CHAPTER 3

THEORETICAL FORMULATION

3.1 PURPOSE OF THE STUDY

From the available literature pertaining to the mechanical characteristics of powder metallurgy composites, it was noted that no research work has been carried out on the workability studies of iron – carbon manganese P/M composites during the cold upsetting operation. Even though the uniaxial load is applied to the preforms during upsetting, due to friction at the contact surfaces, barrelling takes place. This is attributed to the fact that the resultant of the same is that different stress states are developed in the preform which may be either plane or triaxial state of stress.

In this chapter, the theoretical essence of the workability study of ferrous powder metallurgy composite preforms in triaxial stress state condition is discussed elaborately. The workability study of the ferrous powder metallurgy preforms under triaxial stress state condition has been derived based on the values of the contact and bulged diameters of the deforming preforms during cold uniaxial upsetting. Various parameters, like, the Poisson’s ratio, the axial strain, the hoop strain and stresses like the axial, the hoop, the mean or hydrostatic, the effective, the stress formability index ($\beta_{\sigma}$) for triaxial stress state condition have been derived and discussed in detail in this chapter. This formability stress index ($\beta_{\sigma}$) parameters are very useful in determining the workability of the given powder material as described elsewhere Abdel-Rahman and El-Sheikh (1995) and Narayanasamy
et al (2008) respectively. Due to this reason, the theory of plasticity is also dealt in this section in order to evaluate the formability stress parameter.

In order to further enhance the performance of the P/M technology, it is required to obtain a better understanding of the behaviour of metal powders under compaction, sintering, deformation and fracture. Establishing limited conditions in yielding and fracturing is essential to the analysis of plastic deformation during deformation. Determining the various stress parameters is the other way of checking the workability study of the powder metallurgy preforms. This chapter, in addition, discusses the determination of various parameters, like instantaneous strain hardening index ($n_i$), instantaneous strength coefficient ($k_i$), strain hardening index ($n$)and strength coefficient ($k$) that are essentially required for the prediction of workability study of ferrous powder metallurgy preforms.

### 3.1.1. Assumptions for theoretical equations

- The material of the specimen is porous and homogeneous.
- The compacting pressure is applied uniformly throughout the preform.
- The preform was undergone same sintering temperature in all sections.
- In deformation experiment, the load applied is uniformly distributed to all the sections of the preform.
- The hoop stress and the radial stress are assumed as equal.
- Bulging contour of the preform assumes the form of a circular arc.
- For the P/M preforms, mass-constancy principle is assumed before and after the deformation.

### 3.2 STRESS AND STRAIN

The uniaxial stress state condition is exhibited during the upsetting under ideal condition. Only the single stress namely, the axial stress is considered and other stresses are considered to be zero. This pertains to the
ideal stress state condition of uniaxial compression, which never prevails in the actual upsetting operation.

It has been established that while deforming sintered P/M preforms under frictional constrains the average density is enhanced. The presence of the friction enhances densification, while the height reduction at fracture decreases at the same instant.

The axial stress component of powder metallurgy composite preforms can be determined by the following relationship:

True axial stress ($\sigma_z$) = Load / Contact area \hspace{1cm} (3.1)

In the above equation, the contact area ($A_c$) is calculated from the expression cited below:

\[ A_c = \frac{\pi}{4} D_c^2 \] \hspace{1cm} (3.2)

where,

$D_c$ is the average contact diameter = \[ \frac{D_{CT} + D_{CB}}{2} \] \hspace{1cm} (3.3)

$D_{CT}$ is the top contact diameter of the preform and

$D_{CB}$ is the bottom contact diameter of the preform.

The axial strain ($\varepsilon_z$) is calculated using the standard expression as described

\[ \varepsilon_z = -\varepsilon_{eff} = \ln \left( \frac{H_0}{H_f} \right) \] \hspace{1cm} (3.4)

where,  

$H_0$ is the initial height of the preform and,

$H_f$ is the deformed height of the preform.
The hoop strain \( (\varepsilon_\theta) \) and radial strain \( (\varepsilon_r) \) are calculated using the expression given below by Abdel-Rahman and El-Sheikh (1995), since these two parameters are equal (refer 3.20).

\[
\varepsilon_\theta = \ln \left( \frac{D_c}{D_o} \right)
\]  

(3.5)

where, \( D_o \) is the initial diameter of the preform and \( D_c \) is the contact diameter after deformation,

where the axial compression takes place, the final diameter of the preform increases along with the corresponding hoop strain, which is tensile in nature and increases until it reaches the fine creaks on its surface.

### 3.3 RELATIONSHIP BETWEEN THE RELATIVE DENSITY AND STRAINS

According to Narayanasamy and Pandey (1998), the hoop strain \( (\varepsilon_\theta) \) of a solid cylindrical billet, which is subjected to upsetting condition, is determined in the following ways. When a solid cylinder is compressed axially between the top and bottom platen, the work piece material in contact with their surfaces undergoes heterogeneous deformation, resulting in barrelling of cylinder, as shown in Figure 1.5.

The expression for bulging can be written as follows for fully dense metals, under the condition that the bulging follows a circular arc, i.e. the barrelling effect:

\[
\frac{\pi}{4} (D_o^2 H_o) = \frac{\pi}{12} (2D_B^2 + D_c^2) H_f
\]

(3.6)

The above equation is based on the volume – constancy principle. However, the same is not the case for porous preform upsetting. Instead the mass – constancy principle is employed as discussed by
Narayanasamy and Pandey (1998). Applying the mass – constancy principle before and after deformation, Equation (3.6) converts into

\[
\frac{\pi}{4} (D_o^2 H_o) \left( \frac{\rho_o}{\rho_{th}} \right) = \frac{\pi}{12} \left( 2D_B^2 + D_C^2 \right) h_f \left( \frac{\rho_f}{\rho_{th}} \right) 
\]

(3.7)

where,
- \( D_o \) is initial Diameter of the preform before deformation
- \( H_o \) is height of the preform before deformation,
- \( \rho_o \) is the initial preform density of the cylinder,
- \( D_B \) is Bulged Diameter of the preforms after deformation,
- \( D_C \) is Contact Diameter of the preforms after deformation,
- \( H_f \) is the height of the barrelled cylinder after deformation,
- \( \rho_f \) is the density of the preform after deformation and
- \( \rho_{th} \) is the theoretical density of fully dense material.

The above equation can be rewritten as follows:

\[
\left( \frac{\rho_f}{\rho_{th}} \right) = \left( \frac{D_B}{D_C} \right) \left( \frac{H_o}{H_f} \right) \left( \frac{3D_B^2}{2D_B^2 + D_C^2} \right)
\]

(3.8)

Taking natural logarithms on both sides of the equation (3.8), the following expression is obtained:

\[
\ln \left( \frac{\rho_f}{\rho_{th}} \right) = \ln \left( \frac{D_B}{D_C} \right) + \ln \left( \frac{H_o}{H_f} \right) + \ln \left( \frac{3D_B^2}{2D_B^2 + D_C^2} \right)
\]

(3.9)

However, Equation (3.9) can further be simplified as follows:

\[
\left( \frac{\rho_f}{\rho_{th}} \right) = \left( \frac{\rho_o}{\rho_{th}} \right) e^{\varepsilon_z - \varepsilon_0}
\]

(3.10)

where,

\[
\varepsilon_z = \ln \left( \frac{H_o}{H_f} \right)
\]

(3.11)

And the conventional hoop strain
The Equation (3.10) establishes the relationship between strains and density.

### 3.4 TRIAXIAL STRESS STATE CONDITION

Triaxial stress state condition pertaining to actual stresses is developed during cold upsetting of cylindrical preform with friction and barrelling. Due to this reason, triaxial state of stress is to be considered for cold upsetting operation. Here, the axial stress is compressive in nature, the hoop and the radial stresses are tensile in nature. In order to simplify the plasticity theory, the radial stress can be taken as equivalent to the hoop stress. Narayanasamy et al (2005) reported the state of stress in a triaxial stress condition as follows:

The relationships between stress and strain increment for an ideal plastic solid, where the elastic is negligible, are called flow rules or Levy-Mises equations. The relationship between the strain increment and stresses for triaxial stress state condition for a porous material is written as follows by Narayanasamy et al (2005).

\[ d\varepsilon_\theta = \left[ \frac{(2R^2 + \sigma_\theta^2)}{3\sigma_\theta} \right] \]  \hspace{1cm} (3.13)

\[ d\varepsilon_z = \left[ \frac{(2R^2 + \sigma_z^2)}{3\sigma_z} \right] \]  \hspace{1cm} (3.14)

Where, \( d\varepsilon_\theta \) is the plastic strain increment in hoop direction, \( d\varepsilon_z \) is the plastic strain increment in axial direction, \( R \) is the relative density, \( \sigma_\theta \) is the true stress in hoop direction and \( \sigma_z \) is the true stress in axial direction.

Dividing Equation (3.13) by Equation (3.14)
we get,

\[
\frac{d\varepsilon_\theta}{d\varepsilon_z} = \left(\frac{(2 + R^2)\sigma_\theta - R^2(\sigma_z + 2\sigma_\theta)}{(2 + R^2)\sigma_z - R^2(\sigma_z + 2\sigma_\theta)}\right) = \alpha
\]  

(3.15)

\(\alpha\), which is nothing but the ratio of the hoop strain increment to the axial strain increment.

Say,

\[
\alpha = \begin{bmatrix} A \\ B \end{bmatrix}
\]  

(3.16)

where,

\[A = [(2 + R^2)\sigma_\theta - R^2(\sigma_z + 2\sigma_\theta)]\]

\[B = [(2 + R^2)\sigma_z - R^2(\sigma_z + 2\sigma_\theta)]\]

\(R\) is Relative Density
\(\sigma_z\) is the axial stress and
\(\sigma_\theta\) is the hoop stress

The determination of axial stress \((\sigma_z)\) is carried out from the measured values of contact areas and the corresponding load applied.

In triaxial stress condition for the known values of the ratio of the hoop strain increment to the axial strain increment \((\alpha)\), relative density \((R)\) and axial stress \((\sigma_z)\), the Hoop stress component \((\sigma_\theta)\) can be determined as explained by Narayanasamy et al(2005a), is as follows:

\[
\sigma_\theta = \left[\frac{2\alpha + R^2}{2 - R^2 + 2R^2\alpha}\right] \sigma_z
\]  

(3.17)

In the above Equation (3.17), the relative density \((R)\) plays a major role in determining the hoop stress component \((\sigma_\theta)\).
The hydrostatic stress for axi-symmetric upset forming is inferred from Equation (1.3) as below,

\[ \sigma_m = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3} \]  

(3.18)

In terms of cylindrical coordinated, the equation (3.18) become

\[ \sigma_m = \frac{(\sigma_r + \sigma_\theta + \sigma_z)}{3} \]  

(3.19)

Since \( \sigma_r = \sigma_\theta \) for axi-symmetric or cylindrical upsetting, the above Equation can be written as (3.19):

\[ \sigma_m = \frac{(\sigma_r + 2\sigma_\theta)}{3} \]  

(3.20)

The effective stress can be determined from the following expression in terms of cylindrical coordinates for axi-symmetric upset forging condition as explained by Doraivelu et al (1984):

\[
\left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \right] = (2R^2 - 1) \sigma_{\text{eff}}^2
\]

(3.21)

The above Equation (3.21) can be written in terms of cylindrical coordinates as follows:

\[
\sigma_{\text{eff}}^2 = \frac{\left[ \sigma_2^2 + \sigma_\theta^2 + \sigma_r^2 - R^2(\sigma_2 \sigma_\theta + \sigma_\theta \sigma_r + \sigma_r \sigma_r + \sigma_r \sigma_z) \right]}{2R^2 - 1}
\]

(3.22)

Since \( \sigma_\theta = \sigma_r \) for axi-symmetric or cylindrical upsetting, the equation (3.22) can be modified as:
\[
\sigma_{eff} = \left( \frac{[\sigma_x^2 + 2\sigma_y^2 - R^2(\sigma_z^2 + 2\sigma_y^2\sigma_0)]}{2R^2 - 1} \right)^{1/2} (3.23)
\]

In the above Equation (3.23), for the known values of \(\sigma_x, \sigma_y\) and \(R\), the effective stress \(\sigma_{eff}\) can be determined.

### 3.5 FORMABILITY STRESS PARAMETER \((\beta_\sigma)\)

Workability of metals is one of the most important parameters that must be considered in the design of a forming operation. A preform with high relative density (pores of small size) yielded at a relatively greater stress whilst a preform with low relative density (pores of larger size) yielded at a relatively smaller stress. The application of a compressive hydrostatic stress will close the pore and will increase the relative density. Similarly the application of tensile hydrostatic stress will increase the size of the pores and reduce the relative density.

Bulk forming-which is a complicated deformation, is a process that reflects the workability of materials. It is very important to guide the production practice according to the forming limits. The formability stress index parameter \((\beta_\sigma)\) can be determined for triaxial stress state condition as shown in the Figure 3.1 in the form of flow chart. As an evidence of experimental investigation implying the importance of the spherical component of the stress state on fracture, Vujovic and Shabaik (1986), proposed a parameter called a formability stress index \('\beta_\sigma'\) given by,

\[
\beta_\sigma = \left( \frac{3\sigma_m}{\sigma_{eff}} \right) (3.24)
\]

where, \(\sigma_m\) is the mean or hydrostatic stress component and \(\sigma_{eff}\) is the effective stress component can be calculated through the Equations (3.20) and (3.23). This index determines the fracture limit as explained in the reference as
explained by Narayanasamy et al (2005), Using formability stress index ($\beta_\sigma$),
the pore closure rate indices can be determined.

**Figure 3.1** Flow chart for the calculation for the formability stress
index ($\beta_\sigma$) under triaxial stress state condition
3.6 CONSTITUTIVE RELATIONSHIP

3.6.1 The Instantaneous Strain Hardening Index ($n_i$)

The instantaneous strain hardening index ($n_i$) determined by employing the conventional Ludwik’s equation for consecutive compressive loads specified as $1, 2, 3, .. (i-1), i$ is given by the following expression:

$$\sigma_{eff} = k R^A \varepsilon_{eff}^n \dot{\varepsilon}_{eff}^m$$  \hspace{1cm} (3.25)

where,

- $\sigma_{eff}$ is the effective stress
- $\varepsilon_{eff}$ is the effective strain
- $R$ is the relative density
- $A$ is the density coefficient
- $\dot{\varepsilon}_{eff}$ is the effective strain rate
- $m$ is the strain rate sensitivity
- $k$ is the strength coefficient and
- $n$ is the strain hardening exponent

As described by Narayanasamy and Sathyanarayanan (2006), for axi-symmetric upsetting, the effective strain ($\varepsilon_{eff}$) and the effective strain rate ($\dot{\varepsilon}_{eff}$) are as follows:

$$\varepsilon_{eff} = \left[ \frac{2}{3(2 + R^2)} \left[ (2\varepsilon_\theta^2 + 2\varepsilon_z^2) - 4\varepsilon_\theta\varepsilon_z \right] + \frac{(\varepsilon_z + 2\varepsilon_\theta)^2}{3} (1 - R^2) \right]^{1/2}$$  \hspace{1cm} (3.26)

$$\dot{\varepsilon}_{eff} = \left[ \frac{2}{3(2 + R^2)} \left[ (2\dot{\varepsilon}_\theta^2 + 2\dot{\varepsilon}_z^2) - 4\dot{\varepsilon}_\theta\dot{\varepsilon}_z \right] + \frac{(\dot{\varepsilon}_z + 2\dot{\varepsilon}_\theta)^2}{3} (1 - R^2) \right]^{1/2}$$  \hspace{1cm} (3.27)

Now the Equation (3.25) can be rewritten as follows:

$$\sigma_{eff} = C_1 R^A$$  \hspace{1cm} (3.28)
where

\[ C_1 = k \epsilon_{\text{eff}}^n \dot{\epsilon}_{\text{eff}}^m \]

Now, the Equation (3.28) can be written for two consecutive compression loads as follows:

\[ \sigma_1 = C_1 R_1^{A_1} \quad (3.29) \]

\[ \sigma_2 = C_1 R_2^{A_1} \quad (3.30) \]

Here \( \sigma_1 \) and \( \sigma_2 \) refer to the effective stresses and \( R_1 \) and \( R_2 \) refer to the relative perform densities for the compressive loads, namely, 1 and 2 respectively.

Dividing the Equation (3.30) by equation (3.29), the following is obtained:

\[ \frac{\sigma_2}{\sigma_1} = \left( \frac{R_2}{R_1} \right)^{A_1} \quad (3.31) \]

Taking natural logarithm on both sides,

\[ \ln \left( \frac{\sigma_2}{\sigma_1} \right) = \ln \left( \frac{R_2}{R_1} \right)^{A_1} \quad (3.32) \]

The Equation (3.32) can be rewritten as follows:

\[ A_1 = \frac{\ln(\sigma_2/\sigma_1)}{\ln(R_2/R_1)} \quad (3.33) \]

For consecutive compressive loads specified as \( i-1 \) and \( i \), the equation (3.31) becomes:

\[ A_i = \frac{\ln(\sigma_i/\sigma_{i-1})}{\ln(R_i/R_{i-1})} \quad (3.34) \]
In the Equation (3.34), for the known values of \( R \) and \( \sigma \), the instantaneous
density coefficient \( A_i \) can be determined.

Similarly, the Equation (3.25) can be rewritten as follows in order
to determine the strain hardening exponent \( (n) \).

\[
\sigma_{eff} = C_2 \varepsilon_{eff}^n \quad (3.35)
\]

where

\[
C_2 = k R^A \varepsilon_{eff}^m \quad (3.36)
\]

Now, the Equation (3.35) can be written for two consecutive compression
loads as follows:

\[
\sigma_1 = C_2 \varepsilon_1^n \quad (3.37)
\]

\[
\sigma_2 = C_2 \varepsilon_2^n \quad (3.38)
\]

Here \( \sigma_1 \) and \( \sigma_2 \) refer to the effective stresses and \( \varepsilon_1 \) and \( \varepsilon_2 \) refer to the
effective strains for the compressive loads, namely, 1 and 2 respectively.

Dividing the equation (3.38) by equation (3.37) the following is obtained:

\[
\frac{\sigma_2}{\sigma_1} = \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^n \quad (3.39)
\]

Taking natural logarithm on both sides,

\[
\ln \left( \frac{\sigma_2}{\sigma_1} \right) = \ln \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^n \quad (3.40)
\]

The Equation (3.34) can be rewritten as follows:
\[ n_1 = \frac{\ln(\sigma_2/\sigma_1)}{\ln(\varepsilon_2/\varepsilon_1)} \]  

(3.41)

For consecutive compressive loads specified as \( i-1 \), the equation (3.39) becomes as follows

\[ n_i = \frac{\ln(\sigma_i/\sigma_{i-1})}{\ln(\varepsilon_i/\varepsilon_{i-1})} \]  

(3.42)

For the known values of \( \sigma_{\text{eff}} \) and \( \varepsilon_{\text{eff}} \), the instantaneous strain-hardening index \( (n_i) \) can be determined using Equation (3.42). Using the instantaneous strain hardening index \( (n_i) \), the pore closure rate indices can be determined.

### 3.6.2 The Instantaneous Strength Coefficient \((k_i)\)

The instantaneous Strength coefficient \((k_i)\) for consecutive compressive loads are specified as \( 1, 2, 3, \ldots, (i-1), i \). and determined by employing the constitute relationship for the porous powder metallurgy performs is given by the following expression.

\[ \sigma_{\text{eff}} = C_3 \varepsilon_{\text{eff}}^m \]  

(3.43)

Where,

\[ C_3 = k R \varepsilon_{\text{eff}}^n \]  

(3.44)

Now, the Equation (3.43) can be written for two consecutive compression loads which is as follows:

\[ \sigma_1 = C_3 \varepsilon_1^m \]  

(3.45)

\[ \sigma_2 = C_3 \varepsilon_2^m \]  

(3.46)
Here \( \sigma_1 \) and \( \sigma_2 \) refer to the effective stresses and \( \dot{\varepsilon}_1 \) and \( \dot{\varepsilon}_2 \) refer to the effective strain rates for the compressive loads, namely, 1 and 2 respectively.

Dividing the Equation (3.46) by the Equation (3.45), the following is obtained.

\[
\frac{\sigma_2}{\sigma_1} = \left(\frac{\dot{\varepsilon}_2}{\dot{\varepsilon}_1}\right)^{m_1} \tag{3.47}
\]

Taking natural logarithm on both sides,

\[
\ln \left(\frac{\sigma_2}{\sigma_1}\right) = \ln \left(\frac{\dot{\varepsilon}_2}{\dot{\varepsilon}_1}\right)^{m_1} \tag{3.48}
\]

The Equation (3.46) can be rewritten as follows:

\[
m_1 = \frac{\ln (\sigma_2/\sigma_1)}{\ln (\dot{\varepsilon}_2/\dot{\varepsilon}_1)} \tag{3.49}
\]

For consecutive compressive loads specified as \( i-1 \) and \( I \), the Equation (3.49) is as follows:

\[
m_i = \frac{\ln (\sigma_i/\sigma_{i-1})}{\ln (\dot{\varepsilon}_i/\dot{\varepsilon}_{i-1})} \tag{3.50}
\]

For the known values of \( \sigma_{\text{eff}} \) and \( \dot{\varepsilon}_{\text{eff}} \), the strain rate sensitivity \( m_i \) can be determined using the Equation (3.50),

On substitution of all the parameter values in the Equation (3.25), the constant \( k_i \) can be determined using the equation given below:

\[
k_i = \left(\frac{\sigma}{R^A \varepsilon^n \dot{\varepsilon}^m}\right) \tag{3.51}
\]

Based on the experimental data (initial & deformed height, initial & deformed diameter, initial & deformed density etc.,) from upset test and are substituted in the above theoretical formulas to calculate the various parameters like stresses, strains, Poisson’s ratio, workability, strain hardening etc. There are no comparison studies between the theoretical and experimental results as this is a substitution analysis between one another.