APPENDIX

FREQUENCY DOMAIN SEGMENTATION

A.1 INTRODUCTION

Real world signals can be anything that is a collection of numbers, or measurements and the most commonly used signals include images, audio and medical and seismic data. In most digital signal processing applications, the frequency content of the signal is very important. The Fourier transform (FT) is the most popular transform used to obtain the frequency spectrum of a signal. MRI is excellent for showing abnormalities of the brain such as: stroke, hemorrhage, tumour, multiple sclerosis or lesions (Victor Musoko 2005).

The biggest achievement of Fourier's work was the fact, that functions in the frequency domain contain exactly the same information as originals: this means, that people are able to perform analysis of a function from a different point of view. Fast and efficient way of calculating Discrete Fourier Transform, which reduces number of arithmetical computations from $O(N^2)$ to $O(N \log_2 N)$ is called as FFT algorithm (Michal Dobroczynski 2006).

In order to perform good quantitative studies, ROI’s within the brain must be well defined. In traditional methods, a skilled operator manually outlines the ROI’s using a mouse or cursor. More recently, computer-assisted methods have been used for specific tasks such as extraction of MS lesions from MRI brain scans (Zijdenbos et al 1994).
A.2 FOURIER TRANSFORM (FT)

Fourier Transform (FT) is named in the honor of Joseph Fourier (1768-1830), one of greatest names in the history of mathematics and physics. The Fourier transform is a linear operator that maps a functional space to another functions space and decomposes a function into another function of its frequency components.

The Fourier Transform is used to decompose an image into its sine and cosine components which is an important image processing tool. Input image is the spatial domain and the output represents the image in the frequency domain. Bracewell (1999) describes as some applications include in the case of communications, used to predict how a signal behaves when it passes through filters, amplifiers and communications channels by using Fourier Transform (Chowning 1973, Brandenberg and Bosi 1997, Bosi and Goldberg 2003).

In the case of image processing and data analysis it used for smoothing, transformation and sharpening of images and by encoding the time series used to estimate the noise and signal and also used as low-pass, high- pass and band-pass filters (Good 1960, Harris 1978, Zwicker and Fastl 1999, Kailath et al 2000, Gray and Davisson 2003).

A.3 FOURIER TRANSFORM AND INVERSE FOURIER TRANSFORM

As the thesis is only concerned with digital images, discussion is made on Discrete Fourier Transform (DFT). The DFT is the sampled Fourier Transform and therefore does not contain all frequencies forming an image, but only a set of samples which is large enough to fully describe the spatial domain image. The number of frequencies corresponds to the number of
pixels in the spatial domain image, i.e. the image in the spatial and Fourier domain is of the same size (Ballard and Brown 1982).

For a square image of size N×N, the two-dimensional DFT is given by

\[
F(k,l) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(i,j) e^{-i2\pi \frac{ki}{N} + \frac{lj}{N}} \tag{A.1}
\]

Where \( f(i,j) \) is the image in the spatial domain and the exponential term is the basis function corresponding to each point \( F(k,l) \) in the Fourier space. The basis functions are sine and cosine waves with increasing frequencies, i.e. \( F(0,0) \) represents the DC-component of the image which corresponds to the average brightness and \( F(N-1,N-1) \) represents the highest frequency.

The Fourier image can be re-transformed to the spatial domain. The inverse Fourier transform is given by

\[
f(a,b) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k,l) e^{i2\pi \left( \frac{ka}{N} + \frac{lb}{N} \right)} \tag{A.2}
\]

Where \( \frac{1}{N^2} \) is the normalization term in the inverse transformation.

To obtain the result for the above equations, a double sum has to be calculated for each image point. However, because the Fourier Transform is separable, it can be written as

\[
F(k,l) = \frac{1}{N} \sum_{b=0}^{N-1} p(k,b) e^{-i2\pi \frac{lb}{N}} \tag{A.3}
\]
Where

\[ P(k, b) = \frac{1}{N} \sum_{a=0}^{N-1} f(a, b)e^{-\frac{i2\pi ka}{N}} \]  

(A.4)

By using these two formulas, the spatial domain image is first transformed into an intermediate image using \( N \) one-dimensional Fourier Transforms. This intermediate image is then transformed into the final image, again using \( N \) one-dimensional Fourier Transforms. Expressing the two-dimensional Fourier Transform in terms of a series of \( 2N \) one-dimensional transforms decreases the number of required computations.

Even with these computational savings, the ordinary one-dimensional DFT has \( N^2 \) complexity. This can be reduced to \( N \log_2 N \) if Fast Fourier Transform (FFT) has been employed to compute the one-dimensional DFTs. This is a significant improvement, in particular for large images. There are various forms of the FFT and most of them restrict the size of the input image that may be transformed, often to \( N = 2^n \) where \( n \) is an integer (Cooley and Tukey 1965).

The Fourier Transform produces a complex number valued output image which can be displayed with two images, either with the real and imaginary part or with magnitude and phase. In image processing, often only the magnitude of the Fourier Transform is displayed, as it contains most of the information of the geometric structure of the spatial domain image. However, if the Fourier image has to be re-transformed into the correct spatial domain after some processing in the frequency domain, it is necessary to preserve both magnitude and phase of the Fourier image (Smith 2002).
The Fourier domain image has a much greater range than the image in the spatial domain. Hence, to be sufficiently accurate, its values are usually calculated and stored in float values (Gonzales and Woods 1992).

A.4 DISCRETE FOURIER TRANSFORM (DFT)

Discrete Fourier Transform is one of the specific forms of Fourier analysis. It transforms one function into another which is called as frequency domain representation. But DFT requires an input function that is discrete and whose non-zero values have a limited or finite duration. Such inputs are obtained by sampling a continuous function.

In the case of Discrete Time Fourier Transform it only evaluates enough frequency components to reconstruct the finite segment that was analyzed. Its inverse Fourier Transform cannot reproduce the entire time domain unless the input happens to be periodic. Therefore it is often said that DFT is a transform for Fourier analysis of finite domain discrete-time functions. The sinusoidal basic functions of the decomposition have the same properties.

The input function is a finite sequence of real or complex numbers, the DFT is ideal for processing information stored in computers. In particular DFT is widely employed in signal processing and related fields to analyze the frequencies contained in a sampled signal. It can used to solve partial differential equations and for performing other operations such as convolutions. DFT’s are computed efficiently in practice using a FFT algorithm (Heideman et al 1984).

FFT algorithms are employed to compute DFT. DFT’s refers to mathematical transformation regardless of how it is computed while FFT refers to any one of several efficient algorithms for the DFT.
A.5 FAST FOURIER TRANSFORM

Fast Fourier Transforms (FFTs) are efficient algorithms for calculating the Discrete Fourier Transform (DFT). The DFT usually arises as an approximation to the continuous Fourier transform when functions are sampled at discrete intervals in space or time. The naive evaluation of the discrete Fourier transform is a matrix-vector multiplication $Wg$ and would take $O(N^2)$ operations for the N data points.

The general principle of the Fast Fourier Transform algorithms is to use a divide-and-conquer strategy to factorize the matrix $W$ into smaller sub-matrices typically reducing the operation count to $O(N\sum f_i)$ if N can be factorized into smaller integers $N = f_1, f_2, f_3, \ldots, f_n$ (Oran Brigham 1974).

A.6 FAMILIES OF FFT ALGORITHMS

Oran Brigham (1974) discusses about two main families of FFT algorithms: the Cooley-Tukey algorithm and the Prime factor algorithm. These differ in the way they map the full FFT into smaller sub-transforms.

Of the Cooley-Tukey algorithms there are two types of routine in common use: Mixed-Radix (general-N) algorithms and Radix-2 (power of 2) algorithms. Each type of algorithm can be further classified by additional characteristics, such as whether it operates in-place or uses additional scratch space, whether its output is in a sorted or scrambled order, and whether it uses decimation-in-time or-frequency iterations.
A.6.1 Radix-2 FFT

Radix-2 decimation-in-time is the most common form of the Cooley-Tukey algorithm, for any arbitrary size N; Radix-2 DIT divides the size N DFT into two interleaved DFT’s of size N/2, the DFT as defined earlier,

\[ X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn} \]  \hspace{1cm} (A.5)

where \( k = 0,1,2,\ldots,N-1 \). Radix-2 divides the DFT into two equal parts, the first part calculates the Fourier Transform of the even index numbers and other part calculates the Fourier Transform of the odd index numbers and then finally merges them to get the Fourier Transform for the whole sequence. This will reduce the overall time to \( O(N \log_2 N) \). In Figure A.1, a Cooley-Tukey based decimation in frequency for 8-point FFT algorithm is shown.
A.6.2  Radix-4 FFT

When the number of data points $N$ in the DFT is a power of 4 (i.e. $N = 4^V$), Radix-2 algorithm can be used for the computation. However, for this case, it is more efficient computationally to employ a Radix-$r$ FFT algorithm (Cooley and Tukey 1965).

Radix-4 FFT requires less number of multiplications than Radix-2 FFT of the same size, and for the same throughput, it requires less hardware. Hence, quite often, Radix-4 structure is preferred to Radix-2 structure. The analysis assumes fixed-point two's complement arithmetic, with rounding in the case of multiplication, and truncation in the case of scaling. The predicted results of output noise agree closely with the computer simulation results. The different schemes of scaling for preventing arithmetic overflow are compared from the noise-to-signal ratio point of view. In comparison with Radix-2 the Radix-4 FFT has got a marginally better error performance.

The presented method aims to solve the problems of Fourier transform on existing complexity and long time-consuming, and it can reduce the number of data store by conformal symmetry of Fourier transform. FFT algorithms presented by Cooley and Tukey (1965).

The computational redundancy in DFT is exploited by them. They invented a method from which several FFT algorithms can be derived. This method is known as Cooley-Tukey decomposition. DFT of an arbitrary composite size $N=\text{PQ}$ can be re-expressed in terms of sizes $\text{P}$ and $\text{Q}$ recursively. Initial computational complexity of DFT is $N$ power 2. It has been reduced to the order of $N \log N$ by Cooley-Tukey decomposition.

The principle of Cooley-Tukey decomposition is explained as follows for the composite size of $N=\text{PQ}$.
1. Decimate the sequence into P sequences.
2. Compute P- DFTs of size Q.
3. Multiply the resulting sequences by twiddle factors.
4. Re-order the sequences.
5. Compute Q DFTs of size P.

In Radix-4FFT by using the number four as the Radix, computationally efficient forms are evaluated.

By counting the number of real multiplications and additions the computational complexity is evaluated i.e count the number of real multiplications and additions by “real” we mean either fixed-point or floating point operations depending on the specific hardware. Subtraction is considered equivalent to additions and divisions are counted separately.

Radix-4 decimation-in-time FFT algorithm is described as by splitting or decimate the $N$-point input sequence into four subsequences,

$$x(4n), x(4n +1), x(4n +2), x(4n + 3), n = 0,1.....N/4-1$$  \hspace{1cm} \text{(A.6)}

$$X(p,q) = \sum_{l=0}^{3} [W_N^{lq} F(l,q)] W_4^{lq}$$ \hspace{1cm} \text{(A.7)}

$$F(l,q) = \sum_{m=0}^{(N/4)-1} x(l,m) W_N^{mq}$$ \hspace{1cm} \text{(A.8)}

### A.6.3 Mixed Radix FFT

Mixed-Radix algorithms work by factorizing the data vector into shorter lengths. These can then be transformed by small-N FFTs. Typical programs include FFTs for small prime factors, such as 2, 3, 5... which are highly optimized.
The small-N FFT modules act as building blocks and can be multiplied together to make longer transforms. By combining a reasonable set of modules it is possible to compute FFTs of many different lengths. Radix-\(r\) implementations are limited to \(r^n\)-point FFTs, and therefore Radix-2 computation must also be employed for a Radix-4 processor to cover all \(2^n\)-point FFTs. A mixed-Radix FFT (MRFFT) architecture combines multiple radices—typically Radix-2 and Radix-4—and has been utilized in previous FFT processors.

Many digital image-processing systems, especially for advanced military and medical imaging applications, require fast pattern matching for precision image-recognition results. In medical imaging systems, biological cells of a specific type may need to be located in slides containing thousands of such cells. In many of these systems, where high speed is critical, optical correlators have been used to perform the forward and inverse fast Fourier transforms (FFTs) needed in pattern matching.

The procedure performs a fast discrete Fourier transform (FFT) of a complex sequence, \(x\), of an arbitrary length, \(n\). The output, \(y\), is also a complex sequence of length \(n\).

\[
Y(k) = \sum \{x[m] \times \exp(-i \times \pi \times k \times m / n), m=0 \ldots (n-1), k=0, \ldots (n-1)\} \quad (A.9)
\]

The general idea is to factor the length of the DFT, \(n\), into factors that are efficiently handled by the routines. A number of short DFT's are implemented with a minimum of arithmetical operations and using straight line code resulting in very fast execution when the factors of \(n\) belong to this set. Especially Radix-10 is optimized. Prime factors, that are not in the set of short DFT's are handled with direct evaluation of the DFP expression (Jens Joergen Nielsen 2000).
A.7 K-SPACE DATA

A Free Induction Decay (FID) curve is generated as excited nuclei relax. The FID signal is sampled to get the discrete signal which contains all the necessary information to reconstruct an image, called the k-space signal. The signal sampled in Fourier space contains all the low frequency data at the center which gives the information about the contrast change in gray level with highest amplitude values. The uniformly sampled k-space data is shown in Figure A.2. On the other hand, the high frequency component gives information about the spatial resolution of the object (Jon Louis Bentley 1982).

![Figure A.2 Uniformly sampled K-SPACE data](image)

A.8 TWO-DIMENSIONAL MR IMAGE FORMATION, NOISE AND IMAGE PROCESSING

The following Figure A.3 illustrates the principles of 2-D Fourier transform imaging on clinical MR scanners. The variants of this Fourier transform imaging technique are the most widespread MR method for obtaining structural and functional information from the living human body. Noise in MRI enters the data samples in k-space. Here the noise voltage
competes with the NMR signal and is due to random fluctuations in the receiving coil electronics and in the patient body (e.g., Brownian motion of spins).

The variance of this thermal noise can be described as the sum of noise variances from independent stochastic processes, representing the body, the coil and the electronics, i.e.,

$$\sigma_{\text{thermal}}^2(k_x, k_y) = \sigma_{\text{body}}^2(k_x, k_y) + \sigma_{\text{coil}}^2(k_x, k_y) + \sigma_{\text{electronics}}^2$$

(A.10)

The noise removal is obtained by averaging in measurements where noise in k-space is additive, i.e., zero mean Gaussian noise with standard deviation $\sigma$ which yields an improvement in $\sigma = \sigma / \sqrt{n}$ (Marius Lysaker et al 2003).

In magnetic resonance imaging, complex-valued measurements are acquired in time corresponding to spatial frequency measurements in space generally placed on a Cartesian rectangular grid. These complex-valued measurements are transformed into a measured complex-valued image by an image reconstruction method. The most common image reconstruction method is the inverse Fourier transform. It is known that image voxels are spatially correlated. A property of the inverse Fourier transformation is that uncorrelated spatial frequency measurements yield spatially uncorrelated voxel measurements and vice versa. Spatially correlated voxel measurements result from correlated spatial frequency measurements (Haacke et al 1999).
Figure A.3 Principles of MR image formation

In MRI, a real-valued 3D object, $R(x,y,z)$, is imaged, a lattice of volume elements called as voxels. In general, a single slice $R(x,y)$ is considered for processing and it is shown in Figure A.4. In MRI different tissues have different magnetic properties yielding contrast. The need of Fourier Transform arises because of the fact that in MRI/fMRI measurements are not voxel values. Mainly measurements are spatial frequencies. These
spatial frequencies got by applying Gx & Gy magnetic field gradients to encode then complex-valued DFT of the object is measured.

![Figure A.4](image)

**Figure A.4** (a) Volume   (b) Slice

**A.9 POWER SPECTRUM**

How much information is contained at a particular frequency is needed and maybe it is part of the sine or cosine series. Therefore, one is interested in the absolute value of the FFT coefficients. The absolute value will provide with the total amount of information contained at a given frequency, the square of the absolute value is considered the power of the signal. The absolute value of the Fourier coefficients is the distance of the complex number from the origin. The power in the signal at each frequency is commonly called the power spectrum.

**A.10 SPECTRUM ANALYSIS WITH THE FFT AND NEED OF POWER SPECTRUM**

The FFT does not directly give the spectrum of a signal. FFT can vary dramatically depending on the number of points (N) of the FFT, and the number of periods of the signal that are represented. There is another problem as well. The FFT contains information between 0 and $f_s$, however, we know that the sampling frequency must be at least twice the highest frequency
component. Therefore, the signal's spectrum should be entirely below $f_s / 2$, the Nyquist frequency.

The quality of Computed Tomographic (CT) images is usually evaluated in terms of spatial resolution and level of noise. A single-parameter measure of image noise, however, such as the standard deviation over an area, is inadequate to predict the utility of an image for a given task. Riederer et al (1978) showed that the random variation in CT number at one point of an image is not independent of the random variation at other points. These spatial correlations can be fully described by either the autocorrelation function or its Fourier transform of the Noise Power Spectrum (NPS).

### A.11 FFT WINDOWS

FFT based measurements are subject to errors from an effect known as leakage. This effect occurs when the FFT is computed from a block of data which is not periodic. To correct this problem appropriate windowing functions must be applied. The user must choose the appropriate window function for the specific application. An actual plot of a smoothing window shows that the frequency characteristic of the smoothing window is a continuous spectrum with a main lobe and several side lobes shows the spectrum of a typical smoothing window. The following Figure A.5 shows the spectrum of a typical smoothing window.
SPECTRAL LEAKAGE

The FFT computation assumes that a signal is periodic in each data block, that is, it repeats over and over again and it is identical every time. In Figure A.6, because there are an integer number of cycles of the sine wave in the data record. Another type of signal that satisfies the periodic requirement is a transient signal that starts at zero at the beginning of the time window and then rises to some maximum and decays again to zero before the end of the time window.

When the FFT of a non-periodic signal is computed then the resulting frequency spectrum suffers from leakage. Leakage results in the signal energy smearing out over a wide frequency range in the FFT when it should be in a narrow frequency range. Figure A.7 illustrates the effect of leakage. The left-top graph shows a 10 Hz sine wave with amplitude 1.0 that is periodic in the time frame. The resulting FFT (bottom-left) shows a narrow peak at 10 Hz in the frequency axis with a height of 1.0 as expected. Note the dB scale is used to highlight the shape of the FFT at low levels. The right-top graph shows a sine wave that is not periodic in the time frame resulting in leakage in the FFT (bottom-right). The amplitude is less than the expected 1.0
value and the signal energy is more dispersed. The dispersed shape of the FFT makes it more difficult to identify the frequency content of the measured signal (Ramirez 1985).

Figure A.6 Time waveform of sine function (top) and FFT (bottom)

Figure A.7 Comparison of periodic sine wave (left) and FFT to nonperiodic (right) with leakage in the FFT
Figure A.8 shows the effect of applying a Hamming window to a pure sine tone. The left-top plot shows a sine tone that is not periodic in the time window without the windowing function resulting in leakage in the FFT (left-bottom). Figure A.9 shows the Hamming windowing function and its FFT. The highest side lobe is -32 dB. The FFT of a window has a peak at the applied frequency and other peaks, called side lobes, on either side of the applied frequency. The height of the side lobes indicates what affect the windowing function will have on frequencies around the applied frequency. In general, lower side lobes reduce the leakage in the measured FFT but increase the bandwidth of the major lobe (Oppenheim and Schafer 1989).
When the window is implemented in the frequency domain, the FFT of the window function is computed one time and saved in memory and then it is applied to every FFT frequency value correcting the leakage in the FFT. This gives rise to one measure of the window's characteristics, known as the side lobe. Side lobes occur on each side of the main lobe and approach zero at multiples of $f_s/N$ from the main lobe (Harris 1978).

The side lobe characteristics of the smoothing window directly affect the extent to which adjacent frequency components leak into adjacent frequency bins. The side lobe response of a strong sinusoidal signal can overpower the main lobe response of a nearby weak sinusoidal signal.

The FFT of a window has a peak at the applied frequency and other peaks, called side lobes, on either side of the applied frequency. The height of the side lobes indicates what affect the windowing function will have on frequencies around the applied frequency. In general, lower side lobes reduce
the leakage in the measured FFT but increase the bandwidth of the major lobe. In addition, each type of window affects the spectrum in a slightly different way and the best type of window should be chosen for each specific application. Since the Hamming window side-lobe level is more than 40 dB down, it is often a good choice for 1% accurate systems, such as 8-bit signal processing systems (Julius Smith 2011).

A.14 Normal Brain and Pathological Brain MRI

Figure A.9 shows normal brain MRI. The white matter and gray matter portions are shown in as light gray and dark gray respectively. Medical image analysis typically involves heterogeneous data that has been sampled from different underlying anatomic and pathologic physical processes.

In the case of Glioblastoma Multiforme Brain Tumour (GBM), for example, the heterogeneous processes in study are the tumour itself, comprising a necrotic (dead) part and an active part, the edema or swelling in the nearby brain, and the brain tissue itself. Not all GBM tumours have a clear boundary between necrotic and active parts, and some may not have any necrotic parts (Jason Corso et al 2008).
Figure A.9  SPGR MR exam of a normal brain. The white matter is the brain tissue appearing light gray and the gray matter appears dark gray.
Figure A.10  SPGR MR exam of a brain with a glioblastoma multiforme Brain Tumour (GBM) and related pathology

Figure A.10 shows the MR image of a brain containing GBM tumour which outlines the different heterogeneous regions of the brain tumour and labeled them as edema, active, or necrotic.