Chapter 1

Introduction

The notation and terminology used through this thesis is taken mostly from Harary and, Harary and Buckley \[17][22]. For hypergraph we follow the notation and terminology used in Berge \[11][12][13]. New definitions and terminology shall be defined wherever necessary. Unless mentioned otherwise, graphs treated in the thesis are simple and finite in the sense that their vertex sets are finite sets.

The theory of graphs is one of the few fields of mathematics with a definite birth date. Graph theory is considered to have begun in 1736 with the publication of Euler’s solution of the Konigsberg bridge problem. Any mathematical object involving points and connections between them may be called a graph. A graph $G$ consists of a nonempty set $V(G)$ of objects called vertices and a (possibly empty) set $E(G)$ of
two element subsets of $V(G)$, called edges. The set $V(G)$ is called the vertex set of $G$ and $E(G)$ its edge set. The number of vertices in a graph $G$ is called its order, and the number of edges is its size. A graph of order $p$ and size $q$ is called a $(p, q)$-graph.

It has become a tradition to describe graphs by means of diagrams in which each element of the vertex set of the graph is represented by a dot and an edge $e = uv$ is represented by a curve joining the dots that represent the vertices $u$ and $v$.

For example, if we consider the graph $G$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_4v_5\}$. Then a possible diagram for this graphs is shown in Figure 1.1.

![Figure 1.1: A disconnected graph on five vertices](image)

However, although the diagram is the most common way of representing graphs, there are many other ways of representing them. Another very common way is by means of the adjacency matrix. Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_m\}$. Then define its $m \times m$ adjacency
matrix $A = (a_{ij})$ to be one, if $(v_i, v_j)$ is an edge and, zero otherwise. For example the adjacency matrix $A$ for the graph of Figure 1.1 is shown below.

$$A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}$$

A parameter that appears often when studying graphs is the degree of vertex. The *degree* of a vertex $u$ of a graph $G$, denoted by $\text{deg}_G u$, or simply by $\text{deg} u$ or $d(u)$, if the graph $G$ is clear from the context, is defined as $d(u) = | \{ v/uv \in E(G) \} |$. A vertex $v$ of a graph $G$ is called even, if its degree is even and odd, if its degree is odd. Also, if $\text{deg} v = 0$, $v$ is called an isolated vertex, and if $\text{deg} v = 1$, it is called an end vertex. Also, if $e = uv$ is an edge of a graph $G$ such that either $\text{deg} u = 1$ or $\text{deg} v = 1$, then $e$ is called a pendant edge of $G$.

Two graphs are said to be *isomorphic* if they have the same structure, and at the most, they differ in the way their vertices and edges are labeled, or in the way they are drawn. In order to make this idea more precise, we will define two graphs $G_1$ and $G_2$ to be isomorphic if there exists a bijective function $\phi : V(G_1) \rightarrow V(G_2)$ such that
$uv \in E(G_1) \iff \phi(u)\phi(v) \in E(G_2)$. The function $\phi$ is called an isomorphism. If two graphs $G_1$ and $G_2$ are isomorphic, then we write $G_1 \cong G_2$.

Let $G$ be a graph, then a graph $H$ is said to be a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H$ is a subgraph of $G$, then we will write $H \subseteq G$. Some types of subgraphs that appear often when studying graph theory are those obtained by the deletion of a vertex or an edge. If $G$ is any graph with $|V(G)| \geq 2$, and $v \in V(G)$, then the subgraph $G - v$ of $G$ is defined to be the graph with $V(G - v) = V(G) \setminus \{v\}$ and $E(G - v) = E(G) \setminus \{e \in E(G) \mid v \text{ is incident with } e\}$. Also if $e \in E$, then the subgraph $G - e$ is defined to be the graph with $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. If $u$ and $v$ are non adjacent vertices in a graph $G$ and $e = uv$, then the graph $G + e$ is defined to be the graph with $V(G + e) = V(G)$, $E(G + e) = E(G) \cup \{e\}$. Another important type of subgraphs are the induced subgraphs. Let $G$ be a graph, and suppose that $U \subseteq V(G)$ is nonempty. Then the subgraph $\langle U \rangle$ of $G$ induced by $U$ is the graph such that $V(\langle U \rangle) = U$ and $E(\langle U \rangle) = \{xy \in E(G) \mid x, y \in U\}$. Also if $F \subseteq E(G)$ is nonempty, then the subgraph $\langle F \rangle$ of $G$ induced by $F$ is the graph such that $V(\langle F \rangle) = \{u \in V(G) \mid uv \in F \text{ for some } v \in V(G)\}$, $E(\langle F \rangle) = F$. 
Another important concept is the connectedness of graphs. Informally, we say that a graph is *connected* if it is possible to travel from any vertex of that graph to any other vertex of it, using the vertices and edges of the graph. We can make this concept more formal in the following way. A graph $G$ is connected, if given any pair of distinct vertices of $G$, namely $u$ and $v$, there exists a sequence of vertices and edges of $G$ of the form

$$u = x_1, x_1x_2, x_2x_3, x_3, x_3x_4, x_4, \ldots, x_{n-1}, x_{n-1}x_n, x_n = v$$

and is called *disconnected* otherwise.

If $G$ is a disconnected graph, then we define a *component* of $G$ to be a subgraph induced by a set $U \subset V(G)$ such that $\langle U \rangle$ is connected, but if $v \in V(G) \setminus U$, then $\langle U \cup \{v\} \rangle$ is disconnected. The number of components of a graph $G$ is usually denoted by $k(G)$. A *bridge* $e$ of a graph $G$ is any element of $E(G)$ such that $k(G) < k(G - e)$. Similarly, a *cut vertex* $V$ of a graph $G$ is any element of $V(G)$ with the property that $k(G) < k(G - v)$.

In order to conclude the first section, we shall also introduce the very important concept of decomposition. A *decomposition* of a graph $G$ is a collection $\{H_i\}$ of subgraphs of $G$ such that $H_i = \langle E_i \rangle$ for some subset $E_i$ of $E(G)$ and where $\{E_i\}$ is a partition of $E(G)$. If $\{H_i\}$ is a decomposition of $G$, then we can write

$$G \cong H_1 \oplus H_2 \oplus \cdots \oplus H_{|H|} = \bigoplus_{i=1}^{|H|} H_i.$$
The complement $\overline{G}$ of a graph $G$ has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A graph and its complement are shown in Figure 1.2.

![Figure 1.2: A graph and its complement](image)

The graphs $\overline{K_p}$ are called totally disconnected, and are regular of degree 0. A self-complementary graph is one which is isomorphic to its complement. A self-complementary graph is shown in Figure 1.3.

![Figure 1.3: A self-complementary graph](image)

We also discuss here those operations on graphs that are used in this thesis. In all the definitions follows, we assume that, graphs $G_1$ and $G_2$ have disjoint vertex sets $V_1$ and $V_2$ and their edge sets as $E_1$ and
The union of $G_1$ and $G_2$, denoted as $G = G_1 \cup G_2$ has $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Join of $G_1$ and $G_2$, as defined by Zykov [31], denoted $G_1 + G_2$, the vertex set consists of $V = V(G_1) \cup V(G_2)$ and the edge set, all edges obtained by joining $V_1$ with $V_2$. In particular, $K_{m,n} = \overline{K}_m + \overline{K}_n$. These operations, namely union and join of two graphs $G$ and $H$ are illustrated in Figure 1.4 and 1.5 with $G = K_3$ and $H = P_4$. 

![Figure 1.4: The union of two graphs](image)

![Figure 1.5: The join of two graphs](image)
For any connected graph $G$, we write $nG$ for the graph with $n$ components, each isomorphic to $G$. Then every graph can be written in the form $\bigcup_i n_i G_i$ with $G_i$ different from $G_j$ for $i \neq j$. To define the cartesian product $G_1 \times G_2$, consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. The cartesian product of $G_1 \cong P_2$ and $G_2 \cong P_2$ is shown in Figure 1.6.

![Figure 1.6: The cartesian product of two graphs](image)

The corona $G_1 \circ G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. For the graphs $G_1 \cong P_3$ and $G_2 \cong K_3$, the two different coronas $G_1 \circ G_2$ and $G_2 \circ G_1$ are shown in Figure 1.7.

![Figure 1.7](image)

The $n$-cube $Q_n$ is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. Thus $Q_n$ has $2^n$ vertices, which may be labeled $a_1 a_2 a_3 \ldots a_n$, where each $a_i$ is either 0 or 1. Two vertices in $Q_n$ are adjacent if their
binary sequences differ in exactly one place. Figure 1.8 shows the 3-cube.

For $n \geq 3$, the wheel $W_{1,n}$ is defined to be the graph $K_1 + C_n$. The generalized wheel $W_{m,n}$ is the graph $K_m + C_n$. A wheel and a generalized wheel are shown in Figure 1.9.

The square $G^2$ of a graph $G$ has $V(G^2) = V(G)$ with $u, v$ are adjacent in $G^2$ whenever $d(u, v) \leq 2$ in $G$. The higher powers $G^3, G^4, \ldots$ of $G$
are defined similarly. Square of $P_4$ is shown in Figure 1.10.

Let graph $G$ has at least one edge. The line graph $L(G)$ of $G$, has $E(G)$ as its vertex set with two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ are adjacent. The line graph of $K_4$ is shown in Figure 1.11. We write $L^2(G) = L(L(G))$, and in general $L^n(G) = L(L^{n-1}(G))$.

A walk in a graph $G$ is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ such that, every $e_i = v_{i-1}v_i$ is an edge of $G$, $1 \leq i \leq n$. A walk is a path if all of its vertices are distinct.
The distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. A shortest $u - v$ path is called a $u - v$ geodesic. The diameter $d(G)$ of a connected graph $G$ is the length of any longest geodesic.

**Definition 1.0.1.** A hypercube, $H_n$ is the graph with vertex set $V = \{0, 1\}^n$, whose edges are pairs of vectors $(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)$ in $\{0, 1\}^n$ such that $|\{i : x_i \neq y_i\}| = 1$.

**Definition 1.0.2.** A half-hypercube, $\frac{1}{2}H_n$ is the graph with vertex set $V = \{(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n\}$ such that $\sum_{i=1}^n x_i$ is even in even representation and $\sum_{i=1}^n x_i$ is odd in odd representation whose edges are pairs $(x, y) \in \{0, 1\}^n$ such that $|\{i : x_i \neq y_i\}| = 2$.

**Definition 1.0.3.** A cubic lattice $Z_n$ is an infinite graph with vertex set $V = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{Z}\}$ whose edges are
\[(x_1, x_2, \ldots, x_n) (y_1, y_2, \ldots, y_n) \text{ such that } \sum_{1 \leq i \leq n} |x_i - y_i| = 1.\]

**Definition 1.0.4.** A half-cubic lattice \(\frac{1}{2} Z_n\) is an infinite graph with vertex set

\[V = \{(x_1, x_2, \ldots, x_n) \in Z^n : \sum_{i=1}^{n} x_i \text{ is even}\}\]

such that \(\sum_{i=1}^{n} x_i \text{ is even}\) whose edges are \((x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)\) such that \(\sum_{1 \leq i \leq n} |x_i - y_i| = 2.\)

**Definition 1.0.5.** A median graph is an undirected graph in which any three vertices \(a, b\) and \(c\) have a unique median. That is, there exists a vertex \(m(a, b, c)\) that belongs to shortest paths between any two of \(a, b\) and \(c.\)

### 1.1 Hypergraphs

Given a set \(V\) of vertices, an edge of a (simple) graph on \(V\) is a set of two vertices, while an edge of a hypergraph on \(V\) is any subset of \(V\). The theory of hypergraphs, popularized and enriched by many contributions of Berge [11], [12], [13] is the extension of theorems about graphs to hypergraphs. The problem is to find a suitable formulation
of the theorems for hypergraphs in such a way that the results holds good for graphs as a special case.

Precisely, C. Berge [13], defined a hypergraph on a finite set $X = \{x_1, x_2, x_3, \ldots, x_n\}$ as a family $H = (E_1, E_2, E_3, \ldots, E_m)$ of subsets of $X$ such that

\begin{align*}
E_i &\neq \emptyset, \quad (1 \leq i \leq m) \quad (1.1.1) \\
\bigcup_{i=1}^{m} E_i &= X \quad (1.1.2)
\end{align*}

The elements $x_1, x_2, x_3, \ldots, x_n$ of $X$ are called vertices, and the sets $E_1, E_2, E_3, \ldots, E_m$ are the edges of the hypergraph. The order of $H$, denoted by $n(H)$, is the number of vertices and the number of edges is denoted by $m(H)$. In a hypergraph, two vertices are said to be adjacent if there is an edge $E_i$ that contains both of these vertices and two edges are said to be adjacent if their intersection is non-empty.

The incidence matrix of hypergraph $H = (X, \mathcal{E})$ is a matrix $(a^i_j)$ with $m$ rows that represent the edges of $H$ and $n$ columns that represent the vertices of $H$, such that

$$a^i_j = \begin{cases} 
1 & \text{if } x_j \in E_i \\
0 & \text{if } x_j \notin E_i
\end{cases}.$$ 

Each $(0, 1)$ matrix is the incidence matrix of a hypergraph if no row or column contains only zeros. A hypergraph $H = (X, \mathcal{E})$ is simple if it
has no repeated edges. Given a hypergraph $H = (X, \mathcal{E})$ and nonempty subset $S \subseteq X$, the hypergraph $H_S = (S, \{E \in \mathcal{E} : E \subseteq S\})$ is called the subhypergraph induced by $S$ in $H$. The dual of a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on $X$ is a hypergraph $H^* = (X_1, X_2, \ldots, X_n)$ whose vertices corresponds to the edges of $H$, and with the edges $X_i = \{e_j : x_i \in E_j \in H\}$.

In a hypergraph $H$, the rank $r(S)$ of a non-empty set $S \subset X$ is defined to be the positive integer $r(S) = \max |(X \cap E_i)|$. The number $r(X)$ is called the rank of hypergraph $H$. If $E_i = r(X)$ for each $i$, then $H$ is called a uniform hypergraph of rank $r(X)$.

Given an integer $k > 0$, the $k$-section of hypergraph $H$ is defined to be the couple $H_{(k)} = (X, \mathcal{E}_{(k)})$ formed by the set $\mathcal{E}_{(k)} = \{F : F \subset X; 1 \leq |F| \leq k; F \subset E_i \text{ for some } E_i \in \mathcal{E}\}$. The 2-section $H_{(2)}$ of $H$ is a symmetric graph with the same vertices as $H$ and with a loop attached to each vertex and $[H]_2$ denotes this two section without loops. A simple hypergraph $H = (E_1, E_2, \ldots, E_m)$ has the Helly property if every intersecting family of $H$ is a star. That is, for $J \subset \{1, 2, 3, \ldots, m\}$

\[ E_j \cap E_k \neq \emptyset; \quad j, k \in J \quad (1.1.3) \]

implies

\[ \bigcap_{j \in J} E_j \neq \emptyset. \quad (1.1.4) \]
In a hypergraph $H = (X, \mathcal{E})$, a chain of length $q$ is defined to be a sequence $(x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1})$ such that (i) $x_1, x_2, \ldots, x_q$ are all distinct vertices of $H$, (ii) $E_1, E_2, \ldots, E_q$ are all distinct edges of $H$, $x_k, x_{k+1} \in E_k$ for $k = 1, 2, \ldots, q$.

If $q > 1$ and $x_{q+1} = x_1$, then this chain is called a cycle of length $q$.

1.2 Labeling of graphs

Graph labeling, where the vertices and edges are assigned by real numbers or elements of a given set or subsets of a set subject to certain conditions, have often been motivated by practical considerations, but they are also of interest on their own right. Graph labeling was first introduced in the mid 60’s. Most graph labeling methods trace their origin to one introduced by Rosa [26] in 1967. An enormous body of literature has grown around the theme. Over the past 40 years or so, more than 800 papers on various graph labeling methods have been published, identifying several classes of graphs admitting a given type of labeling. Labeled graphs are becoming an increasingly useful family of mathematical models for a wide range of applications such as coding, X-ray, crystallography, radar tracking, remote control, radio-astronomy, communication networks, network flows etc.
Even though the study of graceful graphs and graceful labeling methods were introduced by Rosa [26] in 1967, the term *graceful graph* was used first by Bloom Colomb [15] in 1977. Rosa defined a $\beta$-valuation of a graph $G$, as an injection $f$ from the vertices of $G$ to the set $\{0, 1, \ldots, e\}$, where $e$ is the number of edges in $G$, such that when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. In a graceful labeling of a graph $G$ the resulting edge labels must be distinct and take values $1, 2, \ldots, e$. The study of graceful labeling of a graph is an active research area in graph theory. The graceful labeling problem is to determine which graphs are graceful. Proving a graph $G$ is or is not graceful involves either producing a graceful labeling of $G$ or showing that no such labeling exists. While the graceful labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, gracefully labeled graphs often serve as models in a wide range of applications. Such applications include coding theory and communication network addressing. Bloom Colomb [15] gives a detailed explanation of some of the applications of gracefully labeled graphs.

In general, interpersonal relationships depend on personal attitudes of the individuals in any social group. When attitudes are expressed by the individuals to others in the group, the types of interpersonal interactions get affirmed and/or modified. On the other hand, such
affirmations and/or modifications in various types of interpersonal interaction in the group induce change in attitudes of the persons in the group. The revisory socio-psychological phenomenon motivates a study of set assignments \( h : V(G) \cup E(G) \to 2^X \).

In this thesis, we have considered two kinds of notions; (i) Assignment of real numbers, and (ii) Assignment of subsets of a given set, to the element of a given graph with a variety of constraints.

The concept of set-graceful and topologically set-graceful graphs were introduced by Acharya [1].

**Definition 1.2.1.** [1] Let \( G = (V, E) \) be a graph, \( X \) be a nonempty set and \( 2^X \) denote the set of all subsets of \( X \). A set-indexer of \( G \) is an injective set-valued function \( f : V(G) \to 2^X \) such that the function \( f^\oplus : E(G) \to 2^X - \emptyset \) defined by \( f^\oplus(uv) = f(u) \oplus f(v) \), for every \( uv \in E(G) \) is also injective, where \( \oplus \) denotes the symmetric difference of sets.

**Theorem 1.2.1.** [2] Every graph has a set-indexer.

**Definition 1.2.2.** [1] A graph \( G = (V, E) \) is said to be set-graceful if there exists a set \( X \) and a set-indexer \( f : V(G) \to 2^X \) such that \( f^\oplus(E(G)) = 2^X - \emptyset \)

The following theorem gives a straightforward necessary condition
for a graph $G$ to be set-graceful.

**Theorem 1.2.2.** ([1]) A necessary condition for a graph $G = (V, E)$ to have a set-graceful labeling with respect to a set $X$ of cardinality $n$ is that it be possible to partition $V(G)$ into two subsets $V_e$ and $V_o$ such that the number of edges joining the vertices of $V_o$ with those of $V_e$ is exactly $2^{n-1}$.

**Remark 1.2.1.** If a $(p, q)$-graph is set-graceful then $q = 2^m - 1$ for some positive integer $m$. This implies almost all graphs of order $p$, and hence almost all graphs are not set-graceful. Further, for every positive integer $m$, there exists a set-graceful graph of size $q = 2^m - 1$. However, not all $(p, q)$-graphs with $q = 2^m - 1$ are set-graceful as, for instance, it is not difficult to verify that the path $P_8$ is not set-graceful.

**Theorem 1.2.3.** ([25]) The complete graph $K_n$ is set-graceful if and only if $n \in \{2, 3, 6\}$.

**Theorem 1.2.4.** ([25]) The cycle $C_n$ is set-graceful if and only if $n = 2^m - 1$, for some integer, $m \geq 2$.

The following earlier results are also worth noting.

**Theorem 1.2.5.** ([1]) Every connected set-graceful graph with $q$ edges
and \( q + 1 \) vertices is a tree of order \( p = 2^m \) and for every natural number \( m \), such a tree exists.

**Theorem 1.2.6.** [1] If a tree is set-graceful with respect to a set \( X \) of cardinality \( m \), then its order is \( 2^m \).

In March 1983, Acharya proposed the conjecture that no path with \( 2^m \) vertices is set-graceful. In 1985, Vijayakumar of TIFR, Bombay came up with a proof to disprove this conjecture. Hence the following theorem arises.

**Theorem 1.2.7.** [1] For any integer \( m \geq 2 \), the path \( P_{2^m} \) with \( 2^m \) vertices is not set-graceful.

**Remark 1.2.2.** It is important to note here that not every tree of order \( 2^m \) need be set-graceful.

The following problem is open.

**Problem 1.** Characterize set-graceful graphs.

**Definition 1.2.3.** A caterpillar is a tree having a path that contains at least one vertex of every edge.

**Definition 1.2.4.** A graph \( G \) is said to be bi set-graceful if both \( G \) and its line graph are set-graceful.
Definition 1.2.5. A graph $G$ is said to be set-sequential if there exists a nonempty set $X$ and a bijective set-valued function $f : V(G) \cup E(G) \to 2^X - \emptyset$ such that $f(uv) = f(u) \oplus f(v)$, for every $uv \in E(G)$.

Theorem 1.2.8. If a $(p, q)$-graph is set-sequential, then $p + q = 2^m - 1$, for some positive integer $m$.

For every positive integer $m$, there exists a set-sequential $(p, q)$-graph with $p + q = 2^m - 1$. For instance, take the star $G = K_{1, 2^{m-1} - 1}$ and assign the non-empty subsets of the set $X = \{1, 2, 3, \ldots, m\}$ as follows: Assign $X$ to the central vertex and assign the first $2^{m-1} - 1$ nonempty subsets of $X$ in their natural lexicographic order to the pendant vertices of $G$ in a one-to-one manner. It is easy to verify that this assignment results into a set-sequential labeling of $G$. However the converse of Theorem 1.2.8 is not true as, for instance, the path $P_4$ of length 3 is not set-sequential.

Definition 1.2.6. A topological set-indexer of $G$ is a set-indexer $f : V(G) \to 2^X$ for which $f(V(G))$ is a topology on $X$.

Definition 1.2.7. A set-graceful graph $G = (V, E)$ is topologically set-graceful if the set-indexer on $G$ is a topological set-indexer.

Definition 1.2.8. A topology $\tau$ on a nonempty set $X$ is graceful if there exists a graph $G = (V, E)$ and a set-graceful labeling $f : V \to 2^X$
of $G$ such that $f(V) = \tau$; $G$ is then a realization of $\tau$, denoted $G(\tau)$ as and when found convenient. In particular, \( \tau \) is graceful if $f$ is a graceful set-indexer of some realization of $\tau$.

**Definition 1.2.9.** Two topologies $\tau_1$ and $\tau_2$ on $X$ are said to be isomorphic, if there exists a bijection $f : X \to X$ such that $A \in \tau_1$ if and only if $f(A) \in \tau_2$.

### 1.3 Topology and graph theory

Given a graph $G = (V, E)$, we can relate it to different topological structures. The relation between topology and graph theory is undergone many investigations. In 1967, J.W. Evans et.al [7] proved that there is a one to one correspondence between the set of all topologies with $n$ points and the set of all transitive digraphs with $n$ points. He established the result as follows. Let $V$ be a finite set and $T$ be a topology on $V$. The transitive digraph corresponding to this topology is got by drawing a line from $u$ to $v$ if and only if, $u$ is in every open set containing $v$. Conversely let $D$ be a transitive digraph on $V$. The family $B = \{Q(a) : a \in V\}$ forms a base for a topology on $V$, where $Q(a) = \{a\} \cup \{b \in V : (b, a) \in E(D)\}$. 
In 1968, T.N. Bhargav and T.J. Ahlborn [14] analyzed the topological spaces associated with digraphs. According to them a subset $A$ of $V(D)$ is open if and only if for every pair of points $i, j \in V$ with $j$ in $A$ and $i$ not in $A$, $(i, j)$ is not a line in $D$. Sampathkumar [28] extended this result to the case in which the point set is infinite. Sampathkumar et.al [27] also investigated the topological spaces associated with signed graphs and semigraphs.

Let $S = (V, E, \sigma)$ be a signed graph. A subset $A$ of $V$ is an open set in the positive $E$-topology on $S$ denoted by $\tau^+(S)$ if and only if $u \in A$, $uv \in E^+(S)$ implies that $v \in A$. Similarly, he defined negative $E$-topology $\tau^-(S)$. He defined the topology $\tau_V$ on the vertex set $V(D)$ of a disemigraph $D = (V, E)$ as follows: A subset $S$ of $V(D)$ is open whenever $u \in S$ and $v \in V(D)$ such that $vu$ is a partial arc, then $v \in S$.

In 1983, Acharya [1] established another link between graph theory and point-set topology. He proved that for every graph $G$, there exists a set $X$ and a set-indexer $f : V(G) \rightarrow 2^X$ such that the family $f(V)$ is a topology on $X$.

In 2005, Antoine Vella [30] tried to express combinatorial concepts in topological language. As a part of his investigation he defined the classical topology. Given a hypergraph $H$, the classical topology on $V_H \cup E_H$ is the collection of all sets $U$ such that, if $U$ contains a vertex $v$, then it also contains all hyperedges incident with $v$. 

It is interesting to note that all these topologies are either defined on the vertex set or on the union of vertex set and edge set. Here we study a special case of set-indexed graphs in which both $f(V)$ and $f^\oplus(E) \cup \emptyset$ are topologies on the ground set $X$ of the set-indexer $f$.

1.4 Summary of the thesis

In this thesis, Chapter 1 contains some basic definitions and descriptions about the related works which have been done before. We explain our motivations, discuss the advantages and state the general and specific objectives.

Chapter 2 contains some foundational results on set-valuation of graphs. We define distance compatible set-labeled graphs (dcsl-graphs) as an injective set-assignment $f : V(G) \rightarrow 2^X$, $X$ a nonempty ground set, such that the corresponding induced function $f^\oplus : V(G) \times V(G) \rightarrow 2^X - \emptyset$, defined by $f^\oplus(uv) = f(u) \oplus f(v)$ satisfies $|f^\oplus(uv)| = k_{(u,v)}^f d(u, v)$ for all distinct $u, v \in V(G)$, where $d(u, v)$ is the distance between $u$ and $v$ and $k_{(u,v)}^f$ is a constant, not necessarily an integer and identify the classes of graphs which admit a dcsl. We define dcsl index as the minimum cardinality of the ground set $X$. We establish that every graph has a dcsl-labeling. We define dispersible dcsl-graphs, edge-dispersible dcsl-graphs, $(k, r)$-arithmetic dcsl graphs and $k$-uniform dcsl-graphs.
as special cases of dcsl graphs. We prove that all complete graphs are dispersible and give a characterization of $(k, r)$-arithmetic dcsl complete graphs. We characterize 1-uniform dcsl complete graphs and complete bipartite graphs and prove that all trees admit a 1-uniform dcsl. We find the maximum number of chords to be added to a cycle so that the resulting graph is a 1-uniform dcsl-graph. We also, deals with 1-uniform dcsl index of a graph $G$ and find the 1-uniform dcsl index of paths, stars and trees of diameter less than or equal to three and order less than or equal to six. We also find the 1-uniform dcsl index of even cycles. We prove that paths and even cycles are arbitrarily $k$-uniform dcsl-graphs and establish a necessary condition for an odd cycle to be $k$-uniform dcsl-graph and generalizes this results to all non-bipartite graphs.

We have given the following conjectures and problems in chapter 2 for further scope for investigation. They are listed as below:

Conjecture 1: The star $K_{1,n}$ is dispersive dcsl for all finite values of $n$.

Conjecture 2: 1-uniform dcsl index of an arbitrary tree of order $n$ is $n - 1$.

Problem 1: Characterize dispersible dcsl-graphs.

Problem 2: For any dcsl-graph $G$, the dispersivity $\nu(G)$ of $G$ is the least cardinality of ground set $X$ such that $G$ admits a dispersive dcsl. Find $\nu(K_n)$. 


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Problem 3: Find the minimum number of edges $\rho(m, n)$ to be deleted from a complete bipartite graph $K_{m,n}$ so that the resulting graph is 1-uniform dcsl.

In chapter 3, we mention some of the results of Deza et. al. [18] on $\lambda$ scale embeddable graphs. We establish the relationship between $k$-uniform dcsl graphs and minimum $\lambda$-scale embeddable graphs. We also establish the relationship between 1-uniform dcsl graphs and partial cubes. In the last section we give some of the applications of $k$-uniform dcsl graphs.

In Chapter 4, we define bitopological set-indexer and bitopological graphs. We prove that for every finite set $X$ and, for each topology $\tau$ on $X$, there exist a graph $G = (V, E)$ which is bitopological with $f(V) = \tau$ for a bitopological set-indexer $f$ of $G$. We give a characterization of bitopological complete graphs. We define equi-bitopological graphs and establish some results on equi-bitopological graphs. We define bitopological index $\beta_\tau(G)$ of a finite graph $G$ and find a lower bound for $\beta_\tau(G)$. We prove that path $P_n$, stars $K_1, n$ and complete bipartite graph $K_{m,n}$ are bitopological graphs. We also proved cycle $C_n$ for $n = 3, 4$ are bitopological and for $n = 5, 6$ are not bitopological. We define a set-indexer for triangular book $B_t = K_2 + \overline{K}_t$, which is bitopological. We establish a sufficient condition for the graph $P_3 + \overline{K}_m$ to be bitopological. We depicts the non-isomorphic bitopological spanning
trees of complete graphs $K_n$, for $n \leq 6$ and also find the bitopological index. We define strongly bitopological graphs and establish some results on strongly bitopological graphs. We consider embedding problems of some classes of non-bitopological graphs and discuss about the NP-completeness of the parameters like chromatic number, clique number, independence number and the domination number of bitopological graphs. We have given the following problems and conjectures for further investigation.

Problem 1: Characterize bitopological set-indexed graphs, in particular, characterize bitopological set-indexed trees.

Problem 2: Characterize the transitive digraphs that are bitopological.

Conjecture 1: Cycle $C_n$, $n \geq 5$ is not bitopological.

Conjecture 2: Spanning trees of a complete graph $K_n$, for finite $n$, are bitopological.

Chapter 5 contains some foundational definitions and results on hypergraphs. We extend the concept of 1-uniform dcsl-graphs to hypergraphs. We construct hypergraphs corresponding to dcsl-graphs and study some of the characteristics of these hypergraphs. We find that the hypergraph corresponding to 1-uniform dcsl paths and stars satisfies the coloured edge property. We also prove that the hypergraph corresponding to a 1-uniform dcsl-graph and its dual hypergraph are
isomorphic. We prove that the stability number of a hypergraph corresponding to a \( k \)-uniform dcsl-labeling is the order of \( G \) if and only if \( k = 2 \) and \( G \cong K_n \). We pose the following problems and conjectures for further investigation.

Problem 1: Characterize hypergraphs corresponding to \( k \)-uniform dcsl-graphs.

Problem 2: Characterize hypergraphs corresponding to \((k, r)\)-arithmetic dcsl-graphs.

Problem 3: Characterize \((0, 1)\)-matrix of hypergraphs corresponding to 1-uniform dcsl-graphs.

Problem 4: Find the transversal number and stability number of an optimal hypergraph \( H^f_G \).

Problem 5: Characterize dcsl-graph \( G \) such that \( G \cong L(H^f_G) \).

Conjecture 1: The hypergraphs of dcsl-graphs satisfy the coloured edge property.

In the last chapter we briefly describe some of the major applications of our work and a few topics for further research which emerge from this work, and which, we hope, will be tackled in the future.
1.5 References


31. A. A. Zykov, *On some properties of linear complexes.*, (Russian)