Chapter 6

Dissipativity and passivity analysis for neutral type neural networks

6.1 Introduction

Neutral type phenomenon always appears in the study of automatic control, population dynamics and vibrating masses attached to an elastic bar, etc. Hence, dynamical behaviour for robust neutral type neural networks and systems with time-varying delay have been considered in the recent years [61, 70]. On the other hand, so far, very little attention has been paid to stability analysis of neural networks and systems with time delay in leakage (or “forgetting”) term [52, 54]. In [54] the effect of leakage time-varying delay on stability of nonlinear differential systems has been investigated and several sufficient conditions are obtained for the existence, uniqueness and global asymptotic stability of the equilibrium point by using fixed point theorems. Based on this, the effects of leakage time-varying delays in Markovian jump neural networks with impulse control have been analyzed in [52].

Recently, in order to obtain the less conservative results, the delay decomposition approach was successfully introduced in [130] for the neural networks with constant delay. Followed this, in [19], the problem of stability analysis for neural networks with time-varying delay has been discussed via delay decomposition approach. In [7], a delay
decomposition approach to delay-dependent passivity analysis for interval neural networks with time-varying delay has been taken into account. In order to apply delay decomposition technique, in this section, discrete and neutral delay intervals are splitted into finitely many equidistant subintervals. By choosing symmetric positive definite matrices in each of these subintervals, a new LKF has been constructed with double and triple integral terms such as

\[ \delta \int_{-\bar{\delta}}^{-(j-1)\delta} \int_{t+\theta}^{t} \dot{x}^T(s) H_j \dot{x}(s) ds d\theta \quad \text{and} \quad \frac{\nu}{2} \int_{-\bar{\delta}}^{-(j-1)\delta} \int_{0}^{\theta} \int_{t+\lambda}^{t} \dot{x}^T(s) \hat{S}_j \dot{x}(s) ds d\lambda d\theta, \]

\( j = 1, 2, \ldots, m \) involving lower and upper bounds of time delays so as to formulate some new dissipativity criteria for the concerned neutral type neural networks with leakage time-varying delays. Further, the different weight matrices \( H_j > 0, \hat{S}_j > 0, j = 1, 2, \ldots, m \) are chosen on different sub-intervals which play the key role in the reduction of conservatism.

Motivated by the above, the main objective of this section is to study the dissipativity analysis for a class of neutral-type neural networks with leakage time-varying delay. New LKF has been constructed by dividing the delay interval into multiple subintervals and choosing different weight matrices in each of these subintervals. By employing this new LKF for the case of robust neutral-type neural networks, some new delay-derivative-dependent dissipativity criteria are established in terms of LMIs, which can be easily solved by using MATLAB LMI control toolbox. These dissipativity criteria depend on both the upper and lower bounds of the derivative of the time delay. The main advantage of this work is to calculate maximum allowable upper bounds of the delay by the delay decomposition approach and some integral inequalities. Besides, some examples are presented to demonstrate the feasibility and effectiveness of the proposed criteria.
6.2 A delay decomposition approach for robust dissipativity and passivity analysis of neutral type neural networks with leakage time-varying delay

6.2.1 Problem description

Consider the following uncertain neural networks model of neutral type with leakage time-varying delay

\[
\dot{x}(t) = -(A + \Delta A(t))x(t - \sigma(t)) + (B + \Delta B(t))g(x(t)) + (C + \Delta C(t))g(x(t - \tau(t)))
\]
\[
+ (E + \Delta E(t))\dot{x}(t - h(t)) + u(t),
\]
\[
y(t) = g(x(t)),
\]
(6.1)

where \(x(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^T\) denotes the state vector associated with \(n\) neurons, \(u(t)\) is the input, \(g(x(t)) = [g_1(x_1(t)) \ g_2(x_2(t)) \ \cdots \ g_n(x_n(t))]^T\) denotes the neuron activation function, \(y(t)\) is the output, \(A = \text{diag}\{a_1, a_2, \ldots, a_n\} > 0\), \(B, C\) and \(E\) are known constant matrices of appropriate dimensions. \(\Delta A(t), \Delta B(t), \Delta C(t)\) and \(\Delta E(t)\) are unknown matrices that represent the time-varying parameter uncertainties and are assumed to be of the form

\[
[\Delta E(t) \ \Delta A(t) \ \Delta B(t) \ \Delta C(t)] = [\overline{H}_0G_0(t)\overline{E}_0 \ \overline{H}_1G_1(t)\overline{E}_1 \ \overline{H}_2G_2(t)\overline{E}_2 \ \overline{H}_3G_3(t)\overline{E}_3],
\]
(6.2)

where \(\overline{H}_\kappa\) and \(\overline{E}_\kappa, \kappa = 0, 1, 2, 3\) are known real constant matrices and \(G_\kappa(t), \kappa = 0, 1, 2, 3\) are unknown time-varying matrix functions satisfying

\[
G_\kappa^T(t)G_\kappa(t) \leq I.
\]
(6.3)

In the neural network (6.1), the bounded functions \(\sigma(t), \tau(t)\) and \(h(t)\) represent respectively, the leakage, discrete and neutral time-varying delays that are assumed to
satisfy the following conditions

\begin{align*}
0 \leq \sigma(t) &\leq \sigma < \infty, \quad \dot{\sigma}(t) \leq \sigma_\mu < \infty, \\
0 < \tau(t) &\leq \tau < \infty, \quad \dot{\tau}(t) \leq \mu < \infty, \\
0 < h(t) &\leq h < \infty, \quad \dot{h}(t) \leq \varrho < 1, \quad \forall t \geq 0,
\end{align*}

where \( \sigma(t) \), \( \tau(t) \) and \( h(t) \) are differentiable functions and \( \sigma, \tau, \mu, \varrho \) are scalars.

For the system (6.1), the initial condition is assumed to be \( x(t) = \phi(t), \ t \in [-\bar{\tau}, 0] \), where \( \bar{\tau} = \max\{\sigma, \tau, h\} \) and the norm is defined by \( ||\phi(s)||_{\bar{\tau}} = \max\{\sup_{-\bar{\tau} \leq s \leq 0} ||\phi(s)||, \sup_{-\bar{\tau} \leq s \leq 0} ||\dot{\phi}(s)||\} \).

For convenience, system (6.1) can be rewritten as

\begin{align*}
\dot{x}(t) = -A(t)x(t - \sigma(t)) + B(t)g(x(t)) + C(t)g(x(t - \tau(t))) + E(t)\dot{x}(t - h(t)) \\
&+ u(t),
\end{align*}

where

\begin{align*}
A(t) &= A + \Delta A(t), & B(t) &= B + \Delta B(t), \\
C(t) &= C + \Delta C(t), & E(t) &= E + \Delta E(t).
\end{align*}

Throughout this chapter, the following assumption is used.

**Assumption 6.1.** For any \( \kappa \in \{1, 2, \ldots, n\} \), \( g_\kappa(0) = 0 \) and there exist constants \( F^-_\kappa \) and \( F^+_\kappa \) such that

\[ F^-_\kappa \leq \frac{g_j(\alpha_1) - g_j(\alpha_2)}{\alpha_1 - \alpha_2} \leq F^+_\kappa, \quad \forall \alpha_1, \alpha_2 \in R, \ \alpha_1 \neq \alpha_2. \]

**6.2.2 Dissipativity analysis for neutral type neural networks**

In this section, the dissipativity results for the neural networks (6.1) is discussed. Since system (6.1) is equivalent to system (6.5), it is enough to investigate the dissipativity of system (6.5) instead of (6.1).
Let \( m > 1 \) and \( l > 1 \) be two integers. Now, the delay intervals \([-\tau, 0]\) and \([-h, 0]\) are divided into \( m \) and \( l \) equidistant subintervals, respectively. That is,

\[
[-\tau, 0] = \bigcup_{j=1}^{m} [-j\delta, -(j-1)\delta]
\]

and

\[
[-h, 0] = \bigcup_{q=1}^{l} [-q\rho, -(q-1)\rho]
\]

where \( \delta = \tau/m \) and \( \rho = h/l \). For presentation convenience, in the following, denote

\[
F_1 = \text{diag}\{F_1^-F_1^+, \ldots, F_n^-F_n^+\}, \quad F_2 = \text{diag}\left\{ \frac{F_1^- + F_1^+}{2}, \ldots, \frac{F_n^- + F_n^+}{2} \right\}.
\]

**Theorem 6.1.** For given integers \( m > 1 \) and \( l > 1 \) and given positive scalars \( \sigma, \tau, h, \sigma, \mu \) and \( 0 < \rho < 1 \), the system (6.5) is strictly \( (Q, S, R) \)-\( \gamma \)-dissipative in the sense of Definition 1.3, if there exist positive definite matrices \( P, Z_i, i = 1, 2, 3 \), \( \tilde{T}_i, i = 1, 2, 3 \), \( Q_0, X_0 \), \( T_j, j = 1, 2, \ldots, m \), \( W_q, N_q, Q_q \), \( q = 1, 2, \ldots, l \), appropriately dimensioned matrices \( R_1, R_2 \), \( Y_1, Y_2 \), positive diagonal matrices \( U, V \), and scalars \( \gamma > 0, \epsilon_i > 0, i = 1, 2, 3, \ldots, 8 \), such that the following LMI holds for any \( k \in \{1, 2, \ldots, m\} \) and \( k' \in \{1, 2, \ldots, l\} \),

\[
\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 & \Xi_1 \\ * & \Omega_3 & 0 \\ * & * & \Xi_3 \end{bmatrix} < 0, \text{ for } k' \neq 1, \quad \& \quad \Omega = \begin{bmatrix} \tilde{\Omega}_1 & \tilde{\Omega}_2 & \Xi_1 \\ * & \Omega_3 & 0 \\ * & * & \Xi_3 \end{bmatrix} < 0, \text{ for } k' = 1, \quad (6.7)
\]

where

\[
\begin{align*}
\Omega_1 &= \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ * & \Sigma_4 \end{bmatrix}, \quad \tilde{\Omega}_1 = \begin{bmatrix} \hat{\Sigma}_1 & \Sigma_2 \\ * & \Sigma_3 \end{bmatrix}, \quad \bar{\Sigma}_1 = \tilde{\Sigma}_1 + \Sigma_2, \\
\Sigma_1 &= \begin{bmatrix} \Omega_{11} + \Omega_{11}^k & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & 0 \\ * & * & \Omega_{33} & 0 \\ * & * & * & \Omega_{44} + \Omega_{44}^k \end{bmatrix}, \quad \bar{\Sigma}_1 = \begin{bmatrix} \Omega_{11} + \Omega_{11}^k & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & 0 \\ * & * & \Omega_{33} & 0 \\ * & * & * & \Omega_{44} \end{bmatrix}, \\
\Sigma_2 &= \begin{bmatrix} \Omega_{15} & \Omega_{16} & \Omega_{17} \\ 0 & 0 & \Omega_{27} \\ 0 & 0 & \Omega_{37} \end{bmatrix}, \quad \Sigma_3 = \begin{bmatrix} \Omega_{55} & 0 & 0 \\ * & \Omega_{66} & 0 \\ * & * & \Omega_{77} \end{bmatrix},
\end{align*}
\]
\[ \begin{split}
\Omega_2 &= \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \end{bmatrix}, C_1 = \begin{bmatrix} \mathcal{H}_1^TR_2^T & 0 \end{bmatrix}^T, C_2 = \begin{bmatrix} \mathcal{H}_2^TR_2^T & 0 \end{bmatrix}^T, \\
\Omega_3 &= \text{diag}\{-\epsilon_5I, -\epsilon_6I, -\epsilon_7I, -\epsilon_8I, -\epsilon_1I, -\epsilon_2I, -\epsilon_3I, -\epsilon_4I\}, \Xi_1 = [I_1 \ I_2], \\
I_1 &= \begin{bmatrix} PA\sqrt{\sigma_p} & 0 \end{bmatrix}^T, I_2 = \begin{bmatrix} 0 & A^TPA\sqrt{\sigma_p} \end{bmatrix}^T, \\
\Omega_{11} &= \begin{bmatrix} \phi_1 & H_1 & 0 & \cdots & 0 & 0 & 0 \\
* & \phi_2 & H_2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
* & * & * & \cdots & \phi_m & H_m & 0 \\
* & * & * & \cdots & * & \phi_{m+1} & 0 \\
* & * & * & \cdots & * & * & \hat{\phi} \end{bmatrix}, \Omega_{12} = \begin{bmatrix} \Phi_1 & \Phi_2 \\
0 & \beta \end{bmatrix}, \\
\Omega_{13} &= \begin{bmatrix} P - R_2 & 0 & 0 \\
0 & 0 & 0 \\
R_2E & 0 & 0 \end{bmatrix}, \Omega_{14} = \begin{bmatrix} I_2 \\
0 \\
I_3 = [N_1 \ 0], \Omega_{15} = \begin{bmatrix} \hat{I}_1 \\
0 \end{bmatrix}, \\
\Omega_{16} &= \begin{bmatrix} \bar{I}_1 \\
0 \end{bmatrix}, \hat{I}_1 = [\hat{\delta}_1\hat{S}_1 \ \hat{\delta}_2\hat{S}_2 \ \cdots \ \hat{\delta}_k\hat{S}_k \ \hat{\delta}_{k-1}\hat{S}_{k-1} \ \cdots \ \hat{\delta}_{m-1}\hat{S}_{m-1} \ \hat{\delta}_m\hat{S}_m \ 0], \\
\hat{I}_1 &= \begin{bmatrix} \hat{\rho}_1Q_1 & \hat{\rho}_2Q_2 & \cdots & \hat{\rho}_kQ_k & \hat{\rho}_{k+1}Q_{k+1} \end{bmatrix}^T \begin{bmatrix} \hat{\delta}_1 \ \hat{\delta}_2 \ \cdots \ \hat{\delta}_k \ \hat{\delta}_{k-1} \ \cdots \ \hat{\delta}_{m-1} \ \hat{\delta}_m \ 0 \end{bmatrix}, \\
\Omega_{17} &= \begin{bmatrix} I_4 \\
0 \end{bmatrix}, I_4 = \begin{bmatrix} R_2 & A^TPA & PA - R_2A & 0 \end{bmatrix}, \Omega_{22} = \text{diag}\{\alpha_1, \alpha_2, \cdots, \alpha_{m+1}, \hat{\alpha}\}, \\
\Omega_{23} &= \begin{bmatrix} \mathcal{I}_1 \ 0 \end{bmatrix}, \mathcal{I}_1 = \begin{bmatrix} B^TR_1^T & 0 & C^TR_1^T \end{bmatrix}^T, \Omega_{27} = \begin{bmatrix} \mathcal{I}_2 \\
0 \end{bmatrix}, \mathcal{I}_2 = \begin{bmatrix} -S^T & 0 \end{bmatrix}, \\
\Omega_{33} &= \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \Omega_{37} = \begin{bmatrix} I_5 \\
0 \end{bmatrix}, I_5 = \begin{bmatrix} R_1 & -A^TP & -R_1A & 0 \end{bmatrix}, \\
\Omega_{55} &= \text{diag}\{-\hat{\delta}_1, -\hat{\delta}_2, \cdots, -\hat{\delta}_k, -\hat{\delta}_{k-1}, -\hat{\delta}_{k-2}, \cdots, -\hat{\delta}_m, -\hat{\delta}_{m-1}, -\hat{\delta}_m, -\hat{\delta}_1 \}, \\
\Omega_{66} &= \text{diag}\{-Q_1, -Q_2, -Q_3, -Q_4, -Q_5, -Q_6, -Q_7, -\hat{T}_1 \}, \\
\Omega_{44} &= \begin{bmatrix} v_1 & N_2 & 0 & \cdots & 0 & 0 & 0 \\
* & v_2 & N_3 & \cdots & 0 & 0 & 0 \\
* & * & v_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
* & * & * & \cdots & v_{l-1} & \hat{N}_l & 0 \\
* & * & * & \cdots & * & v_l & 0 \\
* & * & * & \cdots & * & * & 0 \end{bmatrix}, \\
\Omega_{77} &= \begin{bmatrix} -(R - \gamma I) & 0 & 0 \\
* & -Z_1 & -A^TPA & 0 \\
* & * & \hat{Z} & 0 \\
* & * & * & -Z_2 \end{bmatrix}.
\end{split} \]
\[\phi_j = \begin{cases} 
-PA - A^T P + \sigma^2 Z_1 + Z_2 + Z_3 + \tau^2 \hat{T}_1 + h^2 \hat{T}_3 \\
+T_1 - H_1 - N_1 + Q_0 - \sum_{j=1, j \neq k}^m \delta_j \delta_k - 2\delta_k \\
T_j - T_{j-1} - H_j - H_{j-1}, \quad j = 2, \ldots, m \\
-T_m - H_m, \quad j = m + 1 
\end{cases}\]

\[\beta_j = \begin{cases} 
X_1 + X_0 + F_3 U \\
+R_2 B, \quad j = 1 \\
X_j - X_{j-1}, \quad j = 2, \ldots, m \\
-X_m, \quad j = m + 1 
\end{cases}\]

\[\psi_q = \begin{cases} 
\hat{T}_2 + \sum_{j=1}^m \{\delta^2 H_j + \nu_j^2 \delta_j\} + W_1 \\
+\sum_{q=1}^l \{\rho^2 N_q + \eta_q^2 Q_q\} - R_1 - R_1^T, \quad q = 1 \\
W_q - W_{q-1}, \quad q = 2, \ldots, l \\
-W_l, \quad q = l 
\end{cases}\]

\[\nu_q = \begin{cases} 
N_{q+1} - N_q, \quad q = 1, 2, \ldots, l - 1 \\
-N_l, \quad q = l 
\end{cases}\]

\[\tilde{\phi} = -F_1 V - Q_0 (1 - \mu), \quad \tilde{\beta} = F_2 V - X_0 (1 - \mu),\]

\[\tilde{\alpha} = -V - S_0 (1 - \mu) + \epsilon_3 E_3^T E_3 + \epsilon_7 E_3^T E_3,\]

\[\tilde{\psi} = -T_2 (1 - \eta) + \epsilon_4 E_0^T E_0 + \epsilon_8 E_0^T E_0, \quad \tilde{Z} = -Z_3 (1 - \sigma), \quad \epsilon_1 E_1^T E_1 + \epsilon_5 E_1^T E_1,\]

\[\Phi_1 = diag\{\beta_1, \beta_2, \ldots, \beta_{m+1}\}, \quad \Phi_2 = [C^T R_2^T \quad 0]^T, \quad \Phi_1 = diag\{\psi_1, \psi_2, \ldots, \psi_l\},\]

\[\hat{\Phi}_2 = [E^T R_1^T \quad 0]^T, \quad \Xi_3 = diag\{-Y_1, -Y_2\}, \quad \nu_j = (j \tilde{\delta})^2 - ((j - 1) \tilde{\delta})^2,\]

\[\eta_q = (q \tilde{\delta})^2 - ((q - 1) \tilde{\delta})^2, \quad \Omega_{11}^k = (\gamma_{1j})_{\{m+2\} \times \{m+2\}}, \quad \Omega_{44}^k = (\tilde{\delta}_i j_i^l)_{(l+1) \times (l+1)};\]

\[\Sigma_2 = (\omega_{ij})_{(2m+2l+7) \times (2m+2l+7)};\]
\[
\gamma^k_{ij} = \begin{cases} 
-2H_k, & i = j = m + 2 \\
H_k, & i \in \{k, k + 1\}, \ j = m + 2 \\
-H_k, & i = k, \ j = k + 1, \\
0, & \text{otherwise}
\end{cases},
\]
\[
\tilde{\delta}_{ij}^{k'} = \begin{cases} 
-2N_{k'}, & i = j = l + 1 \\
N_{k'}, & i \in \{k' - 1, k'\}, \ j = l + 1 \\
-N_{k'}, & i = k' - 1, \ j = k', \\
0, & \text{otherwise}
\end{cases}
\]
\[
\varpi_{ij} = \begin{cases} 
-2N_1, & i = j = 2m + 2l + 7 \\
N_1, & i \in \{1, 2m + l + 7\}, \ j = 2m + l + 7 \\
-N_1, & i = 1, \ j = 2m + l + 7, \\
0, & \text{otherwise}
\end{cases}
\]

and the remaining coefficients are zero.

**Proof.** Consider the LKF described by

\[
V(t, x(t)) = \sum_{\kappa=1}^{10} V_{\kappa}(t, x(t)),
\]

where

\[
V_1(t, x(t)) = \left[ x(t) - A \int_{t-\sigma(t)}^{t} x(s) ds \right]^T P \left[ x(t) - A \int_{t-\sigma(t)}^{t} x(s) ds \right],
\]

\[
V_2(t, x(t)) = \sigma \int_{-\tau}^{0} \int_{t+\theta}^{t} x^T(s) Z_1 x(s) ds d\theta + \int_{t-\sigma(t)}^{t} x^T(s) Z_2 x(s) ds + \int_{t-\sigma(t)}^{t} x^T(s) Z_3 x(s) ds,
\]

\[
V_3(t, x(t)) = \tau \int_{-\tau}^{0} \int_{t+\theta}^{t} x^T(s) \tilde{T}_1 x(s) ds d\theta \]

\[
+ \int_{t-\tau(t)}^{t} \left[ x(s) \ g(x(s)) \right]^T \left[ \begin{array}{cc} Q_0 & X_0 \\ S_0 & \end{array} \right] \left[ \begin{array}{c} x(s) \\ g(x(s)) \end{array} \right] ds,
\]

and the remaining coefficients are zero.
\[
V_4(t, x(t)) = \int_{t-h(t)}^{t} \dot{x}(t)\hat{T}_2\dot{x}(s)ds + h \int_{h}^{t} \int_{t+\theta}^{t} x^T(s)\hat{T}_3x(s)dsd\theta,
\]
\[
V_5(t, x(t)) = \sum_{j=1}^{m} \int_{-j\delta}^{-(j-1)\delta} \int_{t+\theta}^{t} x^T(s)H_j\dot{x}(s)dsd\theta,
\]
\[
V_6(t, x(t)) = \sum_{j=1}^{m} \tilde{\delta} \int_{-j\delta}^{-(j-1)\delta} \int_{t+\theta}^{t} x^T(s)H_j\dot{x}(s)dsd\theta,
\]
\[
V_7(t, x(t)) = \sum_{q=1}^{l} \int_{-q\rho}^{-(q-1)\rho} \dot{x}^T(t+s)W_q\dot{x}(t+s)ds,
\]
\[
V_8(t, x(t)) = \sum_{q=1}^{l} \tilde{\rho} \int_{-q\rho}^{-(q-1)\rho} \int_{t+\theta}^{t} x^T(s)N_q\dot{x}(s)dsd\theta,
\]
\[
V_9(t, x(t)) = \sum_{q=1}^{l} \frac{V_j}{2} \int_{-j\delta}^{-(j-1)\delta} \int_{t+\lambda}^{t} \dot{x}^T(s)\tilde{S}_j\dot{x}(s)dsd\lambda d\theta,
\]
\[
V_{10}(t, x(t)) = \sum_{q=1}^{l} \frac{\eta_q}{2} \int_{-q\rho}^{-(q-1)\rho} \int_{t+\lambda}^{t} \dot{x}^T(s)Q_q\dot{x}(s)dsd\lambda d\theta.
\]

Taking the derivative of each term in \(V(t, x(t))\) in (6.8) along the trajectory of (6.5), it follows that

\[
\dot{V}_1(t, x(t)) = 2[x(t) - A \int_{t_0-s}^{t} x(s)ds]^T P\dot{x}(t) - Ax(t) + Ax(t - \sigma(t))
\]
\[
- Ax(t - \sigma(t))\dot{\sigma}(t)
\]
\[
\leq 2x^T(t)P\dot{x}(t) - 2x^T(t)PAX(t) + 2x^T(t)PAX(t - \sigma(t))
\]
\[
+ x^T(t)PAY_1^{-1}A^TP\sigma(t)
\]
\[
+ x^T(t - \sigma(t))Y_1x(t - \sigma(t))\sigma(t) - \int_{t_0-s}^{t} x^T(s)dsA^TP\dot{x}(t)
\]
\[
+ 2 \int_{t_0-s}^{t} x^T(s)dsA^TPAx(t) - \int_{t_0-s}^{t} x^T(s)dsA^TPAx(t - \sigma(t))
\]
\[
+ \int_{t_0-s}^{t} x^T(s)dsA^TPAY_2^{-1}APA^T\sigma(t)
\]
\[
+ x^T(t - \sigma(t))Y_2\sigma(t)x(t - \sigma(t)),
\]  
(6.9)
\[
V_2(t, x(t)) \leq x^T(t) [\sigma^2 Z_1 + Z_2 + Z_3] x(t) - \left( \int_{t - \sigma(t)}^t x(s)ds \right)^T Z_1 \left( \int_{t - \sigma(t)}^t x(s)ds \right) \\
- x^T(t - \sigma) Z_2 x(t - \sigma) - x^T(t - \sigma(t)) Z_3 x(t - \sigma(t)) (1 - \sigma), \quad (6.10)
\]

\[
V_3(t, x(t)) \leq \tau^2 x^T(t) \hat{T}_1 x(t) - \left( \int_{t - \tau(t)}^t x(s)ds \right)^T \hat{T}_1 \left( \int_{t - \tau(t)}^t x(s)ds \right) \\
+ \left[ \begin{array}{c}
x(t) \\
g(x(t))
\end{array} \right]^T \left[ \begin{array}{cc}
Q_0 & X_0 \\
* & S_0
\end{array} \right] \left[ \begin{array}{c}
x(t) \\
g(x(t))
\end{array} \right] \\
- \left[ \begin{array}{c}
x(t - \tau(t)) \\
g(x(t - \tau(t))
\end{array} \right]^T \left[ \begin{array}{cc}
Q_0 & X_0 \\
* & S_0
\end{array} \right] \left[ \begin{array}{c}
x(t - \tau(t)) \\
g(x(t - \tau(t)))
\end{array} \right] (1 - \mu), \quad (6.11)
\]

\[
V_4(t, x(t)) \leq \hat{x}^T(t) \hat{T}_2 \hat{x}(t) - \hat{x}^T(t - h(t)) \hat{T}_2 \hat{x}(t - h(t)) (1 - g) + h^2 x^T(t) \hat{T}_3 x(t) \\
- \left( \int_{t - h(t)}^t x(s)ds \right)^T \hat{T}_3 \left( \int_{t - h(t)}^t x(s)ds \right), \quad (6.12)
\]

\[
V_5(t, x(t)) = \sum_{j=1}^m \left[ \frac{x(t - (j - 1)\delta)}{g(x(t - (j - 1)\delta))} \right]^T \hat{T}_3 \left[ \frac{x(t - (j - 1)\delta)}{g(x(t - (j - 1)\delta))} \right] \\
- \sum_{j=1}^m \left[ \frac{x(t - j\delta)}{g(x(t - j\delta))} \right]^T \hat{T}_3 \left[ \frac{x(t - j\delta)}{g(x(t - j\delta))} \right], \quad (6.13)
\]

\[
V_6(t, x(t)) = \delta^2 \sum_{j=1}^m \hat{x}^T(t) H_j \hat{x}(t) - \sum_{j=1}^m \delta \int_{t - j\delta}^{t - (j - 1)\delta} \hat{x}^T(s) H_j \hat{x}(s)ds, \quad (6.14)
\]

\[
V_7(t, x(t)) = \sum_{q=1}^l \hat{x}^T(t - (q - 1)\rho) W_q \hat{x}(t - (q - 1)\rho) - \sum_{q=1}^l \hat{x}^T(t - q\rho) W_q \hat{x}(t - q\rho), \quad (6.15)
\]

\[
V_8(t, x(t)) = \sum_{q=1}^l \hat{x}^T(t) N_q \hat{x}(t) - \sum_{q=1}^l \hat{x}^T(s) N_q \hat{x}(s)ds, \quad (6.16)
\]

\[
V_9(t, x(t)) = \sum_{j=1}^m \frac{\nu_j^2}{4} \hat{x}^T(t) \hat{S}_j \hat{x}(t) - \sum_{j=1}^m \frac{\nu_j^2}{2} \int_{t - j\delta}^{t - (j - 1)\delta} \int_{t - \theta}^t \hat{x}^T(s) \hat{S}_j \hat{x}(s)dsd\theta, \quad (6.17)
\]

\[
V_{10}(t, x(t)) = \sum_{q=1}^l \frac{\eta_q^2}{4} \hat{x}^T(t) Q_q \hat{x}(t) - \sum_{q=1}^l \frac{\eta_q^2}{2} \int_{t - q\rho}^{t - (q - 1)\rho} \int_{t + \theta}^t \hat{x}^T(s) Q_q \hat{x}(s)dsd\theta. \quad (6.18)
\]

For any \( t \geq 0 \), there should exists an integer \( k \in \{1, 2, \cdots, m\} \) such that \( \tau(t) \in [(k - \#)\delta] \).
\[ 1)\overline{\delta}, k\overline{\delta} \]. In this situation, apply Lemma 1.16 to obtain
\[
-\overline{\delta} \int_{t-k\overline{\delta}}^{t-(k-1)\overline{\delta}} \dot{x}^T(s)H_k\dot{x}(s)ds
= -\overline{\delta} \int_{t-k\overline{\delta}}^{t-(k-1)\overline{\delta}} \dot{x}^T(s)H_k\dot{x}(s)ds - \overline{\delta} \int_{t-(k-1)\overline{\delta}}^{t-(k)\overline{\delta}} \dot{x}^T(s)H_k\dot{x}(s)ds
\leq -(k\overline{\delta} - \tau(t)) \int_{t-k\overline{\delta}}^{t-(k-1)\overline{\delta}} \dot{x}^T(s)H_k\dot{x}(s)ds - (\tau(t) - (k-1)\overline{\delta}) \int_{t-(k-1)\overline{\delta}}^{t-(k)\overline{\delta}} \dot{x}^T(s)H_k\dot{x}(s)ds.
\]

For \( j \neq k \), Lemma 1.16 can be used obtain the following:
\[
-\overline{\delta} \int_{t-j\overline{\delta}}^{t-(j-1)\overline{\delta}} \dot{x}^T(s)H_j\dot{x}(s)ds \leq \left[ \begin{array}{c} x(t-(j-1)\overline{\delta}) \\ x(t-j\overline{\delta}) \\ x(t-\tau(t)) \end{array} \right]^T \left[ \begin{array}{ccc} -H_k & 0 & H_k \\ * & -H_k & H_k \\ * & * & -2H_k \end{array} \right] \left[ \begin{array}{c} x(t-(j-1)\overline{\delta}) \\ x(t-j\overline{\delta}) \\ x(t-\tau(t)) \end{array} \right].
\] (6.19)

Similarly, for any \( t \geq 0 \), there should exist an integer \( k' \in \{1, 2, \cdots , t\} \) such that \( h(t) \in [(k' - 1)\overline{\rho}, k'\overline{\rho}] \). In this situation, using Lemma 1.16, it can be obtained that
\[
-\overline{\rho} \int_{t-k'\overline{\rho}}^{t-h(t)} \dot{x}^T(s)N_{k'}\dot{x}(s)ds
= -\overline{\rho} \int_{t-k'\overline{\rho}}^{t-h(t)} \dot{x}^T(s)N_{k'}\dot{x}(s)ds - \overline{\rho} \int_{t-h(t)}^{t-(k'\overline{\rho})} \dot{x}^T(s)N_{k'}\dot{x}(s)ds
\leq -(k'\overline{\rho} - h(t)) \int_{t-k'\overline{\rho}}^{t-h(t)} \dot{x}^T(s)N_{k'}\dot{x}(s)ds - (h(t) - (k' - 1)\overline{\rho}) \int_{t-h(t)}^{t-(k'\overline{\rho})} \dot{x}^T(s)N_{k'}\dot{x}(s)ds.
\]

For \( q \neq k' \), Lemma 1.16 again yields
\[
-\overline{\rho} \int_{t-q\overline{\rho}}^{t-(q-1)\overline{\rho}} \dot{x}^T(s)N_q\dot{x}(s)ds \leq \left[ \begin{array}{c} x(t-(q-1)\overline{\rho}) \\ x(t-q\overline{\rho}) \\ x(t-h(t)) \end{array} \right]^T \left[ \begin{array}{ccc} -N_q & 0 & N_q \\ 0 & -N_{k'} & N_{k'} \\ 0 & 0 & -2N_{k'} \end{array} \right] \left[ \begin{array}{c} x(t-(q-1)\overline{\rho}) \\ x(t-q\overline{\rho}) \\ x(t-h(t)) \end{array} \right].
\] (6.22)
Similarly,

\[-\frac{\nu_k}{2} \int_{-j+\delta}^{-(j-1)\delta} \int_{t+\theta}^t \ddot{x}(s) \dddot{S}_k \ddot{x}(s) ds \leq \]

\[
\begin{bmatrix}
\ddot{S}_k x(t) \\
\int_{t-(j-1)\delta}^{t-t(\tau(t))} x(s) ds \\
\int_{t-j\delta}^{t-\tau(t)} x(s) ds
\end{bmatrix}
\begin{bmatrix}
-2\dddot{S}_k & \dddot{S}_k & \dddot{S}_k \\
* & -\dddot{S}_k & 0 \\
* & * & -\dddot{S}_k
\end{bmatrix}
\begin{bmatrix}
\int_{t-(j-1)\delta}^{t-t(\tau(t))} x(s) ds \\
\int_{t-j\delta}^{t-\tau(t)} x(s) ds
\end{bmatrix}.
\]  (6.23)

For \( j \neq k \), Lemma 1.16 implies

\[-\frac{\nu_j}{2} \int_{-j+\delta}^{-(j-1)\delta} \int_{t+\theta}^t \ddot{x}(s) \dddot{S}_j \ddot{x}(s) ds \leq \]

\[
\begin{bmatrix}
\ddot{S}_j x(t) \\
\int_{t-(j-1)\delta}^{t-t(\tau(t))} x(s) ds \\
\int_{t-j\delta}^{t-\tau(t)} x(s) ds
\end{bmatrix}
\begin{bmatrix}
-2\dddot{S}_j & \dddot{S}_j & \dddot{S}_j \\
* & -\dddot{S}_j & 0 \\
* & * & -\dddot{S}_j
\end{bmatrix}
\begin{bmatrix}
\int_{t-(j-1)\delta}^{t-t(\tau(t))} x(s) ds \\
\int_{t-j\delta}^{t-\tau(t)} x(s) ds
\end{bmatrix}.
\]  (6.25)

For \( q \neq k' \), Lemma 1.16 gives

\[-\frac{\eta_q}{2} \int_{t-q\delta}^{t-(q-1)\delta} \int_{t+\theta}^t \ddot{x}(s) \dddot{S}_q \ddot{x}(s) ds \leq \]

\[
\begin{bmatrix}
\ddot{S}_q x(t) \\
\int_{t-(q-1)\delta}^{t-t(\tau(t))} x(s) ds \\
\int_{t-q\delta}^{t-\tau(t)} x(s) ds
\end{bmatrix}
\begin{bmatrix}
-\dddot{S}_q & \dddot{S}_q & \dddot{S}_q \\
* & -\dddot{S}_q & 0 \\
* & * & -\dddot{S}_q
\end{bmatrix}
\begin{bmatrix}
\int_{t-(q-1)\delta}^{t-t(\tau(t))} x(s) ds \\
\int_{t-q\delta}^{t-\tau(t)} x(s) ds
\end{bmatrix}.
\]  (6.26)

From Assumption 6.1, it is well known that for any \( n \times n \) positive diagonal matrices \( U \) and \( V \), the following inequalities hold:

\[
\begin{bmatrix}
x(t) \\
g(x(t))
\end{bmatrix}
\begin{bmatrix}
F_U & -F_2 U \\
-F_2 U & U
\end{bmatrix}
\begin{bmatrix}
x(t) \\
g(x(t))
\end{bmatrix} \leq 0, 
\]  (6.27)

\[
\begin{bmatrix}
x(t-t(\tau(t))) \\
g(x(t-t(\tau(t))))
\end{bmatrix}
\begin{bmatrix}
F_1 V & -F_2 V \\
-F_2 V & V
\end{bmatrix}
\begin{bmatrix}
x(t-t(\tau(t))) \\
g(x(t-t(\tau(t))))
\end{bmatrix} \leq 0. 
\]  (6.28)

The following equation is true for any matrices \( R_1 \) and \( R_2 \) with appropriate dimensions:

\[
2[R_1 \ddot{x}(t) + R_2 x'(t)][-\ddot{x}(t) - (A + \Delta A(t)) x(t - \sigma(t)) + (B + \Delta B(t)) g(x(t))
\]

\[
+ (C + \Delta C(t)) g(x(t - t(\tau(t)))) + (E + \Delta E(t)) \dot{x}(t - h(t)) + u(t)] = 0. 
\]  (6.29)
Let $e^T(t) = \left[ \hat{\Gamma}_1^T, \hat{\Gamma}_2^T, \hat{\Gamma}_3^T, \hat{\Gamma}_4^T, \hat{\Gamma}_5^T, \hat{\Gamma}_6^T, \hat{\Gamma}_7^T \right]$, where

\begin{align*}
\hat{\Gamma}_1^T &= \left[ x^T(t) \ x^T(t+\delta) \ x^T(t+2\delta) \ \cdots \ x^T(t-m\delta) \ x^T(t-\tau(t)) \right], \\
\hat{\Gamma}_2^T &= \left[ g^T(x(t)) \ g^T(x(t+\delta)) \ g^T(x(t+2\delta)) \ \cdots \ g^T(x(t-m\delta)) \ g^T(x(t-\tau(t))) \right], \\
\hat{\Gamma}_3^T &= \left[ x^T(t) \ \dot{x}^T(t+\rho) \ \dot{x}^T(t+2\rho) \ \cdots \ \dot{x}^T(t-l\rho) \ \dot{x}^T(t-h(t)) \right], \\
\hat{\Gamma}_4^T &= \left[ x^T(t-\rho) \ x^T(t+2\rho) \ \cdots \ x^T(t-l\rho) \ x^T(t-h(t)) \right], \\
\hat{\Gamma}_5^T &= \left[ \int_{t-\delta}^{t} x^T(s)ds \int_{t-2\delta}^{t} x^T(s)ds \ \cdots \ \int_{t-k\delta}^{t} x^T(s)ds \right], \\
\hat{\Gamma}_6^T &= \left[ \int_{t-\rho}^{t} x^T(s)ds \int_{t-2\rho}^{t} x^T(s)ds \ \cdots \ \int_{t-k\rho}^{t} x^T(s)ds \right], \\
\hat{\Gamma}_7^T &= \left[ u^T(t) \int_{t-\sigma(t)}^{t} x^T(s)ds \ x^T(t-\sigma(t)) \ x^T(t-\sigma) \right].
\end{align*}

From (6.9) to (6.29), after simple algebraic manipulations, one can obtain

\begin{equation}
V(t,x(t)) - y^T(t)Qy(t) - 2y^T(t)Su(t) - u^T(t)(\mathcal{R} - \gamma T)u(t) \leq e^T(t)\Omega e(t). \tag{6.30}
\end{equation}

Suppose $\Omega < 0$, it is easy to get

\begin{equation}
y^T(t)Qy(t) + 2y^T(t)Su(t) + u^T(t)Ru(t) \geq V(t,x(t)) + \gamma u^T(t)u(t). \tag{6.31}
\end{equation}

Integrating (6.31) from 0 to $t_p$, under zero initial condition it follows that

\begin{equation}
\mathcal{G}(y,u,t_p) \geq \gamma(u,u)_{t_p} + V(t_p,x(t_p)) - V(0,x(0)) \geq \gamma(u,u)_{t_p} \tag{6.32}
\end{equation}
for all $t_p \geq 0$. Therefore, the neural network (6.1) is strictly $(Q, S, R)$-$\gamma$-dissipative in the sense of Definition 1.3. This completes the proof.

Considering all possibilities of $k$ in the set $\{1, 2, \ldots, m\}$ and $k'$ in the set $\{1, 2, \ldots, l\}$, the condition (6.7) holds for any $k \in \{1, 2, \ldots, m\}$ and $k' \in \{1, 2, \ldots, l\}$, which completes the proof.

**Remark 6.1.** For the system (6.5), if the functions $\sigma(t)$, $\tau(t)$ and $h(t)$ are non differentiable continuous functions satisfying the following conditions

$$0 \leq \sigma(t) \leq \sigma < \infty, 0 < \tau(t) \leq \tau < \infty, 0 < h(t) \leq h < \infty, \forall t \geq 0,$$

then in order to derive the dissipativity condition, modify the LKF (6.8) as

$$\dot{\tilde{V}}(t, x(t)) = V(t, x(t)), \text{ with } Z_3 = 0, \left[ \begin{array}{c} Q_0 \\ X_0 \\ S_0 \end{array} \right] = 0 \text{ and } \tilde{T}_2 = 0. \quad (6.33)$$

Now, by employing the LKF candidate (6.33) and using similar proof of Theorem 6.1, the following corollary is obtained immediately.

**Corollary 6.1.** For given integers $m > 1$ and $l > 1$ and given positive scalars $\sigma$, $\tau$ and $h$, the system (6.5) is strictly $(Q, S, R)$-$\gamma$-dissipative in the sense of Definition 1.3, if there exist positive definite matrices $P$, $Z_i$, $i = 1, 2$, $\hat{T}_i$, $i = 1, 3$, $T_j$, $j = 1, 2, \cdots, m$, $W_q$, $N_q$, $Q_q$, $q = 1, 2, \cdots, l$, appropriately dimensioned matrices $Y_1, Y_2$, $R_1, R_2$, positive diagonal matrices $U, V$, and scalars $\gamma > 0$, $\epsilon_i > 0$, $i = 1, 2, 3, \ldots, 8$, such that the following LMI holds for any $k \in \{1, 2, \ldots, m\}$ and $k' \in \{1, 2, \ldots, l\}$,

$$\bar{\Omega} = \left[ \begin{array}{c} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{array} \right] < 0, \text{ for } k' \neq 1, \hat{\Omega} = \left[ \begin{array}{c} \hat{\Omega}_1 \\ \Omega_2 \\ \Omega_3 \end{array} \right] < 0, \text{ for } k' = 1, \quad (6.34)$$

where the coefficients are same as in Theorem 6.1.

**Proof.** Consider the LKF (6.33) to deal with the dissipativity of system (6.5).
The derivative of $V(t, x(t))$ in (6.33) along the trajectory of (6.5) yields

$$
\dot{V}_1(t, x(t)) = 2 \left[ x(t) - A \int_{t-\sigma(t)}^t x(s) ds \right]^T P \left[ \dot{x}(t) - Ax(t) + Ax(t - \sigma(t)) \right]
$$

\[
\leq 2x^T(t)P\dot{x}(t) - 2x^T(t)PAx(t) + 2x^T(t)PAx(t - \sigma(t))
\]

\[
-2\int_{t-\sigma(t)}^t x^T(s)dsA^T\dot{x}(t) + 2\int_{t-\sigma(t)}^t x^T(s)dsA^TPAx(t)
\]

\[
-2\int_{t-\sigma(t)}^t x^T(s)dsA^TPAx(t - \sigma(t)).
\] (6.35)

Using (6.35) in place of (6.8) and following the same argument as in Theorem 6.1, the dissipativity results as in (6.34) can be obtained for the system (6.5).

**Remark 6.2.** In this section, consider the neural network (6.5) with time-varying parametric uncertainties. By choosing $\sigma(t) = 0$ and $h(t) = 0$, the system (6.5) can be rearranged into the following form as described in [89]:

$$
\dot{x}(t) = -(A + \Delta A(t))x(t) + (B + \Delta B(t))g(x(t)) + (C + \Delta C(t))g(x(t - \tau(t))) + u(t).
$$

(6.36)

Now, by using the similar proof of Theorem 6.1, the following corollary is obtained immediately.

**Corollary 6.2.** For given integers $m > 1$ and given positive scalars $\tau$ and $\mu$, the system (6.36) is strictly dissipative in the sense of Definition 1.3, if there exist positive matrices $P$, $\hat{T}_1$, $\left[ \begin{array}{cc} Q_0 & X_0 \\ * & S_0 \end{array} \right]$, $\hat{T}_j = \left[ \begin{array}{cc} T_j & X_j \\ * & S_j \end{array} \right]$, $H_j$, $S_j$, $j = 1, 2, \ldots, m$, appropriately dimensioned matrices $R_1, R_2$, positive diagonal matrices $U$, $V$, and scalars $\gamma > 0$, $\epsilon_i > 0$, $i = 1, 2, \ldots, 6$, such that the following LMI holds for any $k \in \{1, 2, \ldots, m\}$,

$$
\hat{\Lambda} = \begin{bmatrix}
\Omega_{11} + \Omega^k_{11} & \Omega_{12} & \Lambda_{13} & \Omega_{15} & \Lambda_{17} \\
* & \Omega_{22} & \Lambda_{23} & 0 & \Lambda_{27} \\
* & * & \Lambda_{33} & 0 & \Lambda_{37} \\
* & * & * & \Omega_{55} & 0 \\
* & * & * & * & \Lambda_{77}
\end{bmatrix} < 0,
$$

(6.37)

where

$$
\Lambda_{13} = \begin{bmatrix} P - R_2 \\ 0 \end{bmatrix}, \Lambda_{17} = \begin{bmatrix} R_2 \\ 0 \end{bmatrix}, \Lambda_{23} = B^T R_1^T, \Lambda_{27} = -S^T, \Lambda_{33} = \sum_{j=1}^m \bar{\delta}^2 H_j + \frac{\nu^2}{4} S_j,
$$

$$
-\Lambda_{13} R_1^T, \Lambda_{37} = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \Lambda_{77} = -(R - \gamma I)
$$
Proof. The proof is same as that of Theorem 6.1 and hence it is omitted.

Remark 6.3. Theorem 6.1 provides a robust dissipativity analysis of neural network (6.5). Now, specialize Theorem 6.1 to obtain the passivity analysis of neural network (6.5) in the absence of uncertainty parameters. Hence, the system (6.5) will be reduced to

\[
\dot{x}(t) = -Ax(t - \sigma) + Bg(x(t)) + Cg(x(t - \tau(t))) + u(t). 
\] (6.38)

By choosing \( Q = 0 \), \( S = I \) and \( R = 2\gamma I \), \( \sigma(t) = \sigma \), and \( E(t) = 0 \), the following corollary can be obtained from Theorem 6.1.

Corollary 6.3. For given integers \( m > 1 \) and given positive scalars \( \sigma \), \( \tau \) and \( \mu \) the system (6.38) is passive in the sense of Definition 1.3, if there exist positive definite matrices \( P \), \( Z_i \), \( i = 1, 2, 3 \), \( \tilde{T}_1 \), \( \tilde{T}_j \), \( j = 1, 2, \ldots, m \), positive diagonal matrices \( U \), \( V \), and scalars \( \gamma > 0 \), such that the following LMI holds for any \( k \in \{1, 2, \ldots, m\} \),

\[
\dot{\Omega} = \begin{bmatrix}
\Omega_{11} + \Omega_{11}^k & \Omega_{12} & \Omega_{15} & \Omega_{17} \\
* & \Omega_{22} & 0 & \Omega_{27} \\
* & * & \Omega_{55} & \Omega_{57} \\
* & * & * & \Omega_{77}
\end{bmatrix} < 0, 
\] (6.39)

where

\[
\dot{\Omega}_{11} = \begin{bmatrix}
\dot{\phi}_1 & H_1 & 0 & \ldots & 0 & 0 & 0 \\
* & \dot{\phi}_2 & H_2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
* & * & \ldots & \dot{\phi}_m & H_m & 0 \\
* & * & \ldots & * & \dot{\phi}_{m+1} & 0 \\
* & * & \ldots & * & * & \dot{\phi}
\end{bmatrix},
\]

\[
\dot{\Omega}_{12} = \begin{bmatrix}
\tilde{\omega}_1 \\
\tilde{\omega}_2 \\
\beta
\end{bmatrix}, \dot{\Omega}_{17} = \begin{bmatrix}
\zeta_1 \\
\overset{\cdot}{\zeta}_1 \\
0
\end{bmatrix}, \dot{\Omega}_{22} = \begin{bmatrix}
\dot{\Phi}_1 & \dot{\Phi}_2 \\
* & \dot{\alpha}
\end{bmatrix}, \dot{\Omega}_{27} = \begin{bmatrix}
\dot{\Lambda}_1 \\
0 \\
\dot{\Lambda}_2
\end{bmatrix},
\]

and the remaining coefficients are same as in Theorem 6.1.
In the case of Remark 6.4, and the remaining coefficients are same as in Theorem 6.1.

\[
\dot{\phi}_j = \begin{cases} 
-PA - A^TP + \sigma^2T_1 + T_1 - H_1 + \sigma^2Z_1 + Z_2 + Z_3 + Q_0, \\
- \sum_{j=1, j \neq k}^{m} \delta_j \tilde{S}_j - 2\delta_k \tilde{S}_k - F_1 U + \sum_{j=1}^{m} A^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j)A, & j = 1 \\
T_j - T_{j-1} - H_j - H_{j-1}, & j = 2, 3, \ldots, m \\
-T_m - H_m, & j = m + 1
\end{cases}
\]

\[
\dot{\beta}_j = \begin{cases} 
X_1 + X_0 + F_2 U + PB - \sum_{j=1}^{m} A^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j)B, & j = 1 \\
X_j - X_{j-1}, & j = 2, 3, \ldots, m; \\
-X_m, & j = m + 1
\end{cases}
\]

\[
\dot{\alpha}_j = \begin{cases} 
S_1 + S_0 - U \\
+ \sum_{j=1}^{m} B^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j)B, & j = 1 \\
S_j - S_{j-1}, & j = 2, 3, \ldots, m \\
-S_m, & j = m + 1
\end{cases}
\]

\[
\dot{\phi} = -F_1 V - Q_0(1 - \mu), \quad \dot{\beta} = PC - \sum_{j=1}^{m} A^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j)C, \quad \dot{\beta} = F_2 V - X_0(1 - \mu),
\]

\[
\dot{\alpha} = -V - S_0(1 - \mu) + \sum_{j=1}^{m} C^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j)C, \quad \dot{\alpha} = \sum_{j=1}^{m} B^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j)C,
\]

\[
\dot{\omega}_1 = \text{diag}\{\dot{\beta}_1, \dot{\beta}_2, \ldots, \dot{\beta}_{m+1}\}, \quad \dot{\omega}_2 = [\dot{\beta}^T \ 0_{n,n(n)}]^T, \quad \dot{\beta}_1 = \text{diag}\{\dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\alpha}_{m+1}\},
\]

\[
\Phi_2 = [\dot{\alpha}^T \ 0]^T, \quad \zeta_1 = \left[ P - \sum_{j=1}^{m} A^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j) \quad PA \quad 0 \right]^T,
\]

\[
\dot{\Lambda}_1 = \left[ -I + \sum_{j=1}^{m} B^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j) - PB \quad 0 \right], \quad \dot{\Lambda}_2 = \left[ \sum_{j=1}^{m} C^T(\frac{\nu_j^2}{4}S_j + \tilde{\delta}^2 H_j) - PC \quad 0 \right]^T
\]

and the remaining coefficients are same as in Theorem 6.1.

**Remark 6.4.** In the case of \(\sigma = 0\), the system (6.38) is reduced to the following neural

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For given integers $U$ and $Q$ (6.40) is passive in the sense of Definition 1.3, if there exist positive definite matrices $P$, $\Pi$, and $\tau$, networks:

$$
\dot{x}(t) = -Ax(t) + bg(x(t)) + Cg(x(t - \tau(t))) + u(t),
$$

where $\tau(t)$ is assumed to satisfy $0 \leq \tau(t) \leq \tau < \infty$ with $\dot{\tau}(t) \leq \mu < \infty$. By using Theorem 6.1, one can obtain the passivity criterion for the above neural networks (6.40) as in the following corollary.

**Corollary 6.4.** For given integers $m > 1$ and given positive scalars $\tau$ and $\mu$, the system (6.40) is passive in the sense of Definition 1.3, if there exist positive definite matrices $P$, $\Pi$, and $\tau$, and given positive scalars $\gamma > 0$, and scalar $\gamma > 0$, such that the following LMI holds for any $k \in \{1, 2, \ldots, m\}$,

$$
\Pi = \begin{bmatrix}
\Pi_{11} + \Omega_{11}^k & \Pi_{12} & \Omega_{15} & \Pi_{17} \\
* & \Pi_{22} & 0 & \Pi_{27} \\
* & * & \Omega_{55} & \Pi_{57} \\
* & * & * & \Pi_{77}
\end{bmatrix} < 0,
$$

$$
\Pi_{11} = \begin{bmatrix}
\gamma_1 & H_1 & 0 & \ldots & 0 & 0 & 0 \\
* & \gamma_2 & H_2 & \ldots & 0 & 0 & 0 \\
* & * & \gamma_m & H_m & 0 \\
* & * & * & \ldots & * & \gamma_m+1 & 0 \\
* & * & * & \ldots & * & * & \gamma
\end{bmatrix},
$$

$$
\Pi_{12} = \begin{bmatrix}
\tilde{T}_1 \\
0 \\
\tilde{T}_2 \\
\vdots
\end{bmatrix},
$$

$$
\Pi_{17} = \begin{bmatrix}
P^T - \sum_{j=1}^m A(\frac{\nu_j^2}{4} \tilde{S}_j + \tilde{\delta}_j H_j^T) & 0 \\
\Sigma_{j=1}^m B \tilde{S}_j + \tilde{\delta}_j \Sigma_{j=1}^m H_j & 0 & \Sigma_{j=1}^m C \tilde{S}_j + \tilde{\delta}_j \Sigma_{j=1}^m \text{diag} \{\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_m, \tilde{T}_m+1\}
\end{bmatrix},
$$

$$
\tilde{\psi}_1 = \text{diag} \{\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_m+1\},
$$

$$
\tilde{\psi}_2 = \begin{bmatrix}
\tilde{\omega}_1 \\
\vdots \\
\tilde{\omega}_m+1
\end{bmatrix},
$$

$$
\gamma_j = \begin{cases}
-PA - A^T P + \tau^2 \tilde{T}_1 + T_1 - H_1 + Q_0, \\
- \sum_{j=1, j \neq k}^m \tilde{\delta}_j \tilde{S}_j - 2 \tilde{\delta}_k \tilde{S}_k - F_1 U + \sum_{j=1}^m A^T(\frac{\nu_j^2}{4} \tilde{S}_j + \tilde{\delta}_j H_j)A, & j = 1 \\
T_j - T_{j-1} - H_j - H_{j-1}, & j = 2, 3, \ldots, m \\
-T_m - H_m, & j = m + 1
\end{cases}
$$

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\[\chi_j = \begin{cases} X_1 + X_0 + F_2 U + PB - \sum_{j=1}^{m} A^T \left( \frac{\nu_j^2}{4} S_j + \tilde{\delta}_j^2 H_j \right) B, & j = 1 \\
X_j - X_{j-1}, & j = 2, 3, \ldots, m \\
-X_m, & j = m + 1 \end{cases}\]

\[\tilde{\omega}_j = \begin{cases} S_1 - U + S_0 + \sum_{j=1}^{m} B^T \left( \frac{\nu_j^2}{4} S_j + \tilde{\delta}_j^2 H_j \right) B, & j = 1 \\
S_j - S_{j-1}, & j = 2, 3, \ldots, m \\
-S_m, & j = m + 1 \end{cases}\]

\[\tilde{\xi} = -F_1 V - Q_0 (1 - \mu), \tilde{\chi} = PC - \sum_{j=1}^{m} A^T \left( \frac{\nu_j^2}{4} S_j + \tilde{\delta}_j^2 H_j \right) C, \tilde{\chi} = F_2 V - X_0 (1 - \mu),\]

\[\tilde{\omega} = -V - S_0 (1 - \mu) + \sum_{j=1}^{m} C^T \left( \frac{\nu_j^2}{4} S_j + \tilde{\delta}_j^2 H_j \right) C, \tilde{\omega} = \sum_{j=1}^{m} B^T \left( \frac{\nu_j^2}{4} S_j + \tilde{\delta}_j^2 H_j \right) C\]

and the remaining coefficients are followed from Theorem 6.1.

**Proof.** The proof follows immediately from Theorem 6.1 and hence it is omitted. \(\square\)

### 6.2.3 Numerical examples

In this section, numerical examples are provided to demonstrate the effectiveness and applicability of the proposed dissipativity and passivity results.

**Example 6.1.** Consider uncertain neural networks (6.5) with the following parameters:

\[A = \begin{bmatrix} 1.62 & 0 \\ 0 & 1.62 \end{bmatrix}, B = \begin{bmatrix} 0.5 & -0.1 \\ -0.3 & 0.1 \end{bmatrix}, C = \begin{bmatrix} -0.1 & -0.2 \\ 0.1 & 0.3 \end{bmatrix}, E = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}.\]

In this example, the following matrices are taken

\[Q = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}, \mathcal{R} = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}, \mathcal{S} = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}, \mathcal{P}_i = \mathcal{E}_i = 0.01 I, i = 0, 1, 2, 3.\]

The activation functions are assumed to be \(g_1 = g_2 = \tanh(s)\) with \(F_1^- = 0, F_1^+ = 1, F_2^- = 0, F_2^+ = 1\). Taking \(m = l = 2\) and using MATLAB LMI Toolbox in the Theorem 6.1, it is found that the neural network (6.5) is dissipative for \(h = 0.3, \tau = 0.5, \sigma = 0.1, \sigma_{\mu} = 0.1, \mu = 0.5\) and \(\varrho = 0.1\) and one can find the corresponding solution to the LMI in (6.7) as follows:
Figure 6.1: The state trajectory of system (6.5) with leakage time-varying delay $\sigma(t) = 0.1 + 0.1 \cos(t)$ for Example 6.1.

Figure 6.2: The state trajectory of system (6.5) with leakage time-varying delay $\sigma(t) = 0.6 + 0.1 \cos(t)$ for Example 6.1.

Table 6.1: Maximum allowable upper bounds of $\tau$ for different $\sigma$ for Example 6.1.

<table>
<thead>
<tr>
<th>Theorem 6.1</th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.2$</th>
<th>$\sigma = 0.3$</th>
<th>$\sigma = 0.4$</th>
<th>$\sigma = 0.5$</th>
<th>$\sigma = 0.6$</th>
<th>$\sigma = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum $\tau$</td>
<td>2.5780</td>
<td>2.5695</td>
<td>2.5569</td>
<td>2.5482</td>
<td>2.5321</td>
<td>2.4629</td>
<td>-</td>
</tr>
</tbody>
</table>
Consider the uncertain delayed neural networks as discussed in [89] with the neural network (6.40) as described in [99, 113, 119, 121]

\[ P = \begin{bmatrix} 0.5329 & -0.0091 \\ -0.0091 & 0.5240 \end{bmatrix}, Z_1 = \begin{bmatrix} 3.5559 & -0.0241 \\ -0.0241 & 3.5869 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.0955 & 0.0023 \\ 0.0023 & 0.0949 \end{bmatrix}, \\
Z_3 = \begin{bmatrix} 0.8368 & -0.0026 \\ -0.0026 & 0.8203 \end{bmatrix}, \hat{T}_1 = \begin{bmatrix} 0.2428 & 0.0040 \\ 0.0040 & 0.2418 \end{bmatrix}, \hat{T}_2 = \begin{bmatrix} 0.0392 & -0.0003 \\ -0.0003 & 0.0383 \end{bmatrix}, \]

\[ \gamma = 4.3329, \epsilon_1 = 0.5288, \epsilon_2 = 0.5293, \epsilon_3 = 0.5293, \epsilon_4 = 0.5284, \epsilon_5 = 0.5285, \epsilon_6 = 0.5291, \epsilon_7 = 0.5290, \epsilon_8 = 0.5281. \]

Maximum allowable upper bounds \( \tau \) for different values of \( \sigma \) have been obtained as in Table 6.1 by fixing the corresponding values of \( h = 2.1, \rho = \frac{21}{5}, \sigma, = \mu = \varrho = 0.1 \). Therefore, by Theorem 6.1, the model (6.5) with above given parameters is dissipative in the sense of Definition 1.3. By considering \( u(t) = 0, \tau(t) = 2.4695 + 0.1 \sin(t), h(t) = 2 + 0.1 \sin(t) \) and \( \phi(t) = [2 - 3]T \), the stability behavior of the neural networks (6.5) is depicted in Figure 6.1. Figure 6.2 shows that the neural network (6.5) is unstable for \( \sigma(t) = 0.1 + 2 \cos(t), \tau(t) = 2.3596 + 0.1 \sin(t) \) and \( h(t) = 2 + 0.1 \sin(t) \).

**Example 6.2.** Consider the uncertain delayed neural networks as discussed in [89] with the parameters:

\[ A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, C = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, N_1 = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & -0.1 \end{bmatrix}, \\
N_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, N_3 = \begin{bmatrix} -0.1 & -0.1 \\ 0 & -0.1 \end{bmatrix}, \bar{H}_i = \bar{F}_i = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix}, i = 1, 2, 3, \]

\[ F_1 = \text{diag}\{0.1, 0.1\}, F_2 = \text{diag}\{0.2, 0.2\}. \]

In this example, choose

\[ Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, R = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}, S = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}. \]

The neuron activation functions are assumed to be \( g_i(x_i) = 0.5(\|x_i + 1\| - |x_i - 1|), i = 1, 2 \).

For this example, in [89], it was shown that \( \mu \) is restricted to less than 1 (i.e. \( \mu < 1 \)). By using Corollary 6.2 of this section, one can easily see that the feasibility can also be obtained for \( \mu > 1 \), which indicates the method proposed in this section is more superior than that of [89].

**Example 6.3.** Consider the neural network (6.40) as described in [99, 113, 119, 121] with the following parameters

\[ A = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, B = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix}, C = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix}. \]
Figure 6.3: The state trajectory of system (6.5) with leakage time-varying delay \( \tau(t) = 1.7 + 0.9 \cos(t) \) for Example 6.3.

In this example, choose \( g_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|), \) \( i = 1, 2. \) By using Corollary 6.4 of this section, the upper bounds of the delay are computed and they are compared with the previous existing results [99, 113, 119, 121] in Table 6.2. From Table 6.2, it can be easily seen that the method proposed in this section is much less conservative than the corresponding method in [99, 113, 119, 121]. When \( u(t) = 0, \) one can obtain the state trajectories of the state \( x(t) \) for the delay \( \tau(t) = 0.9 + 1.2 \cos(t) \) which is shown in Figure 6.3.

Remark 6.5. In [99, 113, 119, 121], the LKF contains the integral term as
\[
\int_{t-\tau}^{t} x(\alpha)^T Q_2 x(\alpha) d\alpha,
\]
where the delay interval is divided into \( m \) equal segments and different LKF are chosen in each segments such as
\[
\sum_{j=1}^{m} \int_{-(j-1)\delta}^{-(j-1)\delta} \left[ x(t+s) g(x(t+s)) \right]^T \bar{T}_j \left[ x(t+s) g(x(t+s)) \right] ds.
\]
From Example 6.3, the simulation results show that the passivity results proposed in this section have less conservatism than the result in [99, 113, 119, 121]. Thus, the presented method architecture discussed in this section is much more general and desirable than the results presented in [99, 113, 119, 121].

Example 6.4. Consider the neural network (6.38) as described in [85] with the following
Table 6.2: Maximum allowable upper bounds of $\tau$ for different $\mu$ for Example 6.3.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\mu = 0.5$</th>
<th>$\mu = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[119] Theorem 1</td>
<td>0.7230</td>
<td>0.6791</td>
</tr>
<tr>
<td>[121] Theorem 1</td>
<td>1.3752</td>
<td>1.3027</td>
</tr>
<tr>
<td>[113] Corollary 3</td>
<td>1.8450</td>
<td>1.7647</td>
</tr>
<tr>
<td>[99] Corollary 4</td>
<td>1.8593</td>
<td>1.7765</td>
</tr>
<tr>
<td>Corollary 6.4</td>
<td>2.8963</td>
<td>2.6594</td>
</tr>
</tbody>
</table>

parameters

\[ A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} 1.3 & -1.7 \\ -1.6 & 1.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1.4 & 1.9 \\ 0.6 & -1.2 \end{bmatrix}, \]

\[ F_1 = \text{diag}\{0.1, 0.1\}, F_2 = \text{diag}\{0.2, 0.2\}. \]

For this example, in [85], it has been shown that infeasibility is obtained for $\sigma = 1.8$ with $\tau(t) = 2 + 0.15|\sin(7t)|$. However, by using Corollary 6.3 of this section, one can easily see that the feasibility is obtained for $\tau(t) \in [-3.45, 0]$ with $\sigma = 2.5$, which yields the method proposed in this section is more superior than the existing ones.

Remark 6.6. In [85], the double integral term $\int_{-\tau}^{0} \int_{t+\xi}^{t} \dot{x}^T(s)P_5\dot{x}(s)dsd\xi$ is used in LKF for obtaining the passivity condition of the neural networks (6.38). However, in this section, the double integral term in the LKF is chosen as $\sum_{j=1}^{m} \delta \int_{-j\delta}^{-(j-1)\delta} \int_{t+\theta}^{t} \dot{x}^T(s)H_j\dot{x}(s)dsd\theta$, which is obtained by constructing different LKF in different segments of the delay interval. From Example 6.4, it is clear that Corollary 6.3 shows that the system (6.38) is stable for larger upper bounds of the leakage delays than those used in [85].

6.3 Conclusions and future directions

In this chapter, the problem of robust dissipativity analysis has been investigated for neutral type neural networks with leakage time-varying delay using the delay decomposition technique. A new LKF is constructed by dividing the entire delay interval
into multiple segments and choosing different Lyapunov functionals to different segments in the LKF. By using such improved Lyapunov functional candidate, a delay dependent LMI approach has been developed to derive sufficient conditions under which the addressed system is dissipative and passive, where the conditions are dependent on the length of the time delays. Numerical examples have been given to illuminate the advantages and merits of the proposed theoretic results. In future, it is expected that the results proposed in this chapter can be extended to the problem of state estimation based on dissipativity theory, robust dissipativity analysis of stochastic neural networks and robust dissipativity analysis of Markovian jumping neural networks with leakage time-varying delay via delay decomposition approach.