{vmatrix} \tag{4.17}
\]

\[
H(\omega) = (\omega_1^2 - i\omega\gamma_1) (\omega_2^2 - i\omega\gamma_2) - \sigma^2
\]
i.e. \( H(\omega) = \omega_1^2 \omega_2^2 - i\omega_1^2 \omega_2 \gamma_2 - i\omega_2^2 \omega_1 \gamma_1 + i^2 \omega^2 \gamma_1 \gamma_2 - \sigma^2 \)

i.e. \( H(\omega) = \omega_1^2 \omega_2^2 - i\omega(\omega_1^2 \gamma_2 + \omega_2^2 \gamma_1) - \omega^2 \gamma_1 \gamma_2 - \sigma^2 \)  \hspace{1cm} (4.18)

from equation (4.18) and (4.16), we get -

\[
G^{-1}(\omega) = \omega_1^2 \omega_2^2 - i\omega(\omega_1^2 \gamma_2 + \omega_2^2 \gamma_1) - \omega^2 \gamma_1 \gamma_2 - \sigma^2
\]

\( (\omega_1^2 \text{ and } \omega_2^2 \text{ the real part of the diagonal elements}) \)

\[
\omega \approx \omega_1 \omega_2
\]

i.e. \( G^{-1}(\omega) = \omega^2 - \omega^4 + i\omega(\omega_1^2 \gamma_2 + \omega_2^2 \gamma_1) - \omega^2 \gamma_1 \gamma_2 - \sigma^2 \)  \hspace{1cm} (4.20)

Putting \( \Gamma_1 = \omega \gamma_1 \) and \( \Gamma_2 = \omega \gamma_2 \) in equation (4.20), we get -

\[
G^{-1}(\omega) = \omega^2 - \omega^4 + i\omega \Gamma_2 + i\omega \Gamma_1 - \Gamma_1 \Gamma_2 - \sigma^2
\]

i.e. \( G^{-1}(\omega) = \omega^2 - \omega^4 + i\omega^2 (\Gamma_2 + \Gamma_1) - \Gamma_1 \Gamma_2 - \sigma^2 \)  \hspace{1cm} (4.21)

\( \omega \) in equation (4.21) is replaced by the average value of angular frequencies \( \bar{\omega} \)-

\[
G^{-1}(\bar{\omega}) = \bar{\omega}^2 - \bar{\omega}^4 + i\bar{\omega}^2 (\Gamma_2 + \Gamma_1) - \Gamma_1 \Gamma_2 - \sigma^2
\]

substituting in above equation -

\[
\bar{\gamma} = \frac{\gamma + \gamma_2}{2}
\]

or \( 2\bar{\gamma} = \gamma + \gamma_2 \)

\[
G^{-1}(\bar{\omega}) = \bar{\omega}^2 - \bar{\omega}^4 + 2i\bar{\omega}^2 \bar{\gamma} - \Gamma_1 \Gamma_2 - \sigma^2
\]

we know that \( \bar{\omega} = \sqrt{\frac{(\omega_1^2 + \omega_2^2)}{2}} \)

\[
\bar{\gamma} = \frac{\gamma_1 + \gamma_2}{2}
\]

\( \therefore 42 \therefore \)
\[ f = \bar{\omega} - \frac{i}{2} \bar{\gamma} \quad (4.23) \]

substituting value \( \bar{\omega} \) and \( \bar{\gamma} \) in equation (4.23), we get -

\[ f = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}} - \frac{i}{2} \left( \frac{\gamma_1 + \gamma_2}{2} \right) \]

i.e. \[ f = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}} - \frac{i}{4} \left( \frac{\gamma_1 + \gamma_2}{2} \right) \quad (4.24) \]

and Rabi oscillation,

\[ \Omega = \frac{R}{2\bar{\omega}} = \frac{R}{\sqrt{\frac{\omega_1^2 + \omega_2^2}{2}}} = \frac{R}{\sqrt{2(\omega_1^2 + \omega_2^2)}} \quad (4.25) \]

The fourier transform \( \overline{G_{12}} (t) \) -

\[ \left| \overline{G_{12}} (t) \right|^2 = \frac{\sigma^2}{\omega^2} \left| \sqrt{\varepsilon_+ + \varepsilon_-} \left( \cos \left( \Omega t \right) + i \frac{\sin \left( \Omega t \right)}{\Omega} \right) \right|^2 \quad (4.26) \]

i.e. \[ e^{-ift} = e^{-i\left( \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}} \cdot \frac{i}{4} (\gamma_1 + \gamma_2) \right)t} \]

i.e. \( \bar{\omega}^2 = \left( \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}} \right)^2 = \frac{(\omega_1^2 + \omega_2^2)}{2} \)

i.e. \( \Omega = \frac{R}{\sqrt{2(\omega_1^2 + \omega_2^2)}} \) and \( \frac{\sin \left( \Omega t \right)}{\Omega} = 1 \) (by Laplace property)

substituting these values in equation No. (26), we get -

\[ \left| \overline{G_{12}} (t) \right|^2 \approx \frac{2\sigma^2}{\omega_1^2 + \omega_2^2} \left| \frac{-i\left( \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}} \cdot \frac{i}{4} (\gamma_1 + \gamma_2) \right)}{\sqrt{\varepsilon_+ + \varepsilon_-}} \left( \cos \left( \frac{Rt}{\sqrt{2(\omega_1^2 + \omega_2^2)}} + ift \right) \right) \right|^2 \quad (4.27) \]

where \( \varepsilon_\pm = \bar{\omega}^2 - i\bar{\Gamma} \pm R \)

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i.e. \( \epsilon_+ = \omega^2 i \Gamma + R \)

and \( \epsilon_- = \omega^2 - i \Gamma - R \)

\[
\epsilon_+ \epsilon_- = \left\{ (\omega^2 - i \Gamma) + R \right\} \left\{ (\omega^2 - i \Gamma) - R \right\}
\]

i.e. \( \epsilon_+ \epsilon_- = \left( \omega^2 - i \Gamma \right)^2 - R^2 \)

substituting in above Equation \( \Gamma = \frac{\Gamma_1 + \Gamma_2}{2} \)

i.e. \( \epsilon_+ \epsilon_- = \left( \frac{\omega^2 + \frac{\omega_2^2}{2} - i(\Gamma_1 + \Gamma_2)}{2} \right)^2 - R^2 \)

i.e. \( \epsilon_+ \epsilon_- = \left( \frac{\omega^2 + \omega_2^2 - i(\Gamma_1 + \Gamma_2)}{2} \right)^2 - R^2 \)  \( (4.28) \)

from equation (4.27) and (4.28) we get -

\[
\left| G_{12}(t) \right|^2 \approx \frac{2 \sigma^2}{\omega_1^2 + \omega_2^2} \left| e^{-i \left\{ \sqrt{\left( \omega_1^2 + \omega_2^2 \right)} - \frac{i}{4} \left( y_1 + y_2 \right) \right\} t} \right|^2 \left( \cos \frac{R t}{\sqrt{2(\omega_1^2 + \omega_2^2)}} + i \sqrt{\frac{(\omega_1^2 + \omega_2^2)}{2}} + \frac{(y_1 + y_2)}{4} \right)^2 \]  \( (4.29) \)

\[
\lim_{(s, \delta) \to EP} \left| G_{12}(t) \right|^2 \approx \frac{2 \sigma^2}{\omega_1^2 + \omega_2^2} \left( \frac{t^2}{\left\{ \left( \omega_1^2 + \omega_2^2 - i(\Gamma_1 + \Gamma_2) \right)^2 - R^2 \right\} - R^2} \right)^{1/2} \cdot e^{-\frac{t}{4}(y_1 + y_2)} \]  \( (4.30) \)

we may written equation (4.30)-

\[
\lim_{(s, \delta) \to EP} \left| G_{12}(t) \right|^2 \approx \frac{2 \sigma^2}{\omega_1^2 + \omega_2^2} \left( \frac{t^2}{\left\{ \left( \omega_1^2 + \omega_2^2 - i(\Gamma_1 + \Gamma_2) \right)^2 - R^2 \right\} - R^2} \right)^{1/2} \cdot e^{-\frac{t}{2R}} \]  \( (4.31) \)

we apply in equation (4.16) of Liouville Equation -

\[
\frac{\partial \rho}{\partial t} = \frac{1}{i \hbar} [H, \rho]
\]
\[
\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} \begin{pmatrix}
\omega_1^2 - i\omega_1 \gamma_1 & \sigma \\
\sigma^* & \omega_2^2 - i\omega_2 \gamma_2
\end{pmatrix} \rho \quad (4.32)
\]

Where \(\rho\) is the density matrix element and \(H\) is the interaction Hamiltonian.

The steady state solution for the density matrix element is:

\[
\rho_{11} = \rho_{12} = \frac{1}{2}
\]

\[
\rho_{33} = \left(\frac{\omega_R}{\sigma^*}\right)^2 \left(1 - \frac{\omega_R^2}{\Gamma^*} \frac{\Gamma'}{\gamma_{12} + \Omega_\mu^2}\right)
\]

\[
\rho_{12} = \delta_{12} e^{i(\omega_1 - \omega_2)t} = \frac{-\omega_R^2}{2\Gamma^*} \frac{1}{\Gamma' + i\Omega_\mu^2} e^{i(\omega_1 - \omega_2)t}
\]

frequency difference between two laser fields \((\omega_1 - \omega_2)\).

The atomic hyperfine transition \(\Gamma' = \gamma_2 + \omega_R^2/\Gamma^*\), \(\gamma_2\) being the relaxation rate of the coherence in the ground state. \(\Omega_\mu\) is the fractional population of the groundstate \(\Omega_\mu\), where \(\omega_R\) is the Rabi angular frequency.

The population \(\rho_{33}\) of the excited level has a minimum

where \(\Omega_\mu = 0\)

The echo frequency is:

\[
\Rightarrow \frac{\Omega}{2\pi} = \frac{R}{2\sigma/2\pi} = \frac{R}{4\pi\sigma} \quad \therefore \text{where } \Omega = \frac{R}{2\omega} \quad (4.34)
\]

for echo frequency \(R << \omega\)
Then this condition $\frac{\Omega}{2\pi}$ or echo frequency is very low.

**4.2- Rabi Oscillations of Frequency Emission in Free Space:**

The rms vacuum electric field amplitude $E_{\text{vac}}$ in a mode of frequency -

$$E_{\text{vac}} = \left( \frac{\hbar \omega}{2 \varepsilon_0 \nu} \right)^{1/2} \quad (4.35)$$

we apply the permittivity may also be divided into two parts -

$$\varepsilon = \varepsilon_0 \varepsilon_r \quad (4.36)$$

Where $\varepsilon_0$, the permittivity constant or permittivity of free space and $\varepsilon_r$ is the dimensionless relative permittivity.

from above these two equations-

$$\varepsilon_0 = \frac{\varepsilon_r}{\varepsilon} \quad (4.37)$$

$$E_{\text{vac}} = \left( \frac{\hbar \omega}{2 \varepsilon_r \nu / \varepsilon} \right)^{1/2} = \left( \frac{\hbar \omega \varepsilon}{2 \varepsilon_r \nu} \right)^{1/2} \quad (4.38)$$

In above equation we applied electric field and electric flux density $D$ where $\varepsilon$ is the permittivity of the medium is a dimensional constant -

$$\varepsilon = \frac{D}{E} = \text{farad/meter}$$

$$E_{\text{vac}} = \left( \frac{\hbar \omega D}{2 \varepsilon \varepsilon_r \nu} \right)^{1/2} \quad (4.39)$$

Where $E$ is the electric field in free space -
Where \( V \) is the size of an arbitrary quantization of volume.

The coupling of an atom to a field made is described by the frequency-

\[
\Omega_{ef} = \frac{D_{ef}E_{vac}}{\hbar} \quad (4.40)
\]

from the above two equation -

\[
\Omega_{ef} = \frac{D_{ef}(\hbar\omega_{D}/2E_{e,v})^{1/2}}{\hbar} \quad (4.41)
\]

Where \( D_{ef} \) is the matrix element of electric dipole of the atom and \( \Omega_{ef} \) is the Rabi frequency of the vacuum, where \( \omega \) is the angular velocity of Rabi frequency.

Substituting in above equation \( \omega=2\pi n \) and \( n = \frac{1}{T} \)

\[
\Omega_{ef} = \frac{D_{ef}(\pi D)^{1/2}}{(\hbar E_{e,v}T)^{1/2}} \quad (4.42)
\]

Apply in above equation to fermi golden Rule -

\[
\Gamma_0 = 2\pi\Omega_{ef}^2 \frac{\rho_0(\omega)}{3} = 2\pi D_{ef}^2 \frac{\pi D}{\hbar E_{e,v}T} \frac{\rho_0(\omega)}{3} \quad (4.43)
\]

where \( \Gamma_0 \) the probability of fermi Golden rule -

4.3- Relation Between Saturation Intensity and Rabi Oscillation Frequency:

The saturation intensity \( I_{sat} \), is defined as, such that the laser intensity,

\[
I_0 = \frac{1}{2} c\varepsilon_0 |E|^2 \quad (4.44)
\]
Equals $I_{\text{sat}}$, one quarter of the atoms, will be an expression for excited population in excited state ($f_e = \frac{1}{4}$) can be written as -

$$f_e = \frac{n_e^{2/4}}{n_e^{2/4} + \Delta^2 + \Gamma^2/4} \quad (4.45)$$

where $\Omega_{ef}$ is the Rabi frequency, $\Delta$ is the laser detuning from resonances, and $\Gamma$ is the decay rate -

for zero detuning, $f_e=1/4 \quad (4.46)$

where $\Omega_{ef}^2 = \frac{\Gamma^2}{2} \quad (4.47)$

or $\frac{I_0}{I_{\text{sat}}} = \frac{2\Omega_{ef}^2}{\pi^2} \quad (4.48)$

Recall that the Rabi frequency is defined as -

$$\Omega_{ef}^2 = \frac{(\text{ground} \, |d\text{e}| \text{excited})E}{\hbar} = \frac{dE}{\hbar} \quad (4.49)$$

with this equation we can write an expression for $\frac{I_0}{I_{\text{sat}}}$ -

$$\frac{I_0}{I_{\text{sat}}} = \frac{1}{2}(\varepsilon_0 |E|^2) \frac{d^2E^2}{\hbar^2 \Gamma^2} \quad (4.50)$$

The decay rate for a Transmission can also be related to the expectation value of the dipole moment -

$$\Gamma = \frac{\omega_0^3 d^2}{3 \pi \varepsilon_0 \hbar c^3} \quad (4.51)$$

Then we can write the saturation intensity as -
\[ I_{\text{sat}} = \frac{\hbar c \pi \Gamma}{3 \lambda^3} \quad (4.52) \]

4.4- Three Level Analysis of Hamiltonian:

The effective three-level Hamiltonian we can find exact solutions to the eigen value problem.

\[ H^{(n)} \Psi = E \Psi \quad (4.53) \]

Let \( E = h \omega \), the exact eigen values of \( H^{(n)} \) are given by the roots of cubic equation:

\[ x^3 - (\varepsilon_1 + \varepsilon_2)x^2 + \left( \varepsilon_1 \varepsilon_2 - \frac{1}{4} (\Omega_{01}^2 + \Omega_{12}^2 + \Omega_{02}^2) \right) x \]

\[ + \frac{1}{4} (\varepsilon_1 \Omega_{02}^2 + \varepsilon_2 \Omega_{01}^2 - \Omega_{01} \Omega_{12} \Omega_{02}) = 0 \quad (4.54) \]

This has the exact solution:

\[ x_n = \frac{1}{3} (\varepsilon_1 + \varepsilon_2) + 2 \rho^{1/2} \cos \left( \theta + \frac{2n\pi}{3} \right) \quad (4.55) \]

\((n = 0, 1, 2)\) with

\[ \theta = \frac{1}{3} \arccos \left( -\frac{q}{2\rho^3}\varepsilon \right) \quad (4.56) \]

\[ P = \frac{1}{9} (\varepsilon_1 + \varepsilon_2)^2 - \frac{1}{3} \varepsilon_1 \varepsilon_2 + \frac{1}{12} (\Omega_{01}^2 + \Omega_{12}^2 + \Omega_{02}^2) \quad (4.57) \]

and \( q = \frac{-2}{27} (\varepsilon_1 + \varepsilon_2)^3 + \frac{1}{4} (\varepsilon_1 \Omega_{02}^2 + \varepsilon_2 \Omega_{01}^2 - \Omega_{01} \Omega_{02} \Omega_{12}) \]

\[ + \frac{1}{3} (\varepsilon_1 + \varepsilon_2) \left[ \varepsilon_1 \varepsilon_2 - \frac{1}{4} (\Omega_{01}^2 + \Omega_{12}^2 + \Omega_{02}^2) \right] \quad (4.58) \]
The eigen vectors are:

\[ |v_n> = N \begin{pmatrix} 2(\varepsilon_1 - x_n)\Omega_{02} - \Omega_{01}\Omega_{12} \\ -2x_n\Omega_{12} - \Omega_{01}\Omega_{02} \\ \Omega_{01}^2 + 4(\varepsilon_1 - x_n)x_n \end{pmatrix} \]  \hspace{1cm} (4.59)

with Normalization:

\[ N = \left[ (2(\varepsilon_1 - x_n)\Omega_{02} - \Omega_{01}\Omega_{12})^2 + (-2x_n\Omega_{12} - \Omega_{01}\Omega_{02})^2 + \right. \\
\left. (\Omega_{01}^2 + 4(\varepsilon_1 - x_n)x_n)^2 \right]^{-1/2} \]  \hspace{1cm} (4.60)

using these eigen vectors are can calculated the time dependent amplitudes for a given initial state that is if-

\[ |\Psi(t)> = a_0(t)|0> + a_1(t)|1> + a_2(t)|2> \]  \hspace{1cm} (4.61)

then given \( a_n(0) \), the final amplitudes are-

\[ a_n(t) = \sum_{l,m=0}^{2} e^{-i\Omega_{lm}^R t} <n|v_l><v_l|m|a_m(0) \]  \hspace{1cm} (4.62)

The eigen values \( x_0, x_1 \) and \( x_2 \) can be used to identify the various transitions, and the effective Rabi frequency are given by differences-

\[ \Omega_{R,01} = x_0 - x_2, \hspace{1cm} \Omega_{R,02} = x_2 - x_1 \]  \hspace{1cm} (4.63)

4.5- Intense Fields and Quantum Electrodynamics (Particles in Laser Field):

Quantum electrodynamics in intense fields (IFQED) is based on a semiclassical approximation scheme. All particles whose interaction processes are studied (electrons, non-laser photons).
A plane wave is characterized by a Propagation vector-

\[ K^\mu = \omega (1, \vec{n}) , \omega = \frac{2\pi}{\lambda} \]  \hspace{1cm} (4.64)

where \( \lambda \) is the wavelength.

we know that \( \omega = 2\pi n \), Putting eqn. No. (4.64)

\[ K^\mu = 2\pi n (1, \vec{n}) \] \{where \( n \) is the frequency of the particle\}

or \[ K^\mu = \frac{2\pi}{T} (1, \vec{n}) \] \{where \( T \) is the atomic Time \}

\[ K^2 = 0 \] (and two mutually orthogonal polarization vectors)

\[ e^\mu_i, i = 1, 2; \quad e_i e_j = -\delta ij; \quad k.e_i = 0 \]

The most general \( a_n \)'s at \( z \) for the corresponding vector potential reads-

\[ A^\mu (x) = a \sum_{i=1}^{2} e^\mu_i a_i (\xi), \xi = k.x \]  \hspace{1cm} (4.65)

Corresponding to arbitrary (elliptic) polarization 'a' is an amplitude factor and the arbitrary functions 'a_i' characterize the frequency decomposition. For a monochromatic wave train of infinite extent and circular polarization wave i.e.

\[ a_1 = \cos \xi, \quad a_2 = -\sin \xi \]  \hspace{1cm} (4.66)

Expand equation (4.65) we get-

\[ A^\mu (x) = a \left\{ e_1^\mu a_1 (\xi) + e_2^\mu a_2 (\xi) \right\} \]

or \[ A^\mu (x) = a \left\{ e_1^\mu a_1 (k.x) + e_2^\mu a_2 (k.x) \right\} \]  \hspace{1cm} (4.67)
Substituting value of $a_1$ and $a_2$ from eqn. (4.66) in eqn. (4.67) we get -

$$A^\mu(x) = a \left\{e_1^\mu \cos(k \cdot x) - e_2^\mu \sin(k \cdot x)\right\} \quad (4.68)$$

i.e. $A^\mu(x) = a \left\{e_1^\mu \cos(\xi) - e_2^\mu \sin(\xi)\right\} \quad (4.69)$

For a finite Pulse the $a_i$'s are zero outside a finite domain in $\xi$ and for Polychromatic waves. Consider a spectral superpositions in $\omega$.

The Solution may be written as an infinite sum of Plane waves with wave vectors -

$$p_{(n)} = \vec{P} - nk, n = 0, \pm 1, \pm 2 \quad (4.70)$$

where $\vec{P}$ is an "effective" momentum of the Particle in the field.

$$\vec{P} = P + \frac{v^2}{\rho} k \quad (4.71)$$

Substituting value of $\vec{P}$ from equation (4.71) in equation (4.70) we get-

$$p_{(n)} = P + \frac{v^2}{\rho} - nk \quad (4.72)$$

$P$ is the four momentum vector of a free particle.

$$p^2 = k^2, \quad k = \frac{mc}{\hbar} \quad (4.73)$$

From equation (4.72) and (4.73) we get -

$$p_{(n)} = P + \frac{v^2}{\rho} - \frac{nmc}{\hbar} \quad (7.74)$$

Inverse Compton wave length.
\[ k = \frac{2\pi}{\lambda c} \quad (4.75) \]

substituting value of \( k \) from equations (4.75) in equation (4.72) we, get -

\[ P_{(n)} = P + \frac{v^2}{\rho} - \frac{2\pi n}{\lambda c} \]

or \( P_{(n)} = P + \frac{v^2}{\rho} - \frac{\omega}{\lambda c} \quad (4.76) \)

where \( \omega \) is the angular velocity of the particle and dimensionless parameters \( v \) and \( p \) are characteristic of the modification of the particle’s the properties due to the interaction with the wave field.

They are given by -

\[ v = \frac{ea}{mc^2} \quad (4.77) \]

where \( e \) is the charge of the particle.

squarring both sides of equation No. (7.77) we get-

\[ v^2 = \frac{e^2a^2}{m^2c^4} \quad (4.78) \]

from equation (4.76) and (4.78) we get-

\[ P_{(n)} = P + \frac{e^2a^2}{m^2c^4p} - \frac{\omega}{\lambda c} \quad (4.79) \]

and \( \rho = \frac{2(Pk)}{k^2} \quad (4.80) \)

from equation (4.79) and (4.80) we get-
\[ P_{(n)} = P + \frac{e^2 a^2 k^4}{2m^2 c^4 (Pc)} - \frac{\omega}{\lambda c} \]

or \[ P_{(n)} = P + \frac{e^2 a^2 k}{2m^2 c^4 P} - \frac{\omega}{\lambda c} \] (4.81)

Thus the effective momentum depends on the intensity of the field via the (Classical) Parameter \(\nu\). Thus the laser influences the particle in two ways-

a) The momentum \(P\) is replace by \(\vec{P}\), we have -

\[ \vec{P}^2 = k^{-2} \]

or

\[ \vec{P}^2 = k^2 \left(1 + \nu^2\right) \] (4.82)

(Intensity dependent mass shift)

Equation No. (4.80) may be written as-

\[ \rho = \frac{2 (pk)}{k^2} \]

or \[ k^2 = \frac{2(pk)}{\rho} \] (4.83)

substituting value of \(k^2\) and \(\nu^2\) from equation (4.78) and (4.83) in equation (4.82), we get -

\[ \vec{P}^2 = \frac{2(pk)}{\rho} \left[1 + \frac{e^2 a^2}{m^2 c^4}\right] \] (4.84)

we know that Einstine Relations-

\[ E = mc^2 \] (4.85)

Squarring equation (4.85) of both sides, we get -
\[ E^2 = m^2 c^4 \quad (4.86) \]

from equation (4.84) and (4.85) we, get -

\[ \vec{p}^2 = \frac{2(pk)}{\rho} \left[ 1 + \frac{e^2 a^2}{E^2} \right] \quad (4.87) \]

when sun light intensity is less than the comparison of energy or velocity of sun light.

\[ (e << E \text{ or } e << mc^2 \text{ i.e. } e^2 << E^2) \]

Then from equation No. (4.87) of \( \frac{e^2 a^2}{E^2} \) is negligible, we get -

\[ \vec{p}^2 = \frac{2(pk)}{\rho} \quad (4.88) \]

4.6- Stark Shift in Perturbation:

The Hamiltonian \( \hat{H} \) in the schrodinger equation can be written as the sum of two parts, one of these parts \( \hat{H}^0 \) corresponds to unperturbed system and other part \( \hat{H}' \) corresponds to perturbation effects. Let us write Schrodinger wave Equation-

\[ \hat{H} \Psi = E \Psi \quad (4.89) \]

in which Hamiltonian \( \hat{H} \) represents the operator

\[ \hat{H} = \frac{-\hbar^2}{2} \sum \frac{1}{m_i} \nabla^2_i + V \quad (4.90) \]

Let \( E \) be the Eigen value and \( \Psi \) be the Eigen function of operator \( \hat{H} \). \( \hat{H} \) is the sum of two terms \( \hat{H}^0 \) and \( \hat{H}' \) already defined as -

\[ H = H^0 + H' \quad (4.91) \]
where $H'$ is a small perturbation term.

Let $\Psi_k^0$ and $E_k^0$ be a particular orthonormal eigen function and eigen value of unperturbed Hamiltonian $H^0$ i.e.

$$H^0\Psi_k^0 = E_k^0\Psi_k^0$$

If we consider non-degenerate system for which there is one eigen function corresponding to each eigen values. In the stationary system the Hamiltonian $H$ does not depend upon time and it is possible to expand $H$ in terms of some parameter $\lambda$ yielding the expressions-

$$H = H^0 + \lambda H' + \lambda^2 H'' + \cdots$$ \hspace{1cm} (4.92)

in which $\lambda$ has been choosen in such a way that equation (4.89) for $\lambda=0$ reduces to the form -

$$H^0\Psi^0 - E^0\Psi^0 = 0$$ \hspace{1cm} (4.93)

It is to be remembered that there is one eigen function $\Psi$ and energy level $E^0$ corresponding to operator $H^0$. Equation (4.93) can be directly solved. This equation is said to be the "wave equation of unperturbed system" while the terms, $\lambda H' + \lambda^2 H'' + \cdots$, are called the perturbation terms.

The unperturbed Equation (4.93) has solutions-

$$\Psi_0^0, \Psi_1^0, \Psi_2^0, \cdots, \Psi_k^0, \cdots$$

called the unperturbed eigen function and corresponding eigen values are-
$E_0^0, E_1^0, E_2^0, \ldots, E_k^0, \ldots\ldots$

The function $\Psi_k^0$ form a complete orthonormal set, i.e. they satisfy the condition -

$$\int \Psi_i^0*\Psi_j^0 \, d\tau = \delta_{ij} \quad (4.94)$$

where $\delta_{ij}$ is kronecker delta symbol defined as:

$$\delta_{ij} = 0 \text{ for } i \neq 0$$

$$\delta_{ij} = 1 \text{ for } i = j$$

Now let us consider the effect of perturbation. The application of perturbation does not cause large changes, hence the energy values and wave functions for the perturbed system will be near to these for the unperturbed system. We can expand the energy $E$ and wave functions $\Psi$ for the perturbed system in terms of $\lambda$, so-

$$\Psi_k = \Psi_k^0 + \lambda \Psi_k' + \lambda^2 \Psi_k'' + \ldots \ldots \quad (4.95)$$

$$E_k = E_k^0 + \lambda E_k' + \lambda^2 E_k'' + \ldots \ldots \quad (4.96)$$

If the perturbation is small, then terms of the series (4.95) and (4.96) will become rapidly smaller i.e. the series will be convergent.

Now substituting value from equations (4.94), (4.95) and (4.96) in Eqn. (4.89) we get -

$$= (H^0 + \lambda H' + \lambda^2 H'' + \cdots) (\Psi_k^0 + \lambda \Psi_k' + \lambda^2 \Psi_k'' + \cdots)$$

$$= (E_k^0 + \lambda E_k' + \lambda^2 E_k'' + \cdots) (\Psi_k^0 + \lambda \Psi_k' + \lambda^2 \Psi_k'' + \cdots)$$

on collecting the coefficients of like powers of $\lambda$ -

\[\text{:- 57 :-}\]
\[(H_0 \Psi_k^0 - E_k^0 \Psi_k^0) + (H_0 \Psi_k' + H'\Psi_k^0 - E_k^0 \Psi_k' - E_k' \Psi_k^0)\lambda + (H^0 \Psi_k^0 + H'\Psi_k^0 - E_k^0 \Psi_k' - E_k' \Psi_k^0)\lambda^2 + \ldots = 0 \quad (4.97)\]

If this series is properly convergent i.e. equal to zero for all possible values of \(\lambda\), then coefficients of various powers of \(\lambda\) must vanish separately. These equations will have successively higher order of the perturbation. The coefficient of \(\lambda^0\) gives -

\[ (H^0 - E_k^0)\Psi_k^0 = 0 \quad (4.98 \text{ a}) \]

The coefficient of \(\lambda\) gives the Equation -

\[ (H_0 \Psi_k' + H'\Psi_k^0 - E_k \Psi_k' - E_k' \Psi_k^0) = 0 \]

or \((H^0 - E_k^0)\Psi_k' + (H' - E_k')\Psi_k^0 = 0 \quad (4.98 \text{ b})\]

The coefficient of \(\lambda^2\) gives the equation -

\[ (H^0 \Psi_k^0 + H'\Psi_k^0 + H''\Psi_k^0 - E_k^0 \Psi_k' - E_k' \Psi_k' - E_k'' \Psi_k^0) = 0 \]

\[ (H^0 - E_k^0)\Psi_k^0 + (H' - E_k')\Psi_k' + (H'' - E_k'')\Psi_k^0 = 0 \quad (4.98 \text{ c}) \]

Similarly the Coefficient of \(\lambda^3\) yield -

\[ (H^0 - E_k^0)\Psi_k^0 + (H' - E_k')\Psi_k' + (H'' - E_k'')\Psi_k' + (H''' - E_k''')\Psi_k^0 = 0 \quad (4.98d) \]

But if we limit the total Hamiltonian \(H\) upto \(\lambda H'\), i.e. if we put, \(H = H^0 + \lambda H'\), the equations (4.98) will be magnified as-

\[
\begin{align*}
(H^0 - E_k^0)\Psi_k^0 &= 0 \quad \text{--- (a)}
\end{align*}
\]

\[
\begin{align*}
(H^0 - E_k^0)\Psi_k' + (H' - E_k')\Psi_k^0 &= 0 \quad \text{--- (b)}
\end{align*}
\]

\[
\begin{align*}
(H^0 - E_k^0)\Psi_k'' + (H' - E_k')\Psi_k' + (H'' - E_k'')\Psi_k^0 &= 0 \quad \text{--- (c)}
\end{align*}
\]

\[
\begin{align*}
(H^0 - E_k^0)\Psi_k''' + (H' - E_k')\Psi_k'' + (H'' - E_k'')\Psi_k' + (H''' - E_k''')\Psi_k^0 &= 0 \quad \text{--- (d)}
\end{align*}
\]

\[ (4.99) \]
First Order Perturbation:

Equation (4.99b) is,

\[(H^0 - E^0_k)\Psi'_k + (H' - E'_k)\Psi^0_k = 0\]

To solve this equation we use expansion theorem. As perturbation is very small, the deviations form unperturbed state are small, therefore the first order perturbation correction function \(\Psi'_k\) can be expanded in terms of unperturbed functions \(\Psi^0_1, \Psi^0_2, \ldots, \Psi^0_l, \ldots\). Since \(\Psi^0_l\) from a normalized orthonormal set.

Hence we can write -

\[\Psi'_k = \sum_{l=0}^{\infty} a_l \Psi^0_l \quad (4.100)\]

Substituting \(\Psi'_k\) from (4.100) in (4.99b) we get -

\[(H^0 - E^0_k) \sum_l a_l \Psi^0_l + (H' - E'_k) \Psi^0_k = 0\]

i.e. \[\sum_l a_l H^0 \Psi^0_l - E^0_k \sum_l a_l \Psi^0_l + (H' - E'_k) \Psi^0_k = 0\]

using \(H^0 \Psi^0_l = E^0_l \Psi^0_l\) we get

\[\sum_l a_l E^0_l \Psi^0_l - E^0_k \sum_l a_l \Psi^0_l + (H' - E'_k) \Psi^0_k = 0\]

\[\sum_l a_l (E^0_l - E^0_k) \Psi^0_k = (E^0_k - H') \Psi^0_k \quad (4.101)\]

Multiplying above equation by \(\Psi^0_m\) and integrating over configuration space we get-

\[:: 59 ::\]
\[ \sum a_l \left( E_l^0 - E_k^0 \right) \int \psi_m^{0*} \psi_l^0 d\tau = \int \psi_m^{0*} (E'_k - H') \psi_k^0 d\tau \]

using the condition of orthonormalisation of \( \psi_0 \) is -

\[ \text{i.e. } \int \psi_i^{0*} \psi_j^0 d\tau = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \]

we get

\[ \sum q_i \left( E_l^0 - E_k^0 \right) \delta_{ml} = \int \psi_m^{0*} E_k^0 \psi_k^0 d\tau = \int \psi_m^{0*} H' \psi_k^0 d\tau \]

\[ = E_k' \delta_{mk} - \int \psi_m^{0*} H' \psi_k^0 d\tau \]

using the notations -

\[ \int \psi_m^{0*} H' \psi_k^0 d\tau = \langle m | H' | k \rangle \]

we get

\[ \sum a_i \left( E_l^0 - E_k^0 \right) \delta_{ml} = E_k' \delta_{ml} - \langle m | H' | k \rangle \quad (4.102) \]

Evaluation of first order energy \( E_k' \), setting \( m=k \) in equation (4.102), we observe that -

\[ \sum q_l \left( E_l^0 - E_k^0 \right) \delta_{kl} = 0 \quad \text{always} \]

Since for \( l=k, E_l^0 - E_k^0 = 0 \) and for \( l \neq k, \delta_{kl} = 0 \) so that.

we get

\[ E_k' - \langle k_m | H' | k \rangle = 0 \]

or

\[ E_k' = \langle k | H' | k \rangle = \int \psi_k^{0*} H' \psi_k^0 d\tau \quad (4.103) \]

This expression gives first order perturbation energy correction. Accordingly the "first order perturbation energy correction for a non-degenerate system is just the expectation value
of first order perturbed Hamiltonian (H') over the unperturbed state of the systems”.

Evaluation of first order Correction to wave function-

Equation (4.102) may be expressed as-

\[ a_m(E_m^0 - E_k^0) = E_k' \delta_{mk} - <m|H'|k> \]  \hspace{1cm} (4.104)

Since \( \delta_{ml} = \begin{cases} 0 & \text{for } l \neq m \\ 1 & \text{for } l = m \end{cases} \)

for \( m \neq k \) equation (4.104) gives -

\[ a_m(E_m^0 - E_k^0) = - <m|H'|k> \]

\[ a_m = \frac{-<m|H'|k>}{E_m^0 - E_k^0} = \frac{<m|H'|k>}{E_k^0 - E_m^0} \]

Setting \( m=l, a_l = \frac{<l|H'|k>}{E_k^0 - E_l^0} \)  \hspace{1cm} (4.105)

If we retain only first order correction terms, then -

\[
\begin{align*}
E_k &= E_k^0 + \lambda E_k' \\
\Psi_k &= \Psi_k^0 + \lambda \Psi_k' \\
\end{align*}
\]  \hspace{1cm} (a)  \hspace{1cm} (4.106)

keeping in view equation (4.100) and (4.105) we get from (4.106b)-

\[ \Psi_k = \Psi_k^0 + \lambda \sum_{l} \frac{<l|H'|k>}{E_k^0 - E_l^0} \Psi^0_l + \lambda a_k \Psi_k^0 \]  \hspace{1cm} (4.107)

Where prime (or dash) on summation indicates that the term \( l=m \) has been omitted from the summation.

The value of constant \( a_k \) may be evaluated by requiring that \( \Psi_k \) is normalized i.e.
\[ \int \Psi_k^* \Psi_k \, d\tau = 1 \quad (4.108) \]

substituting \( \Psi_k \) from (4.107) and retaining only first order terms in \( \lambda \)

we get -

\[
\int \Psi_k^0 \Psi_k^0 \, d\tau + \lambda a_k \int \Psi_k^0 \Psi_k^0 \, d\tau + \lambda a_k^* \int \Psi_k^0 \Psi_k^0 \, d\tau + \lambda \sum_l \left\{ \frac{\langle l | H' | k >}{E_k^0 - E_l^0} \right\} \delta_{lk} + \lambda \sum_l \left\{ \frac{\langle l | H' | k >}{E_k^0 - E_l^0} \right\}^* \delta_{lk} = 1
\]

or \( \lambda a_k = \lambda a_k^* = 0 \) i.e. \( a_k + a_k^* = 0 \) \quad (4.109)

This equation indicates that the real part of \( a_k \) is zero and still it leaves an arbitrary choice for the imaginary parts-

Let us take \( a_k = i\gamma \)

The wave function \( \Psi_k \) can be expressed as -

\[
\Psi_k = \Psi_k^0 + \lambda i\gamma \Psi_k^0 + \lambda \sum_l \left\{ \frac{\langle l | H' | k >}{E_k^0 - E_l^0} \right\} \Psi_l^0
\]

\[= \Psi_k^0 (1 + i\lambda \gamma) + \lambda \sum_l \left\{ \frac{\langle l | H' | k >}{E_k^0 - E_l^0} \right\} \Psi_l^0 \quad (4.110)\]

The term containing \( \gamma \) merely gives a phase shift in the unperturbed function \( \Psi_k^0 \) and for normalization, this shift can be put equal to zero, so that equation (4.110) gives-

\[
\Psi_k = \Psi_k^0 + \lambda \sum_l \left\{ \frac{\langle l | H' | k >}{E_k^0 - E_l^0} \right\} \Psi_l^0 \quad (4.111)
\]

The arbitrary \( \lambda \) can put equal to 1 and it may be included in symbols i.e. \( \lambda H' \to H' \); then eigen values and eigen functions of the
system upto first order perturbation correction terms are expressible as:

\[
E_k = E_k^0 + \langle k | H' | k \rangle \quad - - - (a)
\]

and

\[
\Psi_k = \Psi_k^0 + \sum_l \frac{\langle l | H' | k \rangle}{E_k^0 - E_l^0} \psi_l^0 \quad - - - (b)
\]

Second order Perturbation:

We know that, the second order perturbation equation (4.99c) is:

\[
(H^0 - E_k^0)\Psi''_k + (H' - E_k')\Psi'_k - E_k''\Psi_k^0 = 0 \quad (4.113)
\]

Expanding second order wave function \(\Psi''_k\) as a linear combination of unperturbed orthonormal wave functions \(\Psi_m^0\) by expansion theorem i.e.

\[
\Psi''_k = \sum_m b_m \Psi_m^0 \quad (4.114)
\]

substituting \(\Psi'_k = \sum_l \frac{\langle l | H' | k \rangle}{E_k^0 - E_l^0} \psi_l^0\), \(\Psi''_k = \sum_m b_m \Psi_m^0\)

and \(E_k' = \langle k | H' | k \rangle\) in (4.99c) we get

\[
(H^0 - E_k^0) \sum_m b_m \Psi_m^0 + (H' - \langle k | H' | k \rangle) \sum_l \frac{\langle l | H' | k \rangle}{E_k^0 - E_l^0} \psi_l^0 - E_k'' \Psi_k^0 = 0
\]

or

\[
\sum_m b_m \Psi_k^0 - E_k^0 \sum_m b_m \Psi_k^0 + (H' - \langle k | H' | k \rangle) \sum_l \frac{\langle l | H' | k \rangle}{E_k^0 - E_l^0} \psi_l^0 - E_k'' \Psi_k^0 = 0
\]

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using unperturbed schroedinger equation-

\[ H^0 \psi^0_m = E^0_m \psi^0_m, \text{ we get} \]

\[ \sum_m b_m E^0_m \psi^0_k - E^0_k \sum_m b_m \psi^0_k + (H' - \langle k|H'|k \rangle) \sum_l \frac{\langle l|H'|k \rangle}{E^0_k - E^0_l} \psi^0_l - E^0_k \psi^0_k = 0 \]

or \[ \sum_m b_m (E^0_m - E^0_k) \psi^0_m + (H' - \langle k|H'|k \rangle) \sum_l \frac{\langle l|H'|k \rangle}{E^0_k - E^0_l} \psi^0_l - E^0_k \psi^0_k = 0 \]

Multiplying by \( \psi^0_n^* \) and integrating over all space, we get-

\[ \sum_m b_m (E^0_m - E^0_k) \int \psi^0_n^* \psi^0_m d\tau + \int \psi^0_n^* (H' - \langle k|H'|k \rangle) \sum_l \frac{\langle l|H'|k \rangle}{E^0_k - E^0_l} \psi^0_l d\tau 
- E^0_k \int \psi^0_n^* \psi^0_k d\tau = 0 \]

using orthonormal property of unperturbed wave functions \( \psi^0_0, s \) we get-

\[ \sum_m b_m (E^0_m - E^0_k) \delta_{nm} + \sum_l \frac{\langle l|H'|k \rangle}{E^0_k - E^0_l} - \sum_l \frac{\langle k|H'|l \rangle}{E^0_k - E^0_l} \delta_{ml} - E^0_k \delta_{nk} = 0 \quad (4.115) \]

Evaluation of second order energy correction.

setting \( n=k \) in eqn. (4.115) we get-

\[ \sum_m b_m (E^0_m - E^0_k) \delta_{km} + \sum_l \frac{\langle l|H'|k \rangle}{E^0_k - E^0_l} - \sum_l \frac{\langle k|H'|l \rangle}{E^0_k - E^0_l} \delta_{kl} = 0 \quad (4.116) \]

As \( \delta_{kk} = 1 \) and \( \sum_m b_m (E^0_m - E^0_k) \delta_m = 0 \)

for all values of \( m \), Eqn. (4.116) gives-
\[ E''_k = \sum_l \frac{<l|H'|k> <k|H'|l>}{E_k^0 - E_l^0} - \sum_l \frac{<k|H'|k> <l|H'|l>}{E_k^0 - E_l^0} \delta_{kl} \] (4.117)

Considering the second term in Eqn. (4.117) we note that this term is zero since \( \delta_{kl} = 0 \) for all values of \( l \) except for \( l=k \) and this term is not included in the summation. Then equation (4.117) gives-

\[ E''_k = \sum_l \frac{<l|H'|k> <k|H'|l>}{E_k^0 - E_l^0} \]

If we assume that \( H' \) is the Hermitian operator, we may write -

\[ E''_k = \sum_l \left| \frac{<k|H'|l>}{E_k^0 - E_l^0} \right|^2 \] (4.118)

This equation gives second order energy correction term \( E''_k \).

The Prime on summation reminds of the \( l=k \) in the summation evaluation of second order correction to wave function, for \( m \neq n \), Equation (4.115) gives-

\[ b_n (E_n^0 - E_k^0) + \sum_l \frac{<l|H'|k> <n|H'|l>}{E_k^0 - E_l^0} - \sum_l \frac{<l|H'|k> <l|H'|k>}{E_k^0 - E_l^0} \delta_{nl} = 0 \]

or

\[ b_n (E_n^0 - E_k^0) + \sum_l \frac{<l|H'|k> <n|H'|l>}{E_k^0 - E_l^0} - \sum_l \frac{<lk|H'|k> <n|H'|l>}{E_k^0 - E_n^0} = 0 \]

This gives-

\[ b_n = \sum_l \frac{<l|H'|k> <n|H'|l>}{(E_k^0 - E_l^0)} - \sum_l \frac{<lk|H'|k> <n|H'|l>}{(E_k^0 - E_n^0)^2} = 0 \]

setting \( n=m \), we get -
\[ b_m = \sum _l \frac {\langle l|H'|k \rangle \langle m|H'|l \rangle} {(E^0_k - E^0_l)(E^0_k - E^0_m)} - \sum _l \frac {\langle l|H'|k \rangle \langle m|H'|k \rangle} {(E^0_k - E^0_m)^2} \]  

(4.119)

This equation determines all coefficient \( b_m \)'s but not \( b_k \). The coefficient \( b_k \) is determined by the normalization condition for \( \omega_k \) retaining only terms up to second order in \( \lambda \).

\[ \Psi_k = \Psi_k^0 + \lambda \Psi'_k + \lambda^2 \Psi''_k = \Psi_k^0 + \lambda \Psi'_k + \lambda^2 \sum m \, \Psi_m^0 = \Psi_k^0 + \lambda \Psi'_k + \lambda^2 b_k \Psi_k^0 \]

(4.120)

The normalization condition \( \Psi_k \) gives \( \int \Psi_k^0 \Psi_k^0 d\tau = 1 \)

substituting \( \Psi_k \) from (4.120) we get-

\[
\int \Psi_k^0 \Psi_k^0 d\tau + \lambda \int \Psi_k^0 \Psi_k^0 d\tau + \lambda^2 b_k \int \Psi_k^0 \Psi_k^0 d\tau
+ \lambda^2 \sum m \left( \sum l \frac {\langle l|H'|k \rangle \langle k|H'|l \rangle} {(E^0_k - E^0_l)(E^0_k - E^0_m)} - \frac {\langle k|H'|k \rangle \langle m|H'|k \rangle} {(E^0_k - E^0_m)^2} \right) \int \Psi_k^0 \Psi_m^0 d\tau + \lambda \int \Psi_k^0 \Psi_k^0 d\tau
+ \lambda^2 b_k \int \Psi_k^0 \Psi_k^0 d\tau

+ \lambda^2 \sum m \left( \sum l \frac {\langle l|H'|k \rangle \langle k|H'|l \rangle} {(E^0_k - E^0_l)(E^0_k - E^0_m)} - \frac {\langle k|H'|k \rangle \langle m|H'|k \rangle} {(E^0_k - E^0_m)^2} \right) \int \Psi_k^0 \Psi_m^0 d\tau
+ \lambda^2 b_k \int \Psi_k^0 \Psi_k^0 d\tau = 1
\]
\[ dr + 0 + \lambda^2 b_k \]
\[ + \lambda^2 \sum_m \left\{ \sum_l \frac{\langle l | H' | k \rangle \langle k | H' | l \rangle}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \right\} \delta_{mk} + 0 + \lambda^2 b_k^* \]
\[ + \lambda^2 \sum_m \left\{ \sum_l \frac{\langle l | H' | k \rangle^* \langle k | H' | l \rangle^*}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \right\} \delta_{mk} \]
\[ + \lambda^2 \sum_l \sum_m \frac{\langle l | H' | k \rangle^* \langle k | H' | l \rangle}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \int \Psi_l^0 \Psi_m^* d\tau = 1 \]

using \( \Psi_k \) from Eqn. (4.112b) -
\[ \lambda^2 b_k + \lambda^2 b_k^* \lambda^2 \sum_l \sum_m \frac{\langle l | H' | k \rangle^* \langle m | H' | k \rangle}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \delta_{lm} = 0 \]

or \[ \lambda^2 \left[ b_k + b_k^* + \sum_l \frac{\langle l | H' | k \rangle^* \langle l | H' | k \rangle}{(E_k^0 - E_l^0)(E_k^0 - E_l^0)} \right] = 0 \]

As \( \lambda^2 \neq 0 \), therefore we have -
or \[ b_k + b_k^* = -\sum_l \frac{|\langle l | H' | k \rangle|^2}{(E_k^0 - E_l^0)^2} \quad (4.121) \]

The real part of \( b_k \) is fixed by this equation but the imaginary part is arbitrary. The choice of imaginary part simple affects the phase of the unperturbed wave function and it does not affect the
energy of the system. Hence the imaginary part of $b_k$ may be equal to zero. Thus, we have -

$$b_k = - \sum_l \frac{|<l|H'|k>|^2}{2(E_k^0 - E_l^0)^2}$$  \hspace{1cm} (4.122)

Then $\Psi''_k = \sum_m b_m \Psi^0_m$

$$= b_k \Psi^0_k + \sum_m b_m \Psi^0_m$$

$$= - \sum_l \frac{|<l|H'|k>|^2}{2(E_k^0 - E_l^0)^2} \Psi^0_k$$

$$+ \sum_m \left\{ \sum_l \frac{|<l|H'|k>|^*<k|H'|l>|}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \right\} \Psi^0_m$$

$$- \frac{|<k|H'|k>|<m|H'|k>|}{(E_k^0 - E_m^0)^2} \Psi^0_m$$  \hspace{1cm} (4.123)

Thus the complete eigen values and eigen function corrected upto second order perturbation term are given by -

$$E_k = E_k^0 + \lambda E_k' + \lambda^2 E_k''$$

$$= E_k^0 + \lambda <k|H'|k>$$

$$> \lambda^2 \sum_l \frac{|<k|H'|l>|^2}{E_k^0 - E_l^0}$$  \hspace{1cm} (4.124)
and \( \Psi_k = \Psi_k^0 + \lambda \Psi_k' + \lambda^2 \Psi_k'' \)

\[
= \Psi_k^0 + \lambda \sum_l \frac{|< l|H'|k >|^2}{E_k^0 - E_l^0} \Psi_l^0 \\
+ \lambda^2 \left[ - \sum_l \frac{|< l|H'|k >|^2}{2(E_k^0 - E_l^0)^2} \Psi_k^0 \\
+ \sum_m \left\{ \sum_l \frac{|< l|H'|k >|^2}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \right\} \Psi_m^0 \right] 
\]

(4.125)

If choose arbitrary \( \lambda=1 \) or include \( \lambda \) in symbol, i.e. \( \lambda H \rightarrow H' \),

the above equations take the conventional form -

\[
E_k = E_k^0 + |< k|H'|k >|^2 \sum_l \frac{|< l|H'|k >|^2}{(E_k^0 - E_l^0)} 
\]

(4.126)

\[
\Psi_k = \Psi_k^0 + \sum_l \frac{|< l|H'|k >|^2}{2(E_k^0 - E_l^0)} \Psi_l^0 - \sum_l \frac{|< l|H'|k >|^2}{2(E_k^0 - E_l^0)^2} \Psi_k^0 \\
+ \sum_m \left\{ \sum_l \frac{|< l|H'|k >|^2}{(E_k^0 - E_l^0)(E_k^0 - E_m^0)} \right\} \Psi_m^0 
\]

(4.127)

This is the required equation for stark shift in perturbation.

4.7- Material Properties in Atomic Cavity (QED) in Free Space:

The electromagnetic field components are related by the properties of the medium in which they exist-
\[ D = \mu H \quad (4.128) \]
\[ D = \varepsilon E \quad (4.129) \]
\[ J = \sigma E \quad (4.130) \]

Equation (4.128) may be written as-

\[ B = \mu H \]
\[ \mu = \frac{B}{H} = \text{henery/meter} \]

The permeability may be divided into two parts-

\[ \mu = \mu_0 \mu_r \quad (4.131) \]

Equation (4.129) may be written as-

\[ D = \varepsilon E \]
\[ \varepsilon = \frac{D}{E} = \text{farade/meter} \]

The permittivity may be divided into two parts-

\[ \varepsilon = \varepsilon_0 \varepsilon_r \quad (4.132) \]

\( \varepsilon_0 \) is the Permittivity constant \( \varepsilon_r \) is the permittivity in free space.

The rms vacuum electric field amplitude \( E_{\text{vac}} \) in a mode of frequency \( \omega \) is-

\[ E_{\text{vac}} = \left( \frac{\hbar \omega}{2\varepsilon_0 c} \right)^{1/2} \quad (4.133) \]

The coupling of an atom to a field mode is described by the frequency-
\[ \Omega_{ef} = \frac{D_{ef}E_{vac}}{h} \quad (4.134) \]

Substituting value of \( E_{vac} \) from Equation (4.133) in Equation (4.134) we get-

\[ \Omega_{ef} = \frac{D_{ef}}{h} \left( \frac{\hbar \omega}{2 \varepsilon_0 v} \right)^{1/2} \quad (4.135) \]

from eqn. no. (4.132)-

\[ \varepsilon_0 = \frac{\varepsilon}{\varepsilon_r} \quad (4.136) \]

from Equation (4.135) and (4.136) we get-

\[ \Omega_{ef} = \frac{D_{ef}}{h} \left( \frac{\hbar \omega \varepsilon_r}{2 \varepsilon v} \right)^{1/2} \]

or \[ \Omega_{ef} = \frac{D_{ef}}{h} \left( \frac{\hbar \omega \varepsilon_r}{2 \varepsilon v} \right)^{1/2} \quad (4.137) \]

Where \( D_{ef} \) is the matrix elements.

The mode density is given by the expression-

\[ \rho_0(\omega) = \frac{\omega^2 v}{\pi^2 c^3} \quad (4.138) \]

From Equation (4.137) -

\[ \Omega_{ef} = \frac{D_{ef}}{h} \left( \frac{\hbar \omega \varepsilon_r}{2 \varepsilon v} \right)^{1/2} \]

squarring on both sides in above equation we get-

\[ D_{ef}^2 = \frac{D_{ef}^2}{h^2} \frac{\hbar \omega \varepsilon_r}{2 \varepsilon v} \]

or \[ D_{ef}^2 = \frac{D_{ef}^2}{h^2} \frac{\omega \varepsilon_r}{2 \varepsilon v} \]

\[ : 71 : \]
or \( D_{\text{ef}}^2 = \frac{D_{\text{ef}}^2}{h} \frac{2 \pi \varepsilon_r}{2 \varepsilon_v} \)

or \( D_{\text{ef}}^2 = \frac{D_{\text{ef}}^2}{h} \frac{\pi n \varepsilon_r}{\varepsilon_v} \)

or \( D_{\text{ef}}^2 = \frac{D_{\text{ef}}^2 c v \hbar}{\pi n \varepsilon_r} \) (4.139)

Equation (4.138) may be written as -

\[
\rho_0(\omega) = \frac{(2\pi n)^2 v}{\pi^2 c^3}
\]

or \( \rho_0(\omega) = 4\pi^2 n^2 v / \pi^2 c^3 \)

or \( \rho_0(\omega) = 4 n^2 v / c^3 \)

or \( n = \left( \frac{\rho_0(\omega)c^3}{4v} \right)^{1/2} \) (4.140)

substituting value of \( n \) from equation (4.140) in equation (4.139), we get -

\[
D_{\text{ef}}^2 = \frac{\Omega_{\text{ef}}^2 c v \hbar}{\pi \left( \frac{\rho_0(\omega)c^3}{4v} \right)^{1/2} \varepsilon_r}
\]

or \( D_{\text{ef}}^2 = \frac{\Omega_{\text{ef}}^2 c v \hbar \cdot 2v^{1/2}}{\pi (\rho_0(\omega)c^3)^{1/2} \varepsilon_r} \)

or \( D_{\text{ef}}^2 = \frac{2\Omega_{\text{ef}}^2 c v^3 c \hbar}{\pi^2 \sqrt{(\rho_0(\omega)c)^e}} \)

or \( D_{\text{ef}}^2 = \frac{2\Omega_{\text{ef}}^2 c v^{3/2} \hbar}{\pi \sqrt[4]{\rho_0(\omega)c^3 \varepsilon_r}} \) (4.141)
Equation (4.141) may be written as -

\[ \Omega_{ef}^2 = \frac{\pi D_{ef}^2 \sqrt{\rho_0(\omega)c^3\varepsilon_r}}{2\varepsilon_{hv}^{3/2}} \] (4.142)

The probability \( \Gamma_0 \) is given by Fermi’s Golden Role -

\[ \Gamma_0 = 2\pi \Omega_{ef}^2 \frac{\rho_0(\omega)}{3} \] (4.143)

Substituting value of \( \Omega_{ef}^2 \) from equation (4.142) in equation (4.143) we get-

\[ \Gamma_0 = \frac{2\pi D_{ef}^2 \sqrt{\rho_0(\omega)c^3\varepsilon_r}}{2\varepsilon_{hv}^{3/2}} \frac{\rho_0(\omega)}{3} \]

or \[ \Gamma_0 = \frac{\pi^2 D_{ef}^2 c^3 \varepsilon_r (\rho_0(\omega))^{3/2} \varepsilon_r}{2\varepsilon_{hv}^{3/2}} \]

or \[ \Gamma_0 = \frac{\pi^2 D_{ef}^2 \varepsilon_r}{3\varepsilon_h} \sqrt{\frac{(\rho_0(\omega))^3}{v^3}} = \frac{\pi^2 D_{ef}^2 \varepsilon_r}{3\varepsilon_h} \sqrt{\left(\frac{\rho_0(\omega)}{v}\right)^3} \] (4.144)

Equation (4.144) may be written as -

\[ D_{ef}^2 = \frac{3\Gamma_0 \varepsilon_h}{\pi^2 \varepsilon_r} \sqrt{\left(\frac{v}{\rho_0(\omega)}\right)^3} \] (4.145)

we know that the fermi Gloden rule -

\[ \Gamma_0 = \frac{\omega^3}{3\pi \varepsilon_0^3} \frac{|D_{ef}|^2}{\varepsilon_0} \] (4.146)

Eqn. (1.146) may be written as-

or \[ D_{ef}^2 = \frac{3\pi \varepsilon_0^3 \varepsilon_0 \Gamma_0}{\omega^3} \] (4.147)

from equation (4.145) and (4.147) we get-
\[ \frac{3\pi\hbar c^3 \varepsilon_0 \tau_0}{\omega^3} = \frac{3\Gamma_0 \varepsilon_0}{\pi^2 \varepsilon_r} \sqrt{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \]

or \[ \frac{\pi c^3 \varepsilon_0}{\omega^3} = \frac{\varepsilon_0}{\pi^2} \sqrt{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \]

where \( \varepsilon_0 = \varepsilon / \varepsilon_r \)

or \[ \frac{\pi c^3}{\omega^3} = \frac{1}{\pi^2} \sqrt{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \] (4.148)

we know that \( \omega = 2\pi \Omega_{ef} \)

where \( \Omega_{ef} \) is the Rabi frequency.

\[ \omega = \frac{2\pi}{T} \]

Substituting value of \( \omega \) in equation (4.148) we get-

\[ \frac{\pi c^3}{\left( \frac{2\pi}{T} \right)^3} = \frac{1}{\pi^2} \sqrt{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \]

or \[ \frac{\pi c^3 T^3}{8\pi^3} = \frac{1}{\pi^2} \sqrt{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \]

or \[ T^3 = \frac{8}{c^3} \sqrt{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \]

or \[ T = \frac{2}{c} \sqrt[1/3]{\left( \frac{V}{c\rho_0(\omega)} \right)^3} \] (4.149)

This equation is time period and mode density.

The probability \( \rho_e(t) \) of finding an atom still excited at time \( t \) after its preparation in state at \( t=0 \) is -

\[ \rho_e(t) = e^{-\Gamma_0 t} \] (4.150)
Substituting value of \( \tau_0 \) from equation (4.146) in equation (4.150)

we get-

\[
\rho_e(t) = e^{-\frac{\omega^3}{3\pi \hbar c^3} \frac{|D_{ef}|^2}{\epsilon_0} t} \tag{4.151}
\]

Another form, Taking the value of \( D_{ef}^2 \) from equation (4.147)-

\[
D_{ef}^2 = \frac{3\pi \hbar c^3 \epsilon_0 \tau_0}{\omega^3}
\]

\[
\Gamma_0 = \frac{\omega^3 D_{ef}^2}{3\pi \hbar c^3 \epsilon_0} \tag{4.152}
\]

from equation (4.150) and (4.152) we get-

\[
\rho_e(t) = e^{-\frac{\omega^2 D_{ef}^2}{3\pi \hbar c^3 \epsilon_0} t} \tag{4.153}
\]

where \( |D_{ef}|^2 \) is the matrix element.

This equation is called the probability of equation.

4.8- Redfield Theory for Damped Rabi Oscillations:

We know that the vacuum Rabi-Oscillation

\[
\Omega_R(N) = \Omega_{ef} \sqrt{N} \tag{4.154}
\]

The two level system \((n=0, 1)\) is a part of multilevel system-

\[
H^{(s)} = \sum_n |n > E_n < n| \quad (n = 0, 1, 2 \ldots) \tag{4.155}
\]

A set consisting of an infinite number of harmonic oscillators-

\[
H^{(B)} = \sum_\alpha \left( \frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2 x_\alpha^2}{2} \right) \tag{4.156}
\]
for the equation of introduce temperature-

$$H^{(\text{int})} = \Phi \sum_{\alpha} C_{\alpha} x_{\alpha} \quad (4.157)$$

Where, \( \Phi(x_{\alpha}) \) indicates the degree of freedom of the system \( C_{\alpha} \) is a coupling constant between the system and the environment.

for the two level system \((n=0, 1)\) equation (4.154), (4.155), (4.156) and (4.157) may be written as-

i.e. \(H^{(s)} = (|0 > E_0 < 0|) + (|1 > E_1 < 1|)\) \quad (4.158)

i.e. \(H^{(B)} = \left( \frac{p_0^2}{2m_0} + \frac{m_0 \omega_0 x_0^2}{2} \right) + \left( \frac{p_1^2}{2m_1} + \frac{m_1 \omega_1 x_1^2}{2} \right) \) \quad (4.159)

i.e. \(H^{(\text{int})} = \Phi (C_0 x_0 + C_1 x_1)\) \quad (4.160)

and \( \Omega_R(0) = 0 \)

\( \Omega_R(1) = \Omega_{ef} \)

Adding these two-

\(H^{(s)} = \Omega_R(0) + \Omega_R(1) = 0 + \Omega_{ef}\)

or \(H^{(\Omega)} = \Omega_{ef}\) \quad (4.161)

The Hamiltonian for the whole system is written as-

\(H = H^{(s)} + H^{(B)} + H^{(\text{int})} + H^{(\Omega)}\) \quad (4.162)

substituting value of \(H^{(s)}, H^{(B)}, H^{(\text{int})}\) and \(H^{(\Omega)}\) from equation (4.158), (4.159), (4.160) and (4.161) in equation (4.162) we get -
\[ H = \{(|0 > E_0 < 0|) + (|1 > E_1 < 1|)\} + \left\{ \left( \frac{p_0^2}{2m_0} + m_0\omega_0^2x_0^2/2 \right) + \left( \frac{p_1^2}{2m_1} + m_1\omega_1^2x_1^2/2 \right) \right\} + \{ \Phi(C_0x_0 + C_1x_1) \} + \{ \Omega_{ef} \} \]  

(4.163)

Equation No. (4.162) of first and second term is the Hamiltonian of the system \( H^{(S)} \) and its bath \( H^{(B)} \). The third is the interaction Hamiltonian \( H^{(\text{int})} \), the fourth term is the Hamiltonian driving field and \( H^{(\Omega)} \).

According to the Redfield theory, the Liouville equations for a system-

\[ \dot{\rho}_{nm} = -i\omega_{nm}\rho_{nm} + [H^{(\Omega)},\dot{\rho}]_{nm} \frac{i\hbar}{\hbar} - \sum_{k,l} R_{nmkl}\rho_{kl} \]  

(4.164)

where \( \rho_{nm} = <n|\hat{\rho}|m> \)

and \( \omega_{nm} = \frac{(E_n - E_m)}{\hbar} \)

The other terms are defined as-

\[ R_{nmkl} = \delta_{l,m} \sum_{\gamma} \Gamma_{nYYk}^{(+)} + \delta_{n,k} \sum_{\gamma} \Gamma_{lYYm}^{(-)} - \Gamma_{lmnk}^{(+)} - \Gamma_{lmnk}^{(-)} \]  

(4.165)

\[ \Gamma_{lmnk}^{(+)} = \left( \frac{i}{\hbar} \right)^2 \int_0^{\infty} d\tau e^{-i\omega_{nk}\tau} \mathcal{T}_{YB} \left[ \bar{H}_{lm}^{(\text{int})}(\tau)\bar{H}_{nk}^{(\text{int})}(0)\rho^{(B)} \right] \]  

(4.166)

\[ \Gamma_{lmnk}^{(-)} = \left( \frac{i}{\hbar} \right)^2 \int_0^{\infty} d\tau e^{-i\omega_{lm}\tau} \mathcal{T}_{YB} \left[ \bar{H}_{lm}^{(\text{int})}(0)\bar{H}_{nk}^{(\text{int})}(\tau)\rho^{(B)} \right] \]  

(4.167)

\[ \bar{H}_{nm}^{(\text{int})}(\tau) = <n|\bar{H}_{nm}^{(\text{int})}(\tau)|m> \]  

(4.168)

\[ \bar{H}_{nm}^{(\text{int})}(\tau) = e^{i\hbar H^{(B)}\tau} \bar{H}_{nm}^{(\text{int})} e^{-i\hbar H^{(B)}\tau} \]  

(4.169)
for the damped Rabi Oscillations an initial condition

$$\rho_{00} + \rho_{11} = 1$$

And multilevel system divided into two parts (m, n=0,1)

In the case of Probability density into the external system,

i.e. $\rho_{kl}(k, l \neq 0,1) << 1$

during the typical decay time $\tau_1 = \Gamma_1^{-1}$. 

A secular approximation the terms $R_{nmkl}$ is the -

$n-m=k-l$

The density matrix is represented in the two level system as -

$$\dot{\rho}_{nn} = \frac{[H^{(Ω)}, \rho]}{\hbar} - \sum_{k=0,1} R_{nnkk}\rho_{kk}(n = 0,1), \text{ where}$$

$$R_{0000} = 2\Re \left( \Gamma_{0110}^{(+)\dagger} \right) + 2 \sum_{l \neq 0,1} \Re \left( \Gamma_{0l10}^{(+)\dagger} \right), (= \gamma_{01} + \Gamma_0) \quad (4.170)$$

$$R_{1111} = 2\Re \left( \Gamma_{1001}^{(+)\dagger} \right) + 2 \sum_{l \neq 0,1} \Re \left( \Gamma_{1l11}^{(+)\dagger} \right), (= \gamma_{10} + \Gamma_1) \quad (4.171)$$

This is the required solutions.