Chapter 2

Tree structure of $G(n, k)$

In this chapter, we determine the existence of two distinct fundamental constituents in which the trees attached to all their cycle vertices are isomorphic. Furthermore, we examine the connection between the symmetric and semiregularity property of $G(n, k)$ with the tree structure. This chapter is based from our work in [27] and some part of [28].

Notation 2.0.8. Throughout the rest of this chapter, let $n_1 > 1$ and $n_2 \geq 1$ be relatively prime integers such that $n = n_1n_2$.

2.1 Some Preliminary Results

Lemma 2.1.1. Let $P$ and $Q$ be sets of all primes dividing $n_1$ and $n_2$, respectively. Let $G_{P' \cup Q'}^*(n, k)$ and $G_Q^*(n, k)$ be fundamental constituents of $G(n, k)$, for any $Q' \subseteq Q$ and $\emptyset \neq P' \subseteq P$.

(a) Suppose that $n_1$ is an odd square-free integer factorized as $n_1 = p_1p_2\ldots p_s$. Let $P' = \{p_1, p_2, \ldots, p_t\}$ for any integer $t$ such that $1 \leq t \leq s$, and $k \equiv 1 \pmod{\lambda(n_1)}$. Then $G_Q^*(n, k)$ consists of $(p_1 - 1)(p_2 - 1)\ldots(p_t - 1)$ subdigraphs of $G(n, k)$, each isomorphic to $G_{P' \cup Q'}^*(n, k)$.

(b) Suppose that $n_1 = p^e$ where $p$ is an odd prime. If $k \equiv 1 \pmod{p-1}$ and $p^{e-1}|k$, then $G_Q^*(n, k)$ consists of $(p-1)$ subdigraphs of $G(n, k)$, each isomorphic to $G_{\{p\} \cup Q'}^*(n, k)$. 12
We prove the first part. From Lemma 1.6.2, we see that every vertex in $G_1(n_1, k)$ is a fixed point. Also by Lemma 1.6.4 we obtain $A_1(G(n_1, k)) = n_1$, which implies that every vertex in $G(n_1, k)$ is an isolated fixed point.

Let $Q' \subseteq Q$ and $\emptyset \neq P' \subseteq P$. Then using (1.7.3) we can write

$$G_{Q'}^*(n, k) \cong G_0^*(n, k) \times G_{Q'}^*(n_2, k),$$

$$G_{P' \cup Q'}^*(n, k) \cong G_P^*(n, k) \times G_{Q'}^*(n_2, k).$$

From (1.7.1) and Lemma 1.6.6, we see that $G_0^*(n_1, k)$ consists of $\phi(n_1)$ isolated fixed points, and $G_P^*(n_1, k)$ consists of $(p_{t+1} - 1)(p_{t+2} - 1)\ldots(p_s - 1)$ isolated fixed points, for any $t$ such that $1 \leq t \leq s$. Let $m = (p_{t+1} - 1)(p_{t+2} - 1)\ldots(p_s - 1)$. Then, we have

$$G_{Q'}^*(n, k) \cong \bigcup_{i=1}^{\phi(n_1)} F_i(n_1, k) \times G_{Q'}^*(n_2, k),$$

$$G_{P' \cup Q'}^*(n, k) \cong \bigcup_{i=1}^{m} E_j(n_1, k) \times G_{Q'}^*(n_2, k),$$

where $F_i(n_1, k)$ and $E_j(n_1, k)$ are components of $G_0^*(n_1, k)$ and $G_P^*(n_1, k)$, respectively, for $i = 1, 2, \ldots, \phi(n_1)$ and $j = 1, 2, \ldots, m$. Also, from Lemma 1.4.9 we get $F_i(n_1, k) \times G_{Q'}^*(n_2, k) \cong E_j(n_1, k) \times G_{Q'}^*(n_2, k)$ for all $i, j$. Hence, $G_{Q'}^*(n, k)$ consists of $(p_{1} - 1)(p_{2} - 1)\ldots(p_t - 1)$ subdigraphs of $G(n, k)$, for any $t$ such that $1 \leq t \leq s$, each subdigraph is isomorphic to $G_{P' \cup Q'}^*(n, k)$.

We now prove the second part. Assume that $k \equiv 1 \pmod{p-1}$ and $p^{-1}|k$.

From Lemma 1.6.2, we see that every cycle in $G_1(n_1, k)$ is a fixed point. Also, using Lemma 1.6.4, we obtain $A_1(G(n_1, k)) = p$, and since the only cycle vertex in $G_2(n_1, k)$ is the fixed point 0, it follows that every cycle in $G(n_1, k)$ is a fixed point. Let $a_1, a_2, \ldots, a_p$ be the fixed points of $G(n_1, k)$. Then by Theorem 1.5.1 and Lemma 1.5.3, it follows that $\text{indeg}_k^p(a_i) = (p^{-1}(p - 1), k) = p^{e-1}$ for $i = 1, 2, \ldots, p$. This implies that $\sum_{i=1}^{p} \text{indeg}_k^p(a_i) = p^e = n_1$, and so the height of each component of $G(n_1, k)$ is 1. Hence, all the components of $G(n_1, k)$ are isomorphic.
Consider $G^*_0(n_1, k)$ and $G^*_1(n_1, k)$, the fundamental constituents of $G(n_1, k)$. So, $G^*_0(n_1, k)$ consists of $p-1$ isomorphic components and $G^*_1(n_1, k)$ consists of only one component. Let $G^*_Q(n_2, k)$ be a fundamental constituent of $G(n_2, k)$, for any $Q' \subseteq Q$. Then by \((1.7.3)\), we obtain

\[
G^*_Q(n, k) \cong G^*_0(n_1, k) \times G^*_Q(n_2, k) \cong \bigcup_{i=1}^{p-1} O_{e_i}^{\phi_{e_i}} \times G^*_Q(n_2, k),
\]

\[
G^*_{Q'}(n, k) \cong G^*_1(n_1, k) \times G^*_{Q'}(n_2, k) \cong O_{e}^{\phi_e} \times G^*_Q(n_2, k).
\]

Hence, by Lemma 1.4.9, it is seen that $G^*_Q(n, k)$ consists of $p-1$ subdigraphs of $G(n, k)$, each isomorphic to $G^*_{Q' \cup \{Q\}}(n, k)$.

**Lemma 2.1.2.** Suppose that the trees attached to all cycle vertices in $G(p^r, k)$ are isomorphic. Then the trees attached to all cycle vertices in $G(p^r, k^r)$, for any positive integer $r$, are also isomorphic.

**Proof.** If $C$ is a component of $G(p^r, k^r)$, then $|C| \leq |D|$ where $D$ is a component of $G(p^r, k)$ and $D$ contains all the vertices of $C$. Note that, $a$ is a cycle vertex of $G(p^r, k)$ if and only if $a$ is also a cycle vertex of $G(p^r, k^r)$ for any positive integer $r$. Let $a_0$ be a fixed point in $G(p^r, k)$ as well as a fixed point in $G(p^r, k^r)$. If $C$ is a component of $G(p^r, k^r)$ with the fixed point $a_0$, then $|C| = |D|$ where $D$ is a component of $G(p^r, k)$ with the fixed point $a_0$ and containing all the vertices of $C$. That is, the components $C$ and $D$ have the same vertices but the edges may be different.

By hypothesis, there exists a digraph isomorphism $\phi$ from $T(p^r, k, 0)$ onto $T(p^r, k^r, 1)$. It is enough to show that $T(p^r, k^r, 0) \cong T(p^r, k^r, 1)$. Let $a$ and $b$ be two vertices in $T(p^r, k^r, 0)$ such that $a^{k^r} \equiv b \pmod{p^r}$. Then there exists vertices $x_1, x_2, \ldots, x_r = b$ such that $a^{k} \equiv x_1 \pmod{p^r}$, $x_1^{k} \equiv x_2 \pmod{p^r}$, $x_r^{k} \equiv b \pmod{p^r}$. Since $T(p^r, k^r, 0)$ and $T(p^r, k, 0)$ have the same vertices, and $\phi$ is an isomorphism, we have $\phi(a)^k \equiv \phi(x_1) \pmod{p^r}$, $\phi(x_1)^k \equiv \phi(x_2) \pmod{p^r}$, $\phi(x_r^k) \equiv \phi(b) \pmod{p^r}$. Thus, $\phi(a)^{k^r} \equiv \phi(b) \pmod{p^r}$ in $T(p^r, k, 1)$ as well as in $T(p^r, k^r, 1)$. Hence, $T(p^r, k^r, 0) \cong T(p^r, k^r, 1)$. \(\square\)
Section 2.1

Some Preliminary Results

Lemma 2.1.3. Let $0 \neq a = p^e c$, where $p \nmid c$, be a vertex in $G_2(p^e,k)$. Then $\text{indeg}_{k^e}^p(a) > 0$ if and only if $k | s$ and $\text{indeg}_{k^e}^{p^{e-s}}(c) > 0$.

Proof. If $\text{indeg}_{k^e}^p(a) > 0$, it is seen from Lemma 1.5.2 that $s = kt$ for some positive integer $t$. Also, there exists a positive integer $m$ such that $p^s m^k \equiv (p^m t)^k \equiv p^{es} c \pmod{p^e}$ which implies that $m^k \equiv c \pmod{p^{e-s}}$.

The converse is straightforward. \hfill \Box

Proposition 2.1.4. Let $t_1, t_2, \ldots, t_m$ be distinct positive integers. There exist integers $n > 1$ and $k > 1$ such that $G(n,k)$ contains a $t_i$-cycle for all $i$.

Proof. Take $t = t_1 t_2 \ldots t_m$, and choose $M = k^t - 1$, $N_i = k^{t_i} - 1$, where $k > 1$, so we can have $\text{ord}_M(k) = t$ and $\text{ord}_{N_i}(k) = t_i$ for all $i$. By Dirichlet’s Theorem on primes in arithmetic progression, we may choose a prime $p$ such that $p \equiv 1 \pmod{M}$. Then it follows immediately from Lemma 1.6.1 that $G(p^e,k)$ contains a $t_i$-cycle for all $i$ and $e \geq 1$. Thus for any positive integer $n$ with $p^e || n$, $G(n,k)$ has a $t_i$-cycle for all $i$. \hfill \Box

Recall, a prime number $p$ is a Sophie Germain prime if $2p + 1$ is also a prime, and a Fermat prime is a prime number of the form $2^{2^n} + 1$ for some non-negative integer $n$.

Proposition 2.1.5. There exist positive integers $t$, $m$, $n$, and $l$ such $A_t(G_1(m,k)) > A_t(G_2(m,k))$, $A_t(G_1(n,k)) < A_t(G_2(n,k))$, and $A_t(G_1(l,k)) = A_t(G_2(l,k))$.

Proof. Let $t$ be a prime and take $M = k^t - 1$. By Proposition 2.1.4, there exists an integer $m = p_1 p_2$, where $p_1$ and $p_2$ are congruent to 1 modulo $M$, such that both $G_1(m,k)$ and $G_2(m,k)$ has a $t$-cycle. We then compute the number of $t$-cycles in $G(m,k)$, $G_1(m,k)$ and $G_2(m,k)$ and obtained

$$A_t(G_2(m,k)) = \frac{2(M - k + 1)}{t} < \frac{M^2 - (k - 1)^2}{t} = A_t(G_1(m,k)).$$

Next, assume $k$ to be odd and take $n = p_1 (2q_1 + 1) (2q_2 + 1)$, where $q_1$, $q_2$ are Sophie Germain primes. The existence of a $t$-cycle follows from Lemma 2.1.4.
Then after some easy computations we get

\[ A_t(G_2(n, k)) = \frac{5(k^t - k)}{t} > \frac{4(k^t - k)}{t} = A_t(G_1(n, k)). \]

If \( k \) is even, we choose \( n = p_1q_2q_3 \), where \( q_3, q_4 \) are Fermat primes, to obtain

\[ A_t(G_2(n, k)) = \frac{3(k^t - k)}{t} > \frac{(k^t - k)}{t} = A_t(G_1(n, k)). \]

Finally, let \( l = 2p_1 \) and we have \( A_t(G_1(l, k)) = \frac{(k^t - k)}{t} = A_t(G_2(l, k)). \)

**Lemma 2.1.6** (Lemma 3.8, [12]). Let \( \lambda(p^e) = uv \) where \( u \) is the largest divisor relatively prime to \( u \). Then we have \( h(T(p^e, k, 1)) = \min\{i : v|k^i\} \).

**Lemma 2.1.7** (Lemma 3.7, [12]). If \( h \) is the unique positive integer such that \( k^{h-1} < e \leq k^h \), then \( h(T(p^e, k, 0)) = h \).

**Lemma 2.1.8.** Let \( a \neq 0 \) be a vertex in \( G_2(p^e, k) \) such that \( p^{\alpha k^i} || a \), where \( \gcd(k, \alpha) = 1 \), for some positive integer \( i \). Then \( a \) is at height \( h \) if and only if \( h \) is the least positive integer such that \( \alpha k^{h+i-1} < e \leq \alpha k^{h+i} \).

**Proof.** The proof is straightforward. \( \square \)

### 2.2 Trees in Fundamental Constituents

**Theorem 2.2.1.** Let \( l = m_1m_2 \), where \( m_1 \) and \( m_2 \) are relatively prime integers. Let \( G^*_{P_1}(m_1, k) \) and \( G^*_{P_2}(m_1, k) \) be two distinct fundamental constituents of \( G(m_1, k) \) such that the trees attached to all their cycle vertices are isomorphic. Let \( G^*_Q(m_2, k) \) be any fundamental constituent of \( G(m_2, k) \). Then the trees attached to all cycle vertices in the fundamental constituents \( G^*_{Q,P_1}(l, k) \) and \( G^*_{Q,P_2}(l, k) \) of \( G(l, k) \) are isomorphic.

**Proof.** From (1.7.3) we have

\[
G^*_{P_1 \cup Q}(l, k) \cong G^*_{P_1}(m_1, k) \times G^*_Q(m_2, k),
\]

\[
G^*_{P_2 \cup Q}(l, k) \cong G^*_{P_2}(m_1, k) \times G^*_Q(m_2, k).
\]
Let $a_1, a_2, \ldots, a_t$ be the cycle vertices of $G^*_1(m_1, k)$, $b_1, b_2, \ldots, b_s$ be the cycle vertices of $G^*_2(m_1, k)$, and $c_1, c_2, \ldots, c_u$ be the cycle vertices of $G^*_Q(m_2, k)$, for any positive integers $t$, $s$, and $u$. Then by Lemma 1.4.1, the cycle vertices of $G^*_1 \cup Q(l, k)$ and $G^*_2 \cup Q(l, k)$ are of the form $(a_i, c_j)$ and $(b_x, c_j)$, respectively, for all $i$, $j$, $x$ such that $1 \leq i \leq t$, $1 \leq j \leq u$, and $1 \leq x \leq s$. It now suffices to show that the trees attached to cycle vertices $(a_i, c_j)$ and $(b_x, c_j)$ are isomorphic for all $i$, $j$, $x$. In view of Theorem 1.7.3, it is enough to prove that $T(l, k, (a_1, c_1))$ is isomorphic to $T(l, k, (b_1, c_1))$. By hypothesis, there exists a digraph isomorphism $\phi_{ij}$ from $T(m_1, k, a_i)$ onto $T(m_1, k, b_j)$ for all $i$, $j$ such that $1 \leq i \leq t$, $1 \leq j \leq s$. Note that $\phi_{ij}$ maps a vertex at height $h$ in $T(m_1, k, a_i)$ to a vertex at the same height in $T(m_1, k, b_j)$. Now we define a map $F$ from $T(m_1 m_2, k, (a_1, c_1))$ into $T(m_1 m_2, k, (b_1, c_1))$ as $F((u, v)) = (\phi_{11}(u), v)$ for every vertex $(u, v)$ in $T(m_1 m_2, k, (a_1, c_1))$. We first show that $F$ is well-defined. If $(u, v)$ is a cycle vertex in $T(m_1 m_2, k, (a_1, c_1))$, then $F((u, v)) = F((a_1, c_1)) = (\phi_{11}(a_1), c_1) = (b_1, c_1)$, which is a cycle vertex in $T(m_1 m_2, k, (b_1, c_1))$. Suppose that $(u, v)$ is a vertex which is at height $h \geq 1$ in $T(m_1 m_2, k, (a_1, c_1))$. Then $h$ is the least positive integer such that $(u, v)^{k_h} = (a_1, c_1)$. It follows that $u$ is at height $h$ in $T(m_1, k, a_1)$ or $v$ is at height $h$ in $T(m_2, k, c_1)$. If one of $u$ or $v$ is at height $h$, then the other is at height $i$ such that $i \leq h$. Since $\phi_{11}$ is a digraph isomorphism, then

$$[F((u, v))]^{k_h} = (\phi_{11}(u), v)^{k_h} = (\phi_{11}(u)^{k_h}, v^{k_h}) = (\phi_{11}(u^{k_h}), v^{k_h}) = (\phi_{11}(a_1), c_1) = (b_1, c_1).$$

If $1 \leq i < h$, then it follows from Lemma 1.4.1 that $[F((u, v))]^{k_i} = (\phi_{11}(u^{k_i}), v^{k_i})$ is not a cycle vertex in $T(m_1 m_2, k, (b_1, c_1))$. Hence, $F$ maps a vertex at height $h$ in $T(m_1 m_2, k, (a_1, c_1))$ to a vertex at the same height in $T(m_1 m_2, k, (b_1, c_1))$. We now show that $F$ is a digraph isomorphism. The map $F$ is obviously one-one. To show that $F$ is onto, first we note that, $(b_1, c_1) = (\phi_{11}(a_1), c_1) = F((a_1, c_1))$. Let $(u, v)$ be a vertex at height $h \geq 1$ in $T(m_1 m_2, k, (b_1, c_1))$. Assume that $u$ is at height $h$ in $T(m_1, k, b_1)$ and $v$ is at height $i$ such that $i \leq h$ in
$T(m_2, k, c_1)$. Since $\phi_{11}$ is a digraph isomorphism, there exists a vertex $w$ at 
hight $h$ in $T(m_1, k, a_1)$ such that $\phi_{11}(w) = u$. Then $(w, v)$ is at height $h$ in 
$T(m_1m_2, k, (a_1, c_1))$ and $F((w, v)) = (\phi_{11}(w), v) = (u, v)$. Similarly, if $v$ is at 
hight $h$ in $T(m_2, k, c_1)$, then $F((w, v)) = (\phi_{11}(w), v) = (u, v)$ where $w$ is at 
hight $i$ such that $i \leq h$ in $T(m_1, k, a_1)$. Hence, $F$ is onto.

Finally, we show that $F$ preserves direction. Let $(u_1, v_1)$ and $(u_2, v_2)$ be two 
non-cycle vertices in $T(m_1m_2, k, (a_1, c_1))$. Suppose there exists a directed edge 
from $(u_1, v_1)$ to $(u_2, v_2)$. Since $\phi_{11}$ is edge-preserving, we have

$$[F((u_1, v_1))]^k = (\phi_{11}(u_1), v_1)^k = (\phi_{11}(u_1)^k, v_1^k) = (\phi_{11}(u_1^k), v_1^k) = (\phi_{11}(u_2), v_2).$$

**Lemma 2.2.2.** If the trees attached to all cycle vertices in $G(2^e, k)$ are iso-
morphic, then $k$ must be an even integer.

**Proof.** Suppose that the trees attached to all cycle vertices in $G(2^e, k)$ are isomorphic. Since $|T(2^e, k, 1)| = |T(2^e, k, 0)| = 2^{e-1}$ then it follows that 0 and 
1 are the only cycle vertices in $G(2^e, k)$. Now, we observe that $A_1(G(2^e, k)) = \gcd(\lambda(2^e), k - 1) + 1$. If $k$ is odd then $A_1(G(2^e, k)) \geq 3$, which is a contradiction. 
Hence, $k$ must be even. 

**Theorem 2.2.3.** Suppose that $p$ is an odd prime. Then the trees attached to all 
cycle vertices in $G(p^e, k)$ are isomorphic if and only if $\gcd(p^{e-1}(p-1), k) = p^{e-1}$.

**Proof.** Let $h = h(G(p^e, k))$. Then we can write

$$G(p^e, k^h) = G_2(p^e, k^h) \cup G_1(p^e, k^h) = O_1^{p^{e-1}} \cup a_1O_1^m \cup a_2O_2^m \cup \cdots \cup a_tO_t^m,$$

where $a_i = A_{t_i}(G_1(p^e, k^h))$ for all $t_i \in A(G_1(p^e, k^h))$, and $m = \gcd(p^{e-1}(p - 1), k^h)$. If we assume that $\gcd(p - 1, k) = d > 1$ or $p \not| k$ when $e > 1$, then the 
tree $T(p^e, k^h, 1)$ cannot be isomorphic to $T(p^e, k^h, 0)$. In other words, the trees 
attached to cycle vertices in $G_1(p^e, k^h)$ and $G_2(p^e, k^h)$ are not isomorphic. From
Lemma 2.1.2, it follows that the trees attached to cycle vertices in $G_1(p^e, k)$ and $G_2(p^e, k)$ are also not isomorphic. Hence we must have $\gcd(p - 1, k) = 1$ and $p \mid k$ whenever $e > 1$.

Now we assume that $p^r \mid | k$ for some positive integer $r$. From the hypothesis, we can take $h(T(p^e, k, 0)) = h(T(p^e, k, 1)) = h_0$, say. Then from Lemmas 2.1.6 and 2.1.7 we have $k^{h_0} - 1 < e \leq k^{h_0}$ and $p^{e-1} \mid k^{h_0}$, which implies that $r(h_0 - 1) < e - 1 \leq rh_0$. Thus, $p^{r(h_0-1)} \leq k^{h_0} - 1 \leq e - 1 \leq rh_0$ which forces $h_0 = 1$. This means that the height of every component of $G(p^e, k)$ is 1, and we can write

$$G(p^e, k) = O_1^{p^{e-1}} \cup a_1 O_{t_1}^m \cup a_2 O_{t_2}^m \cup \cdots \cup a_t O_{t_t}^m,$$

where $a_i = A_{t_i}(G_1(p^e, k))$ for all $t_i \in A(G_1(p^e, k))$. In particular, $\text{indeg}_{G(p^e, k)}(a) = p^r$ if $a$ is a cycle vertex in $G_1(p^e, k)$, and $\text{indeg}_{G(p^e, k)}(a) = 0$ otherwise. By hypothesis, we get $m = p^r = p^{e-1}$ which implies $r = e - 1$. Hence, the result follows.

Now we prove the converse. Assume that $\gcd(p^{e-1}(p - 1), k) = p^{e-1}$. Then the indegree of every vertex in $G(p^e, k)$ is either 0 or $p^{e-1}$, and in particular, the indegree of all cycle vertices in $G(p^e, k)$ is $p^{e-1}$. From Lemma 1.6.7 we observe that the number of cycle vertices in $G_1(p^e, k)$ is $p - 1$, and so the number of cycle vertices in $G(p^e, k)$ must be $p$. This implies that the height of every component in $G(p^e, k)$ is 1, and the proof is complete.

**Proposition 2.2.4.** Let $n = p_1 p_2 \ldots p_r$, where the $p_i$’s are distinct odd primes. Suppose that for any fundamental constituent $G_p^*(n, k)$ of $G(n, k)$, there exists a distinct fundamental constituent $G_Q^*(n, k)$ such that the trees attached to all cycle vertices in $G_p^*(n, k) \cup G_Q^*(n, k)$ are isomorphic. Then $G(p_i, k)$ consists of only cycles for at least one $i$ such that $1 \leq i \leq r$, and conversely. Moreover, if $G(p_i, k)$ consists of only cycles for all $i$ then the trees attached to all cycle vertices in $G(n, k)$ are isomorphic.

**Proof.** From Lemma 1.6.6, we see that 0 is an isolated fixed point of $G(p_i, k)$ for all $i$. Suppose that every $G(p_i, k)$ contains some nontrivial trees. Then using Lemma 1.7.3, we see that the indegree of all non-zero cycle vertices in $G(p_i, k)$,
for all $i$, is greater than 1. If $a = (a_1, a_2, \ldots, a_r)$ is a cycle vertex in $G(n, k)$, where each $a_i$ is a cycle vertex of $G(p_i, k)$, then we recall from (1.5.1) that

$$\text{indeg}_k^n(a) = \prod_{i=1}^{r} \text{indeg}_k^n(a_i).$$

Thus, $\text{indeg}_k^n(a) > 1$, unless $a = (0, 0, \ldots, 0)$. This means that the only cycle vertex in $G(n, k)$ with indegree 1 is the fixed point 0. Since $G_{\{p_1, p_2, \ldots, p_r\}}(n, k)$ is the only fundamental constituent of $G(n, k)$ with trivial trees, the result follows.

Conversely, assume that $G(p_i, k)$ consists of only cycles for at least one $i$ such that $1 \leq i \leq r$. Then the trees attached to all cycle vertices in the fundamental constituents $G_{\emptyset}(p_i, k)$ and $G_{\{p_i\}}(p_i, k)$ are isomorphic, in fact the trees are trivial. Let $G^*_Q(n, k)$ be any fundamental constituent of $G(n, k)$, and let $m_i = p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_r$. Then by (1.7.3), we have

$$G^*_Q(n, k) \cong G^*_Q(p_i, k) \times G^*_Q(m_i, k),$$

where $Q_1 = \{p \in Q : p|m_i\}$, $Q_2 = \{p \in Q : p|m_i\}$. So there exists a fundamental constituent $G^*_{Q_1}(p_i, k)$ of $G(p_i, k)$ such that the trees attached to all cycle vertices in $G^*_{Q_1}(p_i, k) \cup G^*_{Q_2}(p_i, k)$ are isomorphic. By Theorem 2.2.1, the trees attached to all cycle vertices in $G^*_{Q}(n, k) \cup G^*_{Q_1 \cup Q_2}(n, k)$ are isomorphic.

The second part follows directly from Lemma 1.4.1. \hfill \Box

**Theorem 2.2.5.** (a) Let $n_1 = p_1 p_2 \ldots p_r$, where the $p_i$’s are distinct odd primes and $r \geq 2$. Suppose that $\gcd(\lambda(n), k) = 1$.

(b) Let $n_1 = p^e$, where $p$ is an odd prime and $\gcd(p^{e-1}(p - 1), k) = p^{e-1}$.

(c) Let $n_1 = 2^e$. Suppose that $G(n_1, k)$ satisfies the following conditions:

(i) $e = 5$, $k = 4$.

(ii) $e = 4$, $k = 2$.

(iii) $e \leq 2$, $2^{e-1}|k$.

(iv) $e \leq 2$, $k > 2$, $2^{e-2}|k$.

Then for any fundamental constituent $G^*_p(n, k)$ of $G(n, k)$, there exists a distinct fundamental constituent $G^*_Q(n, k)$ such that the trees attached to all cycle vertices in $G^*_p(n, k) \cup G^*_Q(n, k)$ are isomorphic.

**Proof.** We first show that the trees attached to all cycle vertices in $G(n_1, k)$ are isomorphic. Cases (a) and (b) follow from Lemma 1.6.3 and Theorem 2.2.3, respectively. Now we consider case (c). For parts (i) and (ii), we see from
Figure 2.2 and Figure 2.1, respectively, that $G(16,2)$ and $G(32,4)$ has exactly two isomorphic components, one with the fixed point 0 and the other with the fixed point 1. We now prove parts (iii) and (iv). By Lemma 1.6.2, every cycle of $G_1(2^e, k)$ is a fixed point. Also, the fixed point 0 is the only cycle in $G_2(2^e, k)$. Since $A_1(G(2^e, k)) = 2$, the only cycles in $G(2^e, k)$ are the fixed points 0 and 1. By Theorem 1.5.1 and Lemma 1.5.3, we get $\text{indeg}_k^1 \equiv 1 = \text{indeg}_k^0 = 2^{e-1}$. Thus, $G(2^e, k)$ has exactly two isomorphic components, one component containing the fixed point 0 and the other containing the fixed point 1.

To finish the proof, in all three cases we use Theorem 2.2.1 and equation (1.7.3). Let $G^*_p(n, k)$ be any fundamental constituent of $G(n, k)$. From (1.7.3), we obtain $G^*_p(n, k) \cong G^*_p(n_1, k) \times G^*_p(n_2, k)$, where $P_1 = \{ p \in P : p|n_1 \}$, $P_2 = \{ q \in P : q|n_2 \}$. Then there exists a fundamental constituent $G^*_{Q_1}(n_1, k)$ of $G(n_1, k)$ such that the trees attached to all cycle vertices in $G^*_{Q_1}(n_1, k) \cup G^*_{Q_2}(n_1, k)$ are isomorphic. Now, using equation (1.7.1), consider the fundamental constituent $G^*_{Q_1}(n_1, k) \times G^*_{Q_2}(n_2, k) \cong G^*_{Q_1 \cup Q_2}(n, k)$. Hence, by Theorem 2.2.1, the trees attached to all cycle vertices in $G^*_p(n, k) \cup G^*_{Q_1 \cup Q_2}(n, k)$ are isomorphic.

\[ \square \]

**Corollary 2.2.6.** Let $n = 3p_1^{e_1}p_2^{e_2} \ldots p_r^{e_r}$, where $p_i \neq 3$ are distinct odd primes. Suppose that $k > 2$ is an odd integer. Then for any fundamental constituent $G^*_p(n, k)$ of $G(n, k)$, there exists a distinct fundamental constituent $G^*_Q(n, k)$ such that the trees attached to all cycle vertices in $G^*_p(n, k) \cup G^*_Q(n, k)$ are isomorphic.
Theorem 2.2.7. Let $n$ be an odd integer factorized as $(1.3.1)$. The trees attached to all cycle vertices in $G(n,k)$ are isomorphic if and only if $\gcd(p_i^{e_i-1}(p_i-1), k) = p_i^{e_i-1}$ for all $i$ such that $1 \leq i \leq r$.

Proof. Suppose we have $\gcd(p_i^{e_i-1}(p_i-1), k) = m \neq p_i^{e_i-1}$ for some $i$ such that $1 \leq i \leq r$. Our aim is to show that $\text{indeg}_{G_k}^{p_i^{e_i}}(0) \neq \text{indeg}_{G_k}^{p_i^{e_i}}(1)$. If $p_i \nmid m$, then we are done. So we consider the case when $p_i | m$. By Theorem 1.5.1, we have $\text{indeg}_{G_k}^{p_i^{e_i}}(a) = 0$ or $m$ for any vertex $a$ in $G_1(p_i^{e_i}, k)$. Suppose that $\text{indeg}_{G_k}^{p_i^{e_i}}(0) = \text{indeg}_{G_k}^{p_i^{e_i}}(1)$, then $m = p_i^{e_i-\lceil e_i/k \rceil} = \gcd(p_i^{e_i-1}(p_i-1), k)$. This implies that $k < e_i$. If $G_2(p_i^{e_i}, k)$ is semiregular, then by Theorem 1.8.7, we get a contradiction. Now consider the case when $G_2(p_i^{e_i}, k)$ is not semiregular. By Theorem 4.5.5, we have $e_i \geq k + e_i - \lceil e_i/k \rceil + 2$, and since $e_i - \lceil e_i/k \rceil + 2 \leq p_i^{e_i-\lceil e_i/k \rceil} \leq k$ for any odd prime $p$, we obtain $e_i \geq 2e_i + 4 - 2\lceil e_i/k \rceil$ which is again a contradiction. Thus we can conclude that, if $\gcd(p_i^{e_i-1}(p_i-1), k) \neq p_i^{e_i-1}$ then $\text{indeg}_{G_k}^{p_i^{e_i}}(0) \neq \text{indeg}_{G_k}^{p_i^{e_i}}(1)$. Let $m_i = p_i^{e_1} p_i^{e_2} \cdots p_i^{e_{i-1}} p_i^{e_{i+1}} \cdots p_i^{e_r}$. Then $\text{indeg}_{G_k}^{p_i^{e_i}}(0) \text{indeg}_{G_k}^{m_i}(1) \neq \text{indeg}_{G_k}^{p_i^{e_i}}(1) \text{indeg}_{G_k}^{m_i}(1)$, which by (1.5.1) implies that $\text{indeg}_{G_k}^{p_i^{e_i}}((0,1)) \neq \text{indeg}_{G_k}^{p_i^{e_i}}((1,1))$. Hence, the trees attached to the cycle vertices $(0,1)$ and $(1,1)$ in $G(n,k)$ are not isomorphic.

We now prove the converse. Assume that $\gcd(\lambda(p_i^{e_i}), k) = p_i^{e_i-1}$ for all $i$ such that $1 \leq i \leq r$. Then by Theorem 2.2.3, the trees attached to all cycle vertices in $G(p_i^{e_i}, k)$, for all $i$, are isomorphic. By similar arguments as in the proof of Theorem 2.2.3, we see that the height of each component of $G(p_i^{e_i}, k)$, for all $i$, is 1. Let $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$ be two cycle vertices in $G(n,k)$, and $\phi_i$ is a digraph isomorphism from $T(p_i^{e_i}, k; a_i)$ onto $T(p_i^{e_i}, k; b_i)$ for all $i$ such that $1 \leq i \leq r$. It is enough to show that $T(n,k,a) \cong T(n,k,b)$. For any vertex $u = (u_1, \ldots, u_r) \in T(n,k,a)$, define a map $F : T(n,k,a) \longrightarrow T(n,k,b)$ as $F((u_1, u_2, \ldots, u_r)) = (\phi_1(u_1), \phi_2(u_2), \ldots, \phi_r(u_r))$. If $u = (u_1, u_2, \ldots, u_r)$ is a cycle vertex, then $F(((u_1, u_2, \ldots, u_r)) = (b_1, b_2, \ldots, b_r)$. Suppose that the vertex $u = (u_1, u_2, \ldots, u_r)$ is at height 1 in $T(n,k,a)$. Since $\phi_i$ is an isomorphism, then

$$
[F((u_1, u_2, \ldots, u_r)])^k = (\phi_1(u_1), \ldots, \phi_r(u_r))^k = (\phi_1(u_1^k), \ldots, \phi_r(u_r^k)) = (\phi_1(a_1), \ldots, \phi_r(a_r)) = (b_1, \ldots, b_r).
$$
Thus, $F$ is well-defined. Since the $\phi_i$'s are one-one and onto, it is clear that $F$ is also one-one and onto. Finally, if $u = (u_1, \ldots, u_r)$ is a vertex at height 1 in $T(n, k, a)$, we obtain $[F((u_1, u_2, \ldots, u_r))]^k = (\phi_1(u_1), \ldots, \phi_r(u_r))^k = (\phi_1(u_1^k), \ldots, \phi_r(u_r^k)) = (\phi_1(a_1), \ldots, \phi_r(a_r)) = F(a).

\[ \Box \]

Note 2.2.8. The arguments of the proof of the ‘$\Rightarrow$’ part of Theorem 2.2.7 will also work to prove the ‘$\Rightarrow$’ part of Theorem 2.2.3.

Theorem 2.2.9. The trees attached to all cycle vertices in $G_2(n, k)$ are isomorphic if and only if the trees attached to all cycle vertices in $G(n, k)$ are isomorphic.

2.3 Symmetry

Lemma 2.3.1. Let $n = pq_1q_2 \ldots q_r$, where $p$ and the $q_i$’s are distinct odd primes, such that $\gcd(p - 1, k) = \gcd(q_i - 1, k) = 1$ for all $i$. Suppose that $G(n, k)$ is symmetric of order $p$ and $G(p, k)$ is not symmetric of order $p$. Then $p \mid A_1(G(q_1q_2 \ldots q_r, k))$.

Proof. The proof follows from Theorems 1.8.2 and 1.8.3. \[ \Box \]

Lemma 2.3.2. A digraph $G(2^e, k)$ is symmetric of order 2 if and only if the trees attached to all cycle vertices in $G(2^e, k)$ are isomorphic.

Proof. If $G(2^e, k)$ is symmetric of order 2, or if the trees attached to all cycle vertices in $G(2^e, k)$ are isomorphic, then $G(2^e, k)$ has exactly two isomorphic components, one containing the fixed point 0 and the other containing the fixed point 1. The assertion then follows immediately. \[ \Box \]

Lemma 2.3.3. Suppose that $G(p^e, k)$ is symmetric of order $p$, where $p$ is an odd prime. Then the trees attached to all cycle vertices in $G(p^e, k)$ are isomorphic.

Proof. If $G(p^e, k)$ is symmetric of order $p$ then from Theorem 1.8.3 we obtain $\gcd(p^{e-1}(p - 1), k) = p^{e-1}$. The assertion then follows from Theorem 2.2.7. \[ \Box \]
The converse of Lemma 2.3.3 does not hold in general. For example, the
tress attached to all cycle vertices in $G(49, 35)$ are isomorphic, but $G(49, 35)$
is not symmetric (see Figure 2.3). However, if $k \equiv 1 \pmod{p-1}$ then the
converse is true. This follows from Theorems 2.2.3 and 1.8.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{G4935.png}
\includegraphics[width=0.4\textwidth]{G3511.png}
\caption{G(49,35) \quad G(35,11)}
\end{figure}

**Lemma 2.3.4.** Let $n = p_1p_2 \ldots p_r$, where the $p_i$'s are distinct odd primes. Suppose that $G(n, k)$ is symmetric of order $n$. Then the trees attached to all cycle vertices in $G(n, k)$ are isomorphic

The converse of Lemma 2.3.4 is not always true. For example, the digraph $G(35, 11)$ have trivial trees but is not symmetric (see Figure 2.4).

**Proposition 2.3.5.** Let $n = pq_1q_2 \ldots q_r$, where $p$ and the $q_i$'s are distinct odd primes, such that $\gcd(p - 1, k) = \gcd(q_i - 1, k) = 1$ for all $i$. Consider the fundamental constituent $G_Q^*(q_1q_2 \ldots q_r, k)$, where $Q \subseteq \{q_1, q_2, \ldots, q_r\}$. Suppose that $G(p, k)$ is symmetric of order $p$. Then $G_Q^*(n, k) \cup G_{Q \cup \{p\}}^*(n, k)$, which is a subdigraph of $G(n, k)$, is symmetric of order $p$.

**Proof.** Since $G(p, k)$ is symmetric of order $p$, then from Theorem 1.8.3 we see that $k \equiv 1 \pmod{p - 1}$. If $C$ is a component of $G_Q^*(n, k)$, then by Lemma 2.1.1(ii), there exists distinct components $C_1, C_2, \ldots, C_{p-2}$ of $G_Q^*(n, k)$ and one component $C_{p-1}$ of $G_{Q \cup \{p\}}^*(n, k)$ such that each $C_i$ is isomorphic to $C$, for $i = 1, 2, \ldots, p-1$. Similarly, it is the case if we take $C$ to be a component of $G_{Q \cup \{p\}}^*(n, k)$. Hence, $G_Q^*(n, k) \cup G_{Q \cup \{p\}}^*(n, k)$ is symmetric of order $p$. \hfill \Box

Proposition 2.3.5 does not hold when $G(p, k)$ is not symmetric of order $p$, even though $G(n, k)$ is symmetric of order $p$. Consider the following example:
Example 2.3.6. Consider the digraphs $G(5, 15)$ and $G(29, 15)$. Note that both $G(5, 15)$ and $G(29, 15)$ consist of only cycles. By Theorem 1.8.3, $G(5, 15)$ is not symmetric of order 5. Now, $A_1(G(5, 15)) = 3$, $A_2(G(5, 15)) = 1$, $A_1(G(29, 15)) = 15$, and $A_2(G(29, 15)) = 7$. Also, $A_1(G(5 \times 29, 15)) = 45$ and $A_2(G(5 \times 29, 15)) = 50$. Hence, $(G(5 \times 29, 15))$ is symmetric of order 5.

Consider the fundamental constituents $G_0^*(145, 15)$, $G_{\{5\}}^*(145, 15)$, $G_{\{29\}}^*(145, 15)$ and $G_{\{5,29\}}^*(145, 15)$ of $G(145, 15)$. By (1.7.3), we have

\[
G_0^*(145, 15) \cong G_0^*(5, 15) \times G_0^*(29, 15), \\
G_{\{5\}}^*(145, 15) \cong G_{\{5\}}^*(5, 15) \times G_0^*(29, 15), \\
G_{\{29\}}^*(145, 15) \cong G_0^*(5, 15) \times G_{\{29\}}^*(29, 15), \\
G_{\{5,29\}}^*(145, 15) \cong G_{\{5\}}^*(5, 15) \times G_{\{29\}}^*(29, 15).
\]

Note that $G_0^*(5, 15)$ consists of one 2-cycle and 2 isolated fixed points, and $G_0^*(29, 15)$ consists of 7 2-cycles and 14 isolated fixed points. Then by Lemma 1.4.3 and Theorem 1.4.4, $G_0^*\{145, 15\}$ consists of $14 + 14 + 14 = 42$ number of 2-cycles and 28 isolated fixed points. Similarly, $G_{\{5\}}^*(145, 15)$ consists of 7 number of 2-cycles and 14 isolated fixed points, $G_{\{29\}}^*(145, 15)$ consists of one 2-cycle and 2 isolated fixed points, and $G_{\{5,29\}}^*(145, 15)$ consists of only 1 isolated fixed point. Therefore, the subdigraph $G_P^*(145, 15) \cup G_Q^*(145, 15)$ is not symmetric of order 5, for any $P, Q \subseteq \{5, 29\}$, $P \neq Q$.

Theorem 2.3.7. Let $n_1 = p^e$, where $p$ is an odd prime, such that $G(n_1, k)$ is symmetric of order $p$. Then $G_Q^*(n, k) \cup G_{Q,\{p\}}^*(n, k)$, which is a subdigraph of $G(n, k)$, is symmetric of order $p$.

Proof. By Theorem 2.2.5, the trees attached to all cycle vertices in $G_Q^*(n, k) \cup G_{Q,\{p\}}^*(n, k)$ are isomorphic. Since $G(n_1, k)$ is symmetric of order $p$, then from Theorem 1.8.3 we get $k \equiv 1 \pmod{p-1}$. If $C$ is a component of $G_{Q,\{p\}}^*(n, k)$, then by Lemma 2.1.1(ii), there exists $p - 1$ distinct components of $G_Q^*(n, k)$, each isomorphic to $C$. Similarly, it is the case when $C$ is a component of $G_Q^*(n, k)$. Hence, $G_Q^*(n, k) \cup G_{Q,\{p\}}^*(n, k)$ is symmetric of order $p$. 

\[\square\]
Theorem 2.3.8. Let $J(n_i, k)$ be subdigraphs of $G(n_i, k)$, for $i = 1, 2$, such that $J(n_1, k)$ consists of $M$ isomorphic components, and $J(n_2, k)$ consists of $N$ isomorphic components. Then $J(n_1, k) \times J(n_2, k)$ is a subdigraph of $G(n, k)$ that is symmetric of order $MN$.

Proof. Let $C_i(n_1, k)$, where $i = 1, 2, \ldots, M$, be $M$ isomorphic components of $J(n_1, k)$, and let $D_j(n_2, k)$, where $j = 1, 2, \ldots, N$, be $N$ isomorphic components of $J(n_2, k)$. Since all the $C_i(n_1, k)$'s are isomorphic, each cycle in $C_i(n_1, k)$ is a $t_1$-cycle for some positive integer $t_1$. Let the $M t_1$-cycles in $J(n_1, k)$ be $<a_1, \ldots, a_{t_1}>, <a_{t_1+1}, \ldots, a_{t_1'}>, \ldots, <a_{(M-1)t_1+1}, \ldots, a_{Mt_1}>$. Similarly, each cycle in $D_j(n_2, k)$, for $j = 1, 2, \ldots, N$, is a $t_2$-cycle for some positive integer $t_2$. Let the $N t_2$-cycles of $J(n_2, k)$ be $<b_1, \ldots, b_{t_2}>, <b_{t_2+1}, \ldots, b_{t_2'}>, \ldots, <b_{(N-1)t_2+1}, \ldots, b_{Nt_2}>$. From Theorem 1.4.4, we see that

$$J(n_1, k) \times D_j(n_2, k) = \bigcup_{i=1}^{M} C_i(n_1, k) \times D_j(n_2, k),$$

for each $j$ such that $1 \leq j \leq N$. Thus it follows from Lemma 1.4.3 that there are $M$ disjoint subdigraphs in $J(n_1, k) \times D_j(n_2, k)$, for each $j$ such that $1 \leq j \leq N$, each subdigraph containing $\gcd(t_1, t_2)$ components. We now show that these $M$ subdigraphs are all isomorphic.

For each $j$, it suffices to prove that

$$C_i(n_1, k) \times D_j(n_2, k) \cong C_l(n_1, k) \times D_j(n_2, k)$$

for all positive integers $i, l$ such that $1 \leq i, l \leq M$. By hypothesis, there exists a digraph isomorphism $\phi_{il}$ from $C_i(n_1, k)$ onto $C_l(n_1, k)$, for all $i, l$ such that $1 \leq i, l \leq M$. Then it is clear that for each $j$, the map $F_{il} : C_i(n_1, k) \times D_j(n_2, k) \rightarrow C_l(n_1, k) \times D_j(n_2, k)$ defined by $F_{il}((u, v)) = (\phi_{il}(u), v)$, for any vertex $(u, v) \in C_i(n_1, k) \times D_j(n_2, k)$, is a digraph isomorphism, for all $i, l$ such that $1 \leq i, l \leq M$. Again, by hypothesis, there exists a digraph isomorphism $\psi_{ij}$ from $D_i(n_2, k)$ onto $D_j(n_2, k)$, for all $i, j$ such that $1 \leq i, j \leq N$. Define a map $F'_{ij} : J(n_1, k) \times D_i(n_2, k) \rightarrow J(n_1, k) \times D_j(n_2, k)$ as $F'_{ij}((u, v)) = (u, \psi_{ij}(v))$.
Section 2.4 Semiregularity

for any vertex \((u, v) \in J(n_1, k) \times D_i(n_2, k)\). It is clear that \(F'_{ij}\) is a digraph isomorphism from \(J(n_1, k) \times D_i(n_2, k)\) onto \(J(n_1, k) \times D_j(n_2, k)\), for all \(i, j\) such that \(1 \leq i, j \leq N\).

Then, \(J(n_1, k) \times J(n_2, k)\) consists of \(MN\) isomorphic subdigraphs, each containing \(\gcd(t_1, t_2)\) components. Hence, \(J(n_1, k) \times J(n_2, k)\) contains \(\gcd(t_1, t_2)\) subdigraphs, each containing \(MN\) isomorphic components. This implies that \(J(n_1, k) \times J(n_2, k)\) is symmetric of order \(MN\).

**Theorem 2.3.9.** Suppose that \(G(n_1, k)\) is symmetric of order \(M\) and \(G(n_2, k)\) is symmetric of order \(N\). Then \(G(n_1, k) \times G(n_2, k)\) is symmetric of order \(MN\).

*Proof.* The proof follows immediately from Theorems 1.4.4 and 2.3.8. 

### 2.4 Semiregularity

We know that the subdigraph \(G_1(n, k)\) is always semiregular. The tree \(T(p^e_i, k, 0)\), which is not always semiregular, has a nice simple structure whenever it is semiregular. In this section, we prove new characterizations of semiregular digraphs \(G(n, k)\). We show that a semiregular digraph \(G(n, k)\) has a nice simple structure, and it can be expressed explicitly. In addition, we found an interesting relation between the symmetric and semiregularity property of \(G(n, k)\).

**Theorem 2.4.1.** Let \(p\) be an odd prime. The trees attached to all cycle vertices in \(G(p^e, k)\) are isomorphic if and only if \(G(p^e, k)\) is semiregular.

*Proof.* The proof follows directly from Theorems 2.2.3 and 1.8.7. 

**Theorem 2.4.2.** Let \(n\) be an odd integer. The trees attached to all cycle vertices in \(G(n, k)\) are isomorphic if and only if \(G(n, k)\) is semiregular.

*Proof.* Assume that the trees attached to all cycle vertices in \(G(n, k)\) are isomorphic. Then from Theorem 2.2.3, it follows that the height of each component of \(G(n, k)\) is 1, and the indegree of every cycle vertex in \(G(n, k)\) is \(\prod_{i=1}^{r} p_i^{e_i-1}\). This implies that \(G(n, k)\) is semiregular.
Conversely, we assume that $G(n, k)$ is semiregular. Our aim is to show that $\gcd(p_{i-1}^e - (p_i - 1), k) = p_{i-1}^e - 1$ for all $i$ such that $1 \leq i \leq r$. However, this follows by using similar arguments as in the proof of the ‘$\Rightarrow$’ part of Theorem 2.2.7.

**Remark 2.4.3.** Note that, $T(p^e, k^r, 0)$ is semiregular whenever $T(p^e, k, 0)$ is semiregular. So we can conclude that for any cycle vertex $a$, $T(n, k^r, a)$ is semiregular whenever $T(n, k, a)$ is semiregular.

**Lemma 2.4.4.** Let $n$ be an odd integer. The digraph $G(n, k)$ is semiregular if and only if $T(n, k, 0) \cong T(n, k, 1)$.

**Proof.** In view of Theorem 2.4.2, we prove only the converse. Since

$$\prod_{i=1}^r p_i^{e_i-1} = |T(n, k, 0)| = |T(n, k, 1)| = \gcd(p_i^{e_i-1}(p_i - 1), k^{s_i}),$$

where $s_i = h(T(p_i^{e_i}, k))$, we observe that $k$ must be odd and $\gcd(p_i - 1, k) = 1$ for all $i$. We also have $p_i^{e_i-\lceil \frac{e_i}{k} \rceil} || k$, and the semiregularity of $T(n, k, 0)$ implies $\lceil \frac{e_i}{k} \rceil \leq 2$ for all $i$. If $\lceil \frac{e_i}{k} \rceil = 2$ for some $i$, then $p_i^{e_i-2} \leq k < e$ which is a contradiction. Therefore, $p_i^{e_i - 1} || k$ for all $i$, and Theorem 2.2.7 together with Theorem 2.4.2 implies that $G(n, k)$ is semiregular.

**Lemma 2.4.5.** The tree $T(n, k, 0)$ is semiregular if and only if $T(p_i^{e_i}, k, 0)$ is semiregular for all $i$.

**Proof.** Suppose that $T(p_i^{e_i}, k, 0)$ is not semiregular for some $i$. Then there exists vertices $a$ and $b$ in $T(p_i^{e_i}, k, 0)$ such that $\text{indeg}_{p_i^{e_i}}(a) \neq \text{indeg}_{p_i^{e_i}}(b)$, and it follows that $\text{indeg}_{p_i^{e_i}}(0, 0, \ldots, a, \ldots, 0) \neq \text{indeg}_{p_i^{e_i}}(0, 0, \ldots, b, \ldots, 0)$ in $T(n, k, 0)$. Thus $T(n, k, 0)$ is not semiregular.

The converse is straightforward.
Theorem 2.4.6. Let $n$ be an odd integer. The following statements are equivalent.

(a) $G(n, k)$ is semiregular.

(b) The trees attached to all cycle vertices in $G(n, k)$ are isomorphic.

(c) $\gcd(p_i^{e_i}(p_i - 1), k) = p_i^{e_i-1}$ for all $i$ such that $1 \leq i \leq r$.

(d) $T(n, k, 1) \cong T(n, k, 0)$.

Remark 2.4.7. If $G(n, k)$ is semiregular and $m = \prod_{i=1}^{r} p_i^{e_i-1}$, then we can write $G(n, k)$ more explicitly as:

$$G(n, k) = a_1 O_{t_1}^m \cup a_2 O_{t_2}^m \cdots \cup a_l O_{t_l}^m,$$

where $a_i = A_{t_i}(G(n, k))$ for all $t_i \in A(G(n, k))$.

Theorem 2.4.8. Let $n$ be an odd integer. Then $G(n, k)$ is symmetric of order $\prod_{i=1}^{r} p_i$ if and only if $G(n, k)$ is semiregular and $k \equiv 1 \pmod{\lambda(\prod_{i=1}^{r} p_i)}$.

Proof. Assume that $G(n, k)$ is symmetric of order $\prod_{i=1}^{r} p_i$. Then there exists at least $(\prod_{i=1}^{r} p_i) - 1$ components, say, $C_1, C_2, \ldots, C_{(\prod_{i=1}^{r} p_i) - 1}$, which are isomorphic to $T(n, k, 0)$. Now since $|T(n, k, 0)| = |T(p_1^{e_1}, k, 0)| \times |T(p_2^{e_2}, k, 0)| \times \cdots \times |T(p_r^{e_r}, k, 0)| = \prod_{i=1}^{r} p_i^{e_i} - 1$, we obtain $|C_1| + |C_2| + \cdots + |C_{(\prod_{i=1}^{r} p_i) - 1}| + |T(0)| = n$, and by Theorem 2.4.6, $G(n, k)$ must be semiregular. Also, since $\prod_{i=1}^{r} p_i = A_1(G(n, k)) = \prod_{i=1}^{r} [\gcd(\lambda(p_i^{e_i}), k - 1) + 1]$ it follows that $k \equiv 1 \pmod{\lambda(\prod_{i=1}^{r} p_i)}$.

The converse follows immediately from Theorem 2.3.9. \qed