Chapter 3

Dissipativity and passivity analysis for Markovian jump neural networks with additive delays

3.1 Introduction

NNs is of great importance to researchers in several areas for dissimilar reasons. Electrical engineers discover copious applications of neural networks in signal processing and control theory. It is being well known that these applications greatly depend on the dynamic behavior of neural networks. For example, scientists and engineers in the field of computer science find that neural networks give an idea for different problems in areas such as artificial intelligence and pattern recognition. For applied mathematicians, neural networks is an essential tool for modeling problems for which the relationship between certain variables is unknown. The research on the dynamical behavior of the neural networks has become an important topic in neural network theory.

Generally, in practical situations, signals transmitted from one point to another may experience a few segments of networks, which can possibly induce successive delays with different properties due to the variable network transmission conditions. For instance, in a
state-feedback networked control, the physical plant, controller, sensor, and actuator are located at different places and hence when signals are transmitted from one device to another, two additive time-varying delays will occur: one from sensor to controller and the other from controller to actuator. Because of the network transmission conditions, the two delays are generally time-varying with different properties. Therefore, it is of necessary to consider stability for NNs with two additive time-varying delay components. A great number of research results on additive time-delay systems exist in the recent literature (see [14, 80, 93] and the references therein).

It is well known that stochastic modeling has come to play an important role in many branches of science and industry. Furthermore, the latching phenomenon usually happens in neural networks, which can be controlled effectively by extracting finite state representations from trained network. In other words, the neural networks may have finite modes and the mode may jump from one to another at different times. The jumping between different modes can be governed by a Markov chain. This process is usually characterized as memoryless property. Thus, in the Markov chain the probability distribution of the next state depends only on the current state and not on the sequence of events that preceded it. Applications of this kind of systems can be found in manufacturing systems, networked control systems, economics systems, air intake system, and other practical systems (see references [15, 115, 116, 135]).

Moreover, in the existing literature [35, 89, 119], the dissipativity and passivity problem has been discussed by using only the double integral terms such as \( \int_{-d}^{0} \int_{t+\beta}^{t} x^T(s) S x(s) ds d\beta \). But in this chapter, the double integral terms along with the triple integral terms such as \( \frac{h^2}{2} \int_{-d_{22}}^{-d_{21}} \int_{\theta}^{t} \int_{t+\lambda}^{t+\lambda} \dot{x}^T(s) S_1 \dot{x}(s) ds d\lambda d\theta \) have been considered in the LKF for finding less conservative results over the existing ones. In addition, in order to particularize some less conservative dissipativity conditions for neural networks, several effective approaches have been proposed. To mention a few, one can refer to free-weighting matrix approach, delay decomposition approach and Jensen’s inequality. Therefore, based on the above observations, it is significant to establish some new integral inequality techniques such as the second order reciprocal convex combination technique for solving
the triple integral terms in order to get some less expensive dissipative criteria.

Motivated by the aforementioned discussions, in this chapter, the dissipativity and passivity analysis of stochastic Markovian jumping neural networks with additive time-varying delays have been investigated. By constructing a suitable LKF involving triple and quadruple integral terms and using GFL technique, together with second order reciprocal convex combination technique, some novel delay-dependent criteria for the dissipativity and passivity of the networks have been established in terms of LMIs which can be easily solved by various effective optimization algorithms. Finally, numerical examples are shown to support that the results concerned are less conservative than those of the existing ones.

3.2 Dissipativity and passivity analysis of Markovian jump neural networks with two additive time-varying delays

3.2.1 Problem description

Consider the following neural networks with Markovian jumping parameters, leakage time-varying delay and two additive time-varying delay components:

\[ \dot{x}(t) = -A_i x(t - \sigma(t)) + B_i g(x(t)) + C_i g(x(t - d_1(t) - d_2(t))) + D_i \int_{t-\rho(t)}^{t} g(x(\vartheta))d\vartheta + u(t), \]

\[ y(t) = g(x(t)), \]  

(3.1)

The system (3.1) can be equivalent to

\[ \frac{d}{dt} \left[ x(t) - A_i \int_{t-\sigma(t)}^{t} x(s)ds \right] = -A_i x(t) - A_i x(t - \sigma(t))\dot{\sigma}(t) + B_i g(x(t)) + C_i g(x(t - d_1(t) - d_2(t))) + D_i \int_{t-\rho(t)}^{t} g(x(\vartheta))d\vartheta + u(t), \]

(3.2)

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) is the state vector associated with the \( n \) neurons, \( u(t) \) is the input, \( y(t) \) is the output. The diagonal matrix
$A_i = \text{diag}\{a_{1i}, a_{2i}, \ldots, a_{ni}\}$ has positive entries $a_{ji} > 0$ ($j = 1, 2, \ldots, n$). The matrices $B_i$, $C_i$ and $D_i$ are the interconnection matrices representing the weight coefficients of the neurons. $g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \cdots, g_n(x_n(t))]^T \in \mathbb{R}^n$ is the neuron activation function. Now, for the system (3.2), the initial condition is taken as $x(t) = \phi(t)$, $t \in [-\bar{d}, 0]$, $\bar{d} = \max\{\sigma, d_{12}, d_{22}, \rho\}$.

In the neural network (3.2), the bounded functions $\sigma(t)$, $\rho(t)$, $d_1(t)$ and $d_2(t)$ represent respectively the leakage, distributed and two additive time-varying delays that are assumed to satisfy the following conditions:

$$0 \leq \sigma(t) \leq \sigma < \infty, \quad \dot{\sigma}(t) \leq \sigma_\mu < \infty, \quad 0 \leq \rho(t) \leq \rho,$$

$$0 \leq d_{11} \leq d_1(t) \leq d_{12}, \quad 0 \leq d_{21} \leq d_2(t) \leq d_{22}, \quad \dot{d}_1(t) \leq \tilde{\mu}_1 < 1, \quad \dot{d}_2(t) \leq \tilde{\mu}_2 < 1,$$

(3.3)

where $d_{12} \geq d_{11}$, $d_{21} \geq d_{22}$, $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\sigma$, $\sigma_\mu$ and $\rho$ are known constants with $d_{11}$ and $d_{21}$ not equal to zero. Here, denote $d_1 = d_{11} + d_{21}$, $d_2 = d_{12} + d_{22}$, $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_2$, $h_1 = d_{12} - d_{11}$, $h_2 = d_{22} - d_{21}$.

**Remark 3.1.** In this section, the values of $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are assumed to be less than 1. When $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are greater than or equal to 1, the fast time-varying delay case will cause problems with causality, minimality and inconsistency as indicated in [97, 127]. So this restriction is a reasonable and necessary assumption for proving the main results.

Throughout this chapter, the activation function satisfies the following assumption.

**Assumption 3.1.** The activation function $g(u)$ is bounded and satisfies

$$0 \leq \frac{g_i(\zeta_1) - g_i(\zeta_2)}{\zeta_1 - \zeta_2} \leq L_i, \quad i = 1, 2, \ldots, n.$$  

(3.4)

for any $\zeta_1, \zeta_2 \in \mathbb{R}, \zeta_1 \neq \zeta_2$, where $L_i > 0$ for $i = 1, 2, \ldots, n$. Further, $g_i(0) = 0$, $i = 1, 2, \ldots, n$. 

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3.2.2 Dissipativity criteria for Markovian jump neural networks

In this section, the dissipativity criteria for Markovian jump neural networks with additive time-varying delays is considered. One of the main issues in dissipativity criteria is how to further reduce the possible conservatism induced by the introduction of the Lyapunov functional when dealing with time delays.

**Theorem 3.1.** The neural networks (3.2) is $(Q, S, R) - \gamma$-dissipative, if there exists positive definite matrices $P_i (i \in S)$, $R_s (s = 1, 2, \cdots, 6)$, $Q_1, Q_2$, $Z$, $J_n (n = 1, 2)$, $S_q (q = 1, 2)$ and $M$, any matrices $K_f (f = 1, 2, \cdots, 6)$, $Y_1, Y_2$ and diagonal matrix $U$ and scalar $\gamma > 0$, such that the following LMIs hold for $l = 1, 2$:

$$
\Omega_l = \begin{bmatrix}
\tilde{\phi}_i^{(l)} & P_i A_i \sqrt{\sigma} & A_i^T P_i A_i \sqrt{\sigma} \\
* & -Y_1 & 0 \\
* & * & -Y_2
\end{bmatrix} < 0,
$$

(3.5)

$$
\begin{bmatrix}
J_1 & K_1 \\
* & J_1
\end{bmatrix} \geq 0,
\begin{bmatrix}
J_2 + \frac{h_2^2}{2} S_1 & K_2 \\
* & J_2 + \frac{h_2^2}{2} S_2
\end{bmatrix} \geq 0,
$$

(3.6)

$$
\begin{bmatrix}
2S_1 & 0 & K_3 & 0 \\
* & S_1 & 0 & K_4 \\
* & * & 2S_1 & 0 \\
* & * & * & S_1
\end{bmatrix} > 0,
\begin{bmatrix}
2S_2 & 0 & K_5 & 0 \\
* & S_2 & 0 & K_6 \\
* & * & 2S_2 & 0 \\
* & * & * & S_2
\end{bmatrix} > 0,
$$

(3.7)

where

$$
\tilde{\phi}_i^{(l)} = \phi_i - \Sigma_{1l}(t) \begin{bmatrix} S_1 & K_3 + K_4 \\ * & S_1 \end{bmatrix} \Sigma_{1l}(t) - \Sigma_{2l}(t) \begin{bmatrix} S_2 & K_5 + K_6 \\ * & S_2 \end{bmatrix} \Sigma_{2l}(t)
$$

with

$$
\Sigma_{1l}(t) = \begin{bmatrix}
0_{n,7n} & (d_2(t) - d_{21}) I_n & 0_{n,5n} & -I_n & 0_{n,3n} \\
0_{n,6n} & (d_{22} - d_2(t)) I_n & 0_{n,7n} & -I_n & 0_{n,2n}
\end{bmatrix},
$$

$$
\Sigma_{2l}(t) = \begin{bmatrix}
0_{n,6n} & -(d_2(t) - d_{21}) I_n & 0_{n,6n} & I_n & 0_{n,3n} \\
0_{n,8n} & -(d_{22} - d_2(t)) I_n & 0_{n,5n} & I_n & 0_{n,2n}
\end{bmatrix},
$$

$$
\phi_i = (\phi_{m,n,i})_{17 \times 17},
$$

$$
\phi_{1,1,i} = -P_i A_i - A_i^T P_i + \sum_{j=1}^{N} q_{ij} P_j + R_1 + R_2 + R_3 + R_4 + R_5 + d_{11}^2 Q_1 + d_{12}^2 Q_2
$$

$$
+ \sigma^2 Z + h_2^2 J_1, \phi_{1,3,i} = P_i B_i + LU, \phi_{1,10,i} = P_i C_i, \phi_{1,11,i} = P_i,
$$

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Choose the LKF candidate as follows:

\[
\phi_{1,16,i} = A_i P_i A_i^T - \sum_{j=1}^{N} q_{ij} P_j A_j, \quad \phi_{1,17,i} = P_i D_i, \quad \phi_{2,2,i} = A_i^T C A_i + Y_1 \sigma_\mu + Y_2 \sigma_\mu
\]

\[-R_1(1 - \sigma_\mu), \quad \phi_{2,3,i} = -A_i^T C B_i, \quad \phi_{2,10,i} = -A_i^T C C_i, \quad \phi_{2,11,i} = -A_i^T C, \quad \phi_{2,17,i} = -A_i^T C D_i, \quad \phi_{3,3,i} = B_i^T C B_i + R_6 + \rho^2 M - U - Q, \quad \phi_{3,10,i} = B_i^T C C_i, \quad \phi_{3,11,i} = B_i^T - S, \quad \phi_{3,16,i} = -B_i^T P_i A_i, \quad \phi_{3,17,i} = B_i^T C D_i, \quad \phi_{4,4,i} = -R_2, \quad \phi_{5,5,i} = -R_3, \quad \phi_{6,6,i} = -\bar{R}_4(1 - \bar{\mu}_1), \quad \phi_{7,7,i} = -\bar{R}_5(1 - \bar{\mu}_2) - J_2 + K_2 + J_2 - J_2, \quad \phi_{7,8,i} = J_2 - K_2, \quad \phi_{7,9,i} = -K_2 + J_2, \quad \phi_{8,8,i} = -J_2, \quad \phi_{8,9,i} = K_2, \quad \phi_{9,9,i} = -J_2, \quad \phi_{10,10,i} = C_i^T C C_i
\]

\[-R_6(1 - \bar{\mu}_1 - \bar{\mu}_2), \quad \phi_{10,11,i} = C_i^T C, \quad \phi_{10,16,i} = -C_i^T P_i A_i, \quad \phi_{10,17,i} = C_i^T C D_i, \quad \phi_{11,11,i} = C - \mathcal{R} - \gamma I, \quad \phi_{11,16,i} = -P_i A_i, \quad \phi_{11,17,i} = D_i^T C, \quad \phi_{12,12,i} = -Q_1, \quad \phi_{13,13,i} = -Q_2, \quad \phi_{14,14,i} = -J_1, \quad \phi_{14,15,i} = -K_1, \quad \phi_{15,15,i} = -J_1, \quad \phi_{16,16,i} = -Z + \sum_{j=1}^{N} q_{ij} A_i^T P_j A_j, \quad \phi_{16,17,i} = -D_i^T P_i A_i, \quad \phi_{17,17,i} = D_i^T C D_i - M, \quad C = h_2^2 J_2 + \frac{h_3^2}{4} S_1 + \frac{h_4^2}{4} S_2, \quad L = \text{diag}\{l_1, l_2, \cdots, l_n\}
\]

and the remaining terms of \( \phi_{m,n,i} \) are zero.

**Proof.** Choose the LKF candidate as follows:

\[
V(t, x(t), i) = \sum_{i=1}^{7} V_i(t, x(t), i), \quad (3.8)
\]

where

\[
V_1(t, x(t), i) = \left[ x(t) - A_i \int_{t-\sigma(t)}^{t} x(s) ds \right]^T P_i \left[ x(t) - A_i \int_{t-\sigma(t)}^{t} x(s) ds \right],
\]

\[
V_2(t, x(t), i) = \int_{t-\sigma(t)}^{t} x^T(s) R_1 x(s) ds + \int_{t-d_{11}}^{t} x^T(s) R_2 x(s) ds + \int_{t-d_{12}}^{t} x^T(s) R_3 x(s) ds
\]

\[
+ \int_{t-d_1(t)}^{t} x^T(s) R_4 x(s) ds + \int_{t-d_2(t)}^{t} x^T(s) R_5 x(s) ds
\]

\[
+ \int_{t-d_1(t)-d_2(t)}^{t} g^T(x(s)) R_6 g(x(s)) ds,
\]

\[
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\]
\[ V_3(t, x(t), i) = d_{11} \int_{-d_{11}}^{0} \int_{t+\theta}^{t} x^T(s)Q_1 x(s) ds d\theta + d_{12} \int_{-d_{12}}^{0} \int_{t+\theta}^{t} x^T(s)Q_2 x(s) ds d\theta, \]
\[ V_4(t, x(t), i) = \sigma \int_{-\sigma}^{0} \int_{t+\theta}^{t} x^T(s)Z x(s) ds d\theta, \]
\[ V_5(t, x(t), i) = h_2 \int_{-d_{21}}^{0} \int_{t+\theta}^{t} x^T(s)J_1 x(s) ds d\theta + h_2 \int_{-d_{22}}^{0} \int_{t+\theta}^{t} x^T(s)J_2 \dot{x}(s) ds d\theta, \]
\[ V_6(t, x(t), i) = \frac{h_2^2}{2} \int_{-d_{22}}^{0} \int_{-d_{22}}^{0} \int_{t+\lambda}^{t} \dot{x}^T(s)S_1 \dot{x}(s) ds d\lambda d\theta, \]
\[ V_7(t, x(t), i) = \rho \int_{-\rho}^{0} \int_{t+\theta}^{t} g^T(x(s)) Mg(x(s)) ds d\theta. \]

Setting \( \zeta = (d_2(t) - d_{21}) / h_2 \) and \( \omega = (d_{22} - d_2(t)) / h_2 \) and applying Jensen inequality lemma to the weak infinitesimal generator \( L \) of the random process \( \{x(t), i, t \geq 0\} \), one can obtain
\[
\mathbb{L}V(t, x(t), i) = \sum_{i=1}^{7} \mathbb{L}V_i(t, x(t), i), \tag{3.9}
\]
where
\[
\mathbb{L}V_1(t, x(t), i) = 2 \left[ x(t) - A_i \int_{t-\sigma(t)}^{t} x(s) ds \right]^T P_i \left[ -A_i x(t) - A_i x(t - \sigma(t)) \dot{\sigma}(t) \right.
+ B_i g(x(t)) + C_i g(x(t - d_1(t) - d_2(t))) + D_i \int_{t-\rho(t)}^{t} g(x(s)) ds + u(t) \bigg]
+ \left[ x(t) - A_j \int_{t-\sigma(t)}^{t} x(s) ds \right]^T \sum_{j=1}^{N} q_{ij} P_j \left[ x(t) - A_j \int_{t-\sigma(t)}^{t} x(s) ds \right] = -2x^T(t) P_i A_i x(t) + x^T(t) P_i A_i Y_1^{-1} A_i^T P_i \sigma, \mu x(t) + x^T(t - \sigma(t)) Y_1 x(t - \sigma(t)) \sigma, \mu + 2x^T(t) P_i B_i g(x(t)) + 2x^T(t) P_i C_i g(x(t - d_1(t) - d_2(t))) + 2x^T(t) P_i D_i \int_{t-\rho(t)}^{t} g(x(s)) ds + 2x^T(t) P_i u(t) + 2 \int_{t-\sigma(t)}^{t} x^T(s) ds A_i^T P_i A_i x(t) + \int_{t-\sigma(t)}^{t} x^T(s) ds A_i^T P_i A_i Y_2^{-1} A_i P_i A_i^T \sigma, \mu \int_{t-\sigma(t)}^{t} x(s) ds.
\]
\[+ x^T(t - \sigma(t))Y_2\sigma_p x(t - \sigma(t)) - 2 \int_{t-\sigma(t)}^{t} x^T(s)ds A_i^T P_i B_i g(x(t))
- 2 \int_{t-\sigma(t)}^{t} x^T(s)ds A_i^T P_i C_i g(x(t - d_1(t) - d_2(t)))
- 2 \int_{t-\sigma(t)}^{t} x^T(s)ds A_i^T P_i D_i \int_{t-\rho(t)}^{t} g(x(s))ds - 2 \int_{t-\sigma(t)}^{t} x^T(s)ds A_i^T P_i u(t)
+ \left[ x(t) - A_j \int_{t-\sigma(t)}^{t} x(s)ds \right] \sum_{j=1}^{N} q_{ij} P_j \left[ x(t) - A_j \int_{t-\sigma(t)}^{t} x(s)ds \right],\]
\[(3.10)\]

\[LV_2(t, x(t), i) \leq x^T(t) (R_1 + R_2 + R_3 + R_4 + R_5) x(t) + g(x(t)) R_6 g(x(t))
- x^T(t - \sigma(t)) R_1 x(t - \sigma(t))(1 - \sigma_p) - x^T(t - d_{11}) R_2 x(t - d_{11})
- x^T(t - d_{12}) R_3 x(t - d_{12}) - x^T(t - d_{11}(t)) R_4 x(t - d_{11}(t))(1 - \mu_1)
- x^T(t - d_{21}(t)) R_5 x(t - d_{21}(t))(1 - \mu_2)
- g^T(x(t - d_{11}(t) - d_{21}(t))) R_6 g(x((t - d_{11}(t) - d_{21}(t)))(1 - \mu_1 - \mu_2),\]
\[(3.11)\]

\[LV_3(t, x(t), i) = x^T(t) [d_{11}^2 Q_1 + d_{12}^2 Q_2] x(t) - d_{11} \int_{t-d_{11}}^{t} x^T(s)Q_1 x(s)ds
- d_{12} \int_{t-d_{12}}^{t} x^T(s)Q_2 x(s)ds
\leq x^T(t) [d_{11}^2 Q_1 + d_{12}^2 Q_2] x(t) - \int_{t-d_{11}}^{t} x^T(s)ds Q_1 \int_{t-d_{11}}^{t} x(s)ds
- \int_{t-d_{12}}^{t} x^T(s)ds Q_2 \int_{t-d_{12}}^{t} x(s)ds,\]
\[(3.12)\]

\[LV_4(t, x(t), i) \leq \sigma^2 x^T(t) Z x(t) - \int_{t-\sigma(t)}^{t} x^T(s)ds Z \int_{t-\sigma(t)}^{t} x(s)ds,\]
\[(3.13)\]

\[LV_5(t, x(t), i) = h_2^2 x^T(t) J_1 x(t) + h_2^2 \dot{x}^T(t) J_2 \dot{x}(t) - h_2 \int_{t-d_{21}}^{t} x^T(s)J_1 x(s)ds
- h_2 \int_{t-d_{22}}^{t} \dot{x}^T(s) J_2 \dot{x}(s)ds
= h_2^2 x^T(t) J_1 x(t) + h_2^2 \dot{x}^T(t) J_2 \dot{x}(t) - h_2 \int_{t-d_{21}}^{t} x^T(s)J_1 x(s)ds\]

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\[
- h_2 \int_{t-d_2(t)}^{t-d_2(t)} x^T(s)J_1 x(s) \, ds - h_2 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s)J_2 \dot{x}(s) \, ds \\
- h_2 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s)J_2 \dot{x}(s) \, ds
\]
\[
\leq h_2^2 \dot{x}^T(t)J_1 x(t) + h_2^2 \dot{x}^T(t)J_2 \dot{x}(t) - \frac{1}{\varsigma} \int_{t-d_2(t)}^{t-d_2(t)} x^T(s) \, ds J_1 \int_{t-d_2(t)}^{t-d_2(t)} x(s) \, ds \\
- \frac{1}{\omega} \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s) \, ds J_1 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}(s) \, ds \\
- \frac{1}{\omega} \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s) \, ds J_2 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}(s) \, ds,
\]
\[
\mathbb{L}V_6(t,x(t),i) = \frac{h_2^4}{4} \dot{x}^T(t)S_1 \dot{x}(t) - \frac{h_2^4}{2} \int_{-d_2(t)}^{-d_1(t)} \int_{t+\theta}^{t+d_2(t)} \dot{x}^T(s)S_1 \dot{x}(s) \, ds \, d\theta + \frac{h_2^4}{4} \dot{x}^T(t)S_2 \dot{x}(t)
\]
\[
- \frac{h_2^4}{2} \int_{-d_2(t)}^{-d_1(t)} \int_{t+\theta}^{t+d_2(t)} \dot{x}^T(s)S_2 \dot{x}(s) \, ds \, d\theta
\]
\[
= \frac{h_2^4}{4} \dot{x}^T(t)S_1 \dot{x}(t) - \frac{h_2^4}{2} (d_2(t) - d_1(t)) \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s)S_1 \dot{x}(s) \, ds
\]
\[
- \frac{h_2^4}{2} \int_{-d_2(t)}^{-d_1(t)} \int_{t+\theta}^{t+d_2(t)} \dot{x}^T(s)S_2 \dot{x}(s) \, ds \, d\theta
\]
\[
- \frac{h_2^4}{2} \int_{-d_2(t)}^{-d_2(t)} \int_{t+\theta}^{t+d_2(t)} \dot{x}^T(s)S_1 \dot{x}(s) \, ds \, d\theta + \frac{h_2^4}{4} \dot{x}^T(t)S_2 \dot{x}(t)
\]
\[
\leq \frac{h_2^4}{4} \dot{x}^T(t)S_1 \dot{x}(t) - \frac{h_2^4}{2} \omega \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s) \, ds S_1 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}(s) \, ds \\
- \frac{1}{\varsigma} \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}^T(s) \, ds S_1 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}(s) \, ds \, d\theta \\
- \frac{1}{\omega^2} \int_{t-d_2(t)}^{t-d_2(t)} \int_{t+\theta}^{t+d_2(t)} \dot{x}^T(s) \, ds \, d\theta S_1 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}(s) \, ds \\
- \frac{1}{\omega^2} \int_{t-d_2(t)}^{t-d_2(t)} \int_{t+\theta}^{t+d_2(t)} \dot{x}^T(s) \, ds \, d\theta S_2 \int_{t-d_2(t)}^{t-d_2(t)} \dot{x}(s) \, ds \\
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respectively. So inequalities (3.17) and (3.18) still hold.

From Lemma 1.9, if there exist matrices $K_1$ and $K_2$ such that the inequalities in (3.6) hold, then the following inequalities will be obtained:

$$\mathcal{L}V_\\mathcal{T}(t, x(t), i) \leq \rho^2 g^T(x(t)) M g(x(t)) - \int_{t-\rho(t)}^{t} g^T(x(s)) ds \int_{t-\rho(t)}^{t} g(x(s)) ds,$$

(3.16)

and

$$- \frac{1}{\varsigma} \int_{t-\rho(t)}^{t} x^T(s) ds J_1 \int_{t-\rho(t)}^{t} x(s) ds - \frac{\omega}{\varsigma} \int_{t-\rho(t)}^{t} x^T(s) ds J_2 \int_{t-\rho(t)}^{t} x(s) ds \leq \mathcal{L}V_\\mathcal{T}(t, x(t), i) \leq \rho^2 g^T(x(t)) M g(x(t)) - \int_{t-\rho(t)}^{t} g^T(x(s)) ds \int_{t-\rho(t)}^{t} g(x(s)) ds,$$

(3.17)

Note that if $d_2(t) = d_{21}$ or $d_2(t) = d_{22}$, then

$$\int_{t-d_{21}}^{t} x(s) ds = \int_{t-d_{21}}^{t} \dot{x}(s) ds = 0 \quad \text{or} \quad \int_{t-d_{22}}^{t} x(s) ds = \int_{t-d_{22}}^{t} \dot{x}(s) ds = 0,$$

respectively. So inequalities (3.17) and (3.18) still hold.

Similarly, by applying lemma 1.9 to (3.15), the following inequalities hold:

$$\mathcal{L}V_\\mathcal{T}(t, x(t), i) \leq \rho^2 g^T(x(t)) M g(x(t)) - \int_{t-\rho(t)}^{t} g^T(x(s)) ds \int_{t-\rho(t)}^{t} g(x(s)) ds,$$

(3.19)
and

\[- \frac{1}{\omega^2} \int_{t-d_2(t)}^{t} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta S_2 \int_{t-d_2(t)}^{t} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta \]

\[- \frac{1}{\omega^2} \int_{-d_2(t)}^{-d_2(t)} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta S_2 \int_{-d_2(t)}^{-d_2(t)} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta \]

\[\leq - \left[ \int_{t-d_2(t)}^{t} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta \right]^T \left[ \begin{array}{cc} S_2 & K_5 + K_6 \\ * & S_2 \end{array} \right] \left[ \int_{t-d_2(t)}^{t} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta \right]. \tag{3.20} \]

It should be noted that when \(d_2(t) = d_{21}\) or \(d_2(t) = d_{22}\), it is easy to see that

\[\int_{t-d_2(t)}^{t} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta = \int_{-d_2(t)}^{-d_2(t)} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta = 0 \quad \text{or} \quad \int_{t-d_2(t)}^{t} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta = \int_{-d_2(t)}^{-d_2(t)} \int_{t-d_2(t)}^{t+\theta} \dot{x}(s)dsd\theta = 0, \]

respectively. So the relations (3.19) and (3.20) still hold.

Furthermore, there exist positive diagonal matrix \(U\) such that the following inequalities hold based on Assumption 3.1:

\[-2g^T(x(t))Ug(x(t)) + 2x^T(t)ULg(x(t)) \geq 0. \tag{3.21} \]

Substituting (3.10)-(3.20) in (3.9) and then adding with (3.21), and after simple algebraic manipulations, one can obtain

\[\mathbb{E}\{LV(t, x(t), i) - y^T(t)Qy(t) - 2y^T(t)Su(t) - u^T(t)(\mathcal{R} - \gamma I)u(t)\} \leq \mathbb{E}\{\psi^T(t)\Omega^T\psi(t)\}, \tag{3.22} \]

where

\[\psi^T(t) = \begin{bmatrix} x^T(t) & x^T(t - \sigma(t)) & g^T(x(t)) & x^T(t - d_1(t)) & x^T(t - d_1(t)) \\ x^T(t - d_2(t)) & x^T(t - d_2(t)) & x^T(t - d_2(t)) & g^T(x(t - d_1(t)) - d_2(t)) & u^T(t) \end{bmatrix} \]

\[\int_{t}^{t} x^T(s)ds \int_{t-d_1(t)}^{t} x^T(s)ds \int_{t-d_2(t)}^{t} x^T(s)ds \int_{t-d_2(t)}^{t} x^T(s)ds \int_{t}^{t} g^T(x(s))ds. \]

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Thus (3.22) can be treated non-conservatively by two corresponding boundary LMIs given in (3.5): The first case is $d_2(t) = d_{21}$ and the second will be $d_2(t) = d_{22}$.

Suppose $\Omega < 0$, it is easy to get

$$
\mathbb{E}\{y^T(t)Qy(t) + 2y^T(t)Su(t) + u^T(t)Ru(t)\} \geq \mathbb{E}\{LV(t,x(t),i) + \gamma u^T(t)u(t)\}.
$$

(3.23)

Integrating (3.23) from 0 to $t_p$ and using zero initial conditions it can be proved that

$$
\mathbb{E}\{\mathcal{G}(y,u,t_p)\} \geq \mathbb{E}\{\gamma\langle u,u\rangle t_p + V(t_p,x(t_p),i) - V(0,x(0),i)\} \geq \mathbb{E}\{\gamma\langle u,u\rangle t_p\}
$$

for all $t_p \geq 0$. Therefore, the Markovian jump neural network (3.2) is strictly $(Q,S,R)\gamma$-dissipative in the sense of Definition 1.3. This completes the proof. □

**Remark 3.2.** The passivity conditions for the system (3.2) can be derived by substituting $Q = 0, S = I$ and $R = 2\gamma I$ in Theorem 3.1. In this case, the following corollary is obtained from Theorem 3.1.

**Corollary 3.1.** The neural networks (3.2) is passive in the sense of Definition 1.4, if there exists positive definite matrices $P_i(i \in S), R_s(s = 1,2,\cdots, 6), Q_1, Q_2, Z, J_n(n = 1, 2), S_q(q = 1,2), \text{ any matrices } K_f(f = 1,2,\cdots, 6), Y_1, Y_2 \text{ and diagonal matrix } U \text{ and scalar } \gamma > 0, \text{ such that the following LMIs hold for } l = 1, 2:

$$
\tilde{\Omega}_l^T = \begin{bmatrix}
\tilde{\phi}_i^{(l)} & P_iA_i\sqrt{\sigma_\mu} & A_i^TP_iA_i\sqrt{\sigma_\mu}
\end{bmatrix} < 0,
$$

(3.24)

$$
\begin{bmatrix}
J_1 & K_1 \\
* & J_1
\end{bmatrix} \geq 0,
\begin{bmatrix}
J_2 & K_2 \\
* & J_2
\end{bmatrix} \geq 0,
$$

(3.25)

$$
\begin{bmatrix}
2S_1 & 0 & K_3 & 0 \\
* & S_1 & 0 & K_4 \\
* & * & 2S_1 & 0
\end{bmatrix} > 0,
\begin{bmatrix}
2S_2 & 0 & K_5 & 0 \\
* & S_2 & 0 & K_6 \\
* & * & 2S_2 & 0
\end{bmatrix} > 0,
$$

(3.26)

where

$$
\tilde{\phi}_i^{(l)} = (\tilde{\phi}_{p,q,i})_{17 \times 17}, \tilde{\phi}_{p,q,i} = \phi_{p,q,i} \ \forall p,q = 1,2,\cdots, 17,
$$

except

$$
\tilde{\phi}_{3,3,i} = B_i^T C B_i + R_6 + \rho^2 M - U, \ \tilde{\phi}_{3,11,i} = B_i^T C - I, \ \tilde{\phi}_{11,11,i} = h_2^2 J_2 + \frac{h_4^2}{4} S_1 + \frac{h_4^2}{4} S_2 - \gamma I.
$$

The remaining coefficients are same as in Theorem 3.1.
Proof. The proof is same as that of Theorem 3.1 and hence it is omitted.

Remark 3.3. In the absence of leakage and distributed delays, the system (3.2) without Markovian jump parameters is reduced to the following neural networks:

\[
\begin{align*}
\dot{x}(t) &= -Ax(t) + Bg(x(t)) + Cg(x(t - d_1(t) - d_2(t))) + u(t), \\
y(t) &= g(x(t)),
\end{align*}
\]  
(3.27)

where \(d_1(t)\) and \(d_2(t)\) are assumed to satisfy \(0 \leq d_1(t) \leq d_{12}\) with \(\dot{d}_1(t) \leq \bar{\mu}_1 < 1\) and \(0 \leq d_2(t) \leq d_{22}\) with \(\dot{d}_2(t) \leq \bar{\mu}_2 < 1\). By using Theorem 3.1, one can obtain the passivity criterion for the above NNs (3.27) as in the following corollary.

Corollary 3.2. The neural networks (3.27) is passive in the sense of Definition 1.4, if there exists positive definite matrices \(P, R_3, R_4, R_5, R_6, Q_2, J_n (n = 1, 2), S_q (q = 1, 2)\), diagonal matrix \(U\) and a scalar \(\gamma > 0\), such that the following LMI holds:

\[
\begin{align*}
\Xi &= (\Xi_{ij})_{10 \times 10} < 0, \\
\Xi_{1,1} &= -PA - A^TP + R_3 + R_4 + R_5 + d_{12}^2Q_2 + d_{22}^2J_1 - J_2 + A^TCA, \\
\Xi_{1,2} &= PB - A^T CB + LU, \\
\Xi_{1,3} &= PC - A^T CC, \\
\Xi_{1,7} &= J_2, \\
\Xi_{1,8} &= P - A^TC, \\
\Xi_{2,2} &= B^T CB + R_6 - 2U, \\
\Xi_{2,3} &= B^T CC, \\
\Xi_{2,8} &= B^T C - I, \\
\Xi_{3,3} &= -R_6(1 - \bar{\mu}_1 - \bar{\mu}_2) + C^T CC, \\
\Xi_{3,8} &= C^T C, \\
\Xi_{4,4} &= -R_3, \\
\Xi_{5,5} &= -R_4(1 - \bar{\mu}_1), \\
\Xi_{6,6} &= -R_5(1 - \bar{\mu}_2), \\
\Xi_{7,7} &= -J_2, \\
\Xi_{8,8} &= C - \gamma I, \\
\Xi_{9,9} &= -Q_2, \\
\Xi_{10,10} &= -J_1, \\
C &= d_{22}^2J_2 + \frac{d_{12}^4}{4} S_1 + \frac{d_{22}^4}{4} S_2
\end{align*}
\]  
(3.28)

and the remaining coefficients are zero.

Proof. By putting \(R_1 = R_2 = Q_1 = Z = M = 0\) in the LKF (3.8) and using the similar arguments of Theorem 3.1, one can obtain the passivity results for the system (3.27).
Remark 3.4. when \( d_1(t) = 0, d_2(t) = d(t) \) and \( d_{22} = d \), the system (3.27) reduces to the following form with the single delay

\[
\dot{x}(t) = -Ax(t) +Bg(x(t)) + Cg(x(t-d(t))) + u(t), \\
y(t) = g(x(t)),
\] (3.29)

where \( d(t) \) satisfies \( 0 \leq d_1(t) \leq d, \dot{d}(t) \leq \hat{\mu} < 1 \). The passivity criterion for delayed neural network (3.29) can be derived as follows:

**Corollary 3.3.** The neural networks (3.29) is passive in the sense of Definition 1.4, if there exists positive definite matrices \( P, R_5, J_n(n = 1, 2), S_1 \), diagonal matrices \( U \) and \( V \) and scalars \( \gamma > 0 \), such that the following LMI holds:

\[
\Phi = (\Phi_{ij})_{7 \times 7} < 0,
\] (3.30)

\[
\Phi_{1,1} = -PA - A^T P + R_5 + d^2 J_1 - J_1 - d^2 S_1 + A^T H A, \quad \Phi_{1,2} = PB - A^T H B + U L, \\
\Phi_{1,3} = PC - A^T H C, \quad \Phi_{1,4} = dS_1, \quad \Phi_{1,5} = P - A^T H, \quad \Phi_{1,7} = J_1, \quad \Phi_{2,2} = B^T H B - 2U, \\
\Phi_{2,3} = B^T H C, \quad \Phi_{2,5} = B^T H + I, \quad \Phi_{3,3} = C^T H C - 2V, \quad \Phi_{3,5} = C^T H, \quad \Phi_{3,6} = LV, \\
\Phi_{4,4} = -J_1 - S_1, \quad \Phi_{5,5} = H - \gamma, \quad \Phi_{6,6} = -R_5(1 - \hat{\mu}), \quad \Phi_{7,7} = -J_1, \quad H = d^2 J_2 + \frac{d^4}{4} S_1
\]

and the remaining coefficients are zero.

**Proof.** From Assumption 3.1, there exists a diagonal matrix \( V \) such that the following inequality holds:

\[
-2g^T(x(t - d(t)))V g(x(t - d(t))) + 2x^T(t - d(t))V L g(x(t - d(t))) \geq 0
\] (3.31)

By adding (3.31) with (3.23) and proceeding as in the proof of Theorem 3.1 with \( R_1 = R_2 = R_3 = R_4 = R_6 = Q_1 = Q_2 = S_2 = 0 \), the passivity results for (3.29) can be obtained.

Remark 3.5. In [51], the problem of second-order reciprocally convex approach is discussed to study the stability of systems with interval time-varying delays. Followed this, in this section, by utilizing the Jensen inequality, the double integral terms are partitioned into single integral terms so as to find a second order reciprocally convex combination of positive functions involving the inverses of squared convex parameters.
Remark 3.6. Different from [89, 119], in this section, two triple integral terms
\[ \frac{h^2}{2} \int_{-d_2}^{d_2} \int_{-d_2}^{d_2} \int_{t}^{t+\lambda} \hat{x}^T(s)S_1\hat{x}(s)dsd\lambda d\theta \quad \text{and} \quad \frac{h^2}{2} \int_{-d_2}^{d_2} \int_{-d_2}^{d_2} \int_{t}^{t+\theta} \hat{x}^T(s)S_2\hat{x}(s)dsd\lambda d\theta \]
are included in \( V_0(x(t), t, i) \) which plays an important role in reducing conservatism of the obtained results. Those two triple integral terms provide the double integral terms
\[ -\frac{h^2}{2} \int_{-d_2}^{d_2} \int_{t+\theta}^{t+d_2} \hat{x}^T(s)S_1\hat{x}(s)dsd\theta \quad \text{and} \quad -\frac{h^2}{2} \int_{-d_2}^{d_2} \int_{t+\theta}^{t+d_2} \hat{x}^T(s)S_1\hat{x}(s)dsd\theta \]
in \( LV_0(t, x(t), i) \), respectively. These two double integral terms are further reduced into terms
with three integral parts such as \(-\frac{h^2}{2} (d_2(t) - d_2(t)) \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}^T(s)S_1\hat{x}(s)ds \)
\[-\frac{h^2}{2} \int_{t-d_2(t)}^{t} \hat{x}^T(s)S_1\hat{x}(s)dsd\theta \quad \text{and} \quad \frac{h^2}{2} \int_{t-d_2(t)}^{t} \hat{x}^T(s)S_2\hat{x}(s)dsd\theta \]
- \[ \frac{h^2}{2} \int_{t-d_2(t)}^{t} \hat{x}^T(s)S_2\hat{x}(s)dsd\theta, \] respectively. Further, applying Jensen’s inequality to these terms will lead to less conservative dissipativity results which can be seen through numerical example.

Remark 3.7. To reduce the conservatism, the lower bound lemma is used to deal with the derivative of \( V_0(t, x(t), i) \), i.e., by using the relations \( \xi = -1 + \frac{1}{\xi} \) and \( \omega = -1 + \frac{1}{\omega} \) in the following inequality, the inequality (3.18) is obtained by using Lemma 1.9.

\[
0 \geq \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right]^T \left[ J_2 + \frac{h^2}{2} S_1 \right] \left[ \frac{\sqrt{2}}{\omega} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right].
\]

Remark 3.8. To obtain the inequalities (3.19) and (3.20), in this section, use the relation
\[ \frac{1}{\xi^2} = \frac{(c+\xi)^2}{c^2} \quad \text{and} \quad \frac{1}{\omega^2} = \frac{(c+\omega)^2}{c^2} \] in the following inequalities:

\[
0 \geq \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right]^T \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right] - \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right]^T \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right] + \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right]^T \left[ \frac{\sqrt{2}}{\xi} \int_{t-d_2(t)}^{t-d_2(t)} \hat{x}(s)ds \right].
\]
and

\[
0 \geq - \begin{bmatrix}
\sqrt{\frac{g}{\xi}} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \dot{x}(s) ds d\theta \\
\frac{g}{\xi} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \dot{x}(s) ds d\theta \\
-\sqrt{\frac{g}{\xi}} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \dot{x}(s) ds d\theta \\
-\frac{g}{\xi} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \dot{x}(s) ds d\theta
\end{bmatrix}^T 
\begin{bmatrix}
2S_2 & 0 & K_5 & 0 \\
* & S_2 & 0 & K_6 \\
* & * & 2S_2 & 0 \\
* & * & * & S_2
\end{bmatrix}
\times 
\begin{bmatrix}
\sqrt{\frac{g}{\xi}} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \ddot{x}(s) ds d\theta \\
\frac{g}{\xi} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \ddot{x}(s) ds d\theta \\
-\sqrt{\frac{g}{\xi}} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \ddot{x}(s) ds d\theta \\
-\frac{g}{\xi} \int_{-d_2(t)}^{t}\int_{-d_2(t)}^{t+\theta} \ddot{x}(s) ds d\theta
\end{bmatrix}.
\]

The above approach is very effective in reducing the conservativeness of the dissipativity criterion, which will be shown through numerical examples in the subsequent section.

3.2.3 Numerical examples

In this section, three numerical examples are demonstrated to show the effectiveness of the established theories.

**Example 3.1.** Consider the system (3.2) with the following parameters:

\[
A_1 = \begin{bmatrix} 2.3 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 & 0.5 \\ -0.2 & 0.1 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 0.2 & -0.3 \\ 0.4 & 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},
\]

\[
\Gamma = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix}, \quad Q = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \quad R = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}, \quad S = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}.
\]

For this model, the nonlinear activation functions are assumed as \(g_1(x) = 0.4 \tanh(x)\) and \(g_2(x) = 0.8 \tanh(x)\). It is easy to see that these activation functions satisfy the Assumption 3.1 with \(L = \text{diag}\{0.4, 0.8\}\).

Setting \(\hat{\mu} = 0.5\), \((i.e. \hat{\mu}_1 = 0.2 \text{ and } \hat{\mu}_2 = 0.3)\), \(\rho = 0.2\), \(\sigma_\mu = 0.01\), \(d_{11} = 0.3\), \(d_{21} = 0.4\) and \(d_{12} = 0.7\), the maximum allowable upper bounds \(d_{22}\) for various \(\sigma\) are obtained as listed in Table 3.1. Moreover, by choosing \(\hat{\mu} = 0.5\), \((i.e. \hat{\mu}_1 = 0.2, \hat{\mu}_2 = 0.3)\), \(\sigma = 0.03\), \(\sigma_\mu = 0.01\), \(\rho = 0.2\), \(d_{11} = 0.3\) and \(d_{21} = 0.4\), the maximum allowable upper bound for \(d_{22}\) are computed for various \(d_{12}\) and are listed in Table 3.2. Furthermore, Figure 3.1 shows the state trajectories of variable \(x(t)\) with the initial condition \([-0.2, 0.2]^T\) for the additive
Figure 3.1: The state trajectories of system (3.2) with additive time-varying delay $d_1(t) = 0.02 + 0.01 \sin(t)$, $d_2(t) = 2.6185 + 0.3 \cos(t)$ for Example 3.1.

Figure 3.2: The state trajectories of system (3.2) with additive time-varying delay $d_1(t) = 0.2 + 0.1 \sin(t)$, $d_2(t) = 2.6762 + 0.3 \cos(t)$ for Example 3.1.

Figure 3.3: Unstable behavior of system (3.2) with additive time-varying delay $\sigma(t) = 0.01 + 0.5 \sin(t)$ for Example 3.1.
Table 3.1: Maximum allowable upper bounds of $d_{22}$ for different $\sigma$ when $d_{11} = 0.3$, $d_{21} = 0.4$, $d_{12} = 0.7$, $\tilde{\mu} = 0.5$, $\sigma_\mu = 0.01$ and $\rho = 0.2$ for Example 3.1.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{22}$</td>
<td>2.9327</td>
<td>2.9236</td>
<td>2.9185</td>
<td>2.9073</td>
<td>2.8937</td>
<td>2.8832</td>
<td>2.8796</td>
<td>2.8625</td>
</tr>
</tbody>
</table>

Table 3.2: Maximum allowable upper bounds of $d_{22}$ for different $d_{12}$ when $d_{11} = 0.3$, $d_{21} = 0.4$, $\tilde{\mu} = 0.5$, $\sigma = 0.03$, $\sigma_\mu = 0.01$ and $\rho = 0.2$ for Example 3.1.

<table>
<thead>
<tr>
<th>$d_{12}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{22}$</td>
<td>2.9875</td>
<td>2.9802</td>
<td>2.9762</td>
<td>2.9652</td>
<td>2.9532</td>
<td>2.9518</td>
<td>2.9485</td>
<td>2.9436</td>
</tr>
</tbody>
</table>

delays $d_1(t) = 0.5 + 0.2 \sin(t)$ and $d_2(t) = 2.6185 + 0.3 \cos(t)$ in the case of $\sigma(t) = 0.02 + 0.01 \sin(t)$. The state trajectories of variable $x(t)$ for $\sigma(t) = 0.02 + 0.01 \sin(t)$, $d_1(t) = 0.2 + 0.1 \sin(t)$ and $d_2(t) = 2.6762 + 0.3 \cos(t)$ are depicted in Figure 3.2. Figure 3.3 predicts the unstable behavior for the state trajectories of variable $x(t)$ with the delay $\sigma(t) = 0.01 + 0.5 \sin(t)$.

Example 3.2. Consider system (3.27) with two additive time-varying delay components as in [14, 39, 80, 93, 132] with the following matrices:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}. $$

In this example, the activation functions are assumed to be $g_1(x) = 0.4 \tanh(x)$ and $g_2(x) = 0.8 \tanh(x)$. It is easy to check that they satisfy the Assumption 3.1 with $L = \text{diag} \{0.4, 0.8\}$. When $\tilde{\mu} = 0.8$ ($\tilde{\mu}_1 = 0.7$, $\tilde{\mu}_2 = 0.1$) and $\check{\mu} = 0.9$ ($\check{\mu}_1 = 0.7$, $\check{\mu}_2 = 0.2$), the corresponding upper bounds for $d_{22}$ for various values of $d_{21}$ are calculated by Corollary 3.2 and are listed in Tables 3.3 and 3.4 in order to compare with the results obtained in [14, 39, 80, 93, 132]. Tables 3.3 and 3.4 show that the method proposed in this section is much less conservative and more superior than the corresponding methods used in [14, 39, 80, 93, 132]. When $u(t) = 0$, one can obtain the state trajectories of the state $x(t)$ for the delay $d_1(t) = 0.1 + 0.7 \sin(t)$ and $d_2(t) = 2.7325 + 0.1 \cos(t)$ with the initial value $[-0.2, 0.2]^T$.
Figure 3.4: The state trajectories of system (3.27) with additive time-varying delay $d_1(t) = 0.1 + 0.7 \sin(t)$ and $d_2(t) = 2.7325 + 0.1 \cos(t)$ for Example 3.2.

Figure 3.5: The state trajectories of system (3.27) with additive time-varying delay $d_1(t) = 0.1 + 0.7 \sin(t)$ and $d_2(t) = 2.1982 + 0.2 \cos(t)$ for Example 3.2.
Table 3.3: Maximum allowable upper bounds of $d_{22}$ for different $d_{21}$ when $\tilde{\mu}_1 = 0.7$ and $\tilde{\mu}_2 = 0.1$ for Example 3.2.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$d_{21} = 0.8$</th>
<th>$d_{21} = 1.0$</th>
<th>$d_{21} = 1.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[39]</td>
<td>0.8831</td>
<td>0.6831</td>
<td>0.4831</td>
</tr>
<tr>
<td>[132]</td>
<td>0.8831</td>
<td>0.6832</td>
<td>0.4843</td>
</tr>
<tr>
<td>[80]</td>
<td>1.5666</td>
<td>1.3668</td>
<td>1.1664</td>
</tr>
<tr>
<td>[93]</td>
<td>2.0164</td>
<td>1.8203</td>
<td>1.6197</td>
</tr>
<tr>
<td>[14]</td>
<td>2.0961</td>
<td>1.8961</td>
<td>1.6961</td>
</tr>
<tr>
<td>Corollary 3.2</td>
<td>2.8325</td>
<td>2.6175</td>
<td>2.3329</td>
</tr>
</tbody>
</table>

**Figure 3.4.** In addition, for $d_1(t) = 0.1 + 0.7\sin(t)$ and $d_2(t) = 2.1982 + 0.2\cos(t)$, and taking the initial value as $[-0.2, 0.2]^T$, the response of the state trajectories are drawn in Figure 3.5.

**Remark 3.9.** In [132], the double integral terms $\int_{-\tilde{\alpha}}^{\tilde{\alpha}} \int_{\beta}^{0} \hat{z}(t + \alpha)Z_1 \hat{z}(t + \alpha) \, d\alpha d\beta$, $\int_{-\tilde{\alpha}}^{\tilde{\alpha}} \int_{\beta}^{0} \hat{z}(t + \alpha)Z_2 \hat{z}(t + \alpha) \, d\alpha d\beta$, and $\int_{-\tilde{\alpha}}^{\tilde{\alpha}} \int_{\beta}^{0} \hat{z}(t + \alpha)M \hat{z}(t + \alpha) \, d\alpha d\beta$ are considered in the LKF and the Jensen inequality is employed to get the derived results. Even though, the similar double integral terms are considered in this section, a new kind of linear convex combination approach is used from Lemma 1.9 by making use of positive functions weighted by the inverses of squared convex parameters, which improves the results proposed in this section. In [93] and [14], by taking the triple integral term $\int_{-\tilde{\alpha}}^{0} \int_{-\alpha}^{0} \int_{t+\lambda}^{0} \hat{z}(s)Z_5 \hat{z}(s) \, ds d\alpha d\beta$ in the LKF, the stability results are obtained without using Lemma 1.9. However, in this section, based on second order reciprocally convex approach, the Lemma 1.9 is used to handle several kinds of function combinations arised from the derivation of triple integral terms considered in $V_6(t, x(t), i)$. It should be pointed out that the results obtained in Theorem 3.1 in this section by making use of second order reciprocally convex combination approach are much more better results than those obtained in [93] and [14] which can be easily seen via numerical example.
Table 3.4: Maximum allowable upper bounds of $d_{22}$ for different $d_{21}$ when $\tilde{\mu}_1 = 0.7$ and $\tilde{\mu}_2 = 0.2$ for Example 3.2.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$d_{21} = 0.8$</th>
<th>$d_{21} = 1.0$</th>
<th>$d_{21} = 1.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[39]</td>
<td>0.3493</td>
<td>0.1493</td>
<td>Infeasible</td>
</tr>
<tr>
<td>[132]</td>
<td>0.3942</td>
<td>0.2637</td>
<td>0.1671</td>
</tr>
<tr>
<td>[80]</td>
<td>0.8515</td>
<td>0.6596</td>
<td>0.4616</td>
</tr>
<tr>
<td>[93]</td>
<td>0.8703</td>
<td>0.6713</td>
<td>0.4715</td>
</tr>
<tr>
<td>[14]</td>
<td>1.1046</td>
<td>0.9046</td>
<td>0.7046</td>
</tr>
<tr>
<td>Corollary 3.2</td>
<td>2.3982</td>
<td>2.1786</td>
<td>2.0831</td>
</tr>
</tbody>
</table>

Figure 3.6: The state trajectories of system (3.29) with time-varying delay $d(t) = 5.7305 + 0.2 \sin^2(t)$ for Example 3.3.
Table 3.5: Maximum allowable upper bounds of $d$ for different $\mu$ for Example 3.3

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\hat{\mu} = 0.4$</th>
<th>$\hat{\mu} = 0.45$</th>
<th>$\hat{\mu} = 0.50$</th>
<th>$\hat{\mu} = 0.55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[84]</td>
<td>3.9972</td>
<td>3.2760</td>
<td>3.0594</td>
<td>2.9814</td>
</tr>
<tr>
<td>[88]</td>
<td>4.3814</td>
<td>3.6008</td>
<td>3.3377</td>
<td>3.2350</td>
</tr>
<tr>
<td>[92]</td>
<td>5.2420</td>
<td>4.4301</td>
<td>4.1055</td>
<td>3.9231</td>
</tr>
<tr>
<td>[82]</td>
<td>5.4036</td>
<td>4.6017</td>
<td>4.3121</td>
<td>4.1582</td>
</tr>
<tr>
<td>Corollary 3.3</td>
<td>6.1305</td>
<td>5.8231</td>
<td>5.6357</td>
<td>5.3208</td>
</tr>
</tbody>
</table>

**Example 3.3.** Consider the neural networks (3.29) as discussed in [82, 84, 88, 92] with the following parameters:

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}.$$  

The activation functions are assumed to satisfy Assumption 3.1 with $g_1(x) = 0.3 \tanh(x), g_2(x) = 0.8 \tanh(x)$ and hence $L = \text{diag}\{0.3, 0.8\}$. By using Corollary 3.3 and solving MATLAB LMI tool box, the corresponding results for the maximum allowable upper bounds of the time-varying delay $d(t)$ are computed as given in Table 3.5 for different $\mu$'s. Further, the computed upper bounds are compared with the existing ones [82, 84, 88, 92]. Figure 3.6 shows the state curves for the delay $d(t) = 5.7305 + 0.2 \sin^2(t)$ with $\mu = 0.4$, and the initial condition $[-0.1, 0.1]^T$.

**Remark 3.10.** It should be noted in [84] that the free weighting matrix method is used for obtaining the theoretical results. By using this approach, any model transformation and bounded technique for cross-terms have not been applied and it may lead to more computational complexity. Further, in [88], the integral term $-\int_{t-h}^{t} z^T(s)S_1z(s)ds$ and $-\int_{t-h}^{t} \dot{z}^T(s)S_1\dot{z}(s)ds$ considered in the LKF are splitted into two integral term such as $-\int_{t-\tau(t)}^{t} z^T(s)S_1z(s)ds$ and $-\int_{t-h}^{t-\tau(t)} z^T(s)S_1z(s)ds$, $-\int_{t-\tau(t)}^{t} \dot{z}^T(s)S_2\dot{z}(s)ds$ and $-\int_{t-h}^{t-\tau(t)} \dot{z}^T(s)S_2\dot{z}(s)ds$, respectively. By using these integral terms and by introducing the relationship among $\tau(t), h - \tau(t)$ and $h$, the stability results are derived for the neural networks in [88] and the obtained results are further shown to be less conservative than
those obtained in [84]. Different from [84] and [88], in [92], the new type of double integral term $\frac{1}{2} \int_{-\frac{1}{2}}^{0} \int_{t+\theta}^{t} \hat{z}(s) Q_1 \hat{z}(s) ds d\theta$ is taken into account in the LKF which is further reduced to the single integral term $-\frac{1}{2} \int_{t-\frac{1}{2}}^{t} \hat{z}(s) Q_1 \hat{z}(s) ds$ after taking the derivative. Further, the single integral term is reduced by a convex optimization approach in [92]. This approach produces less conservatism compared with [84] and [88]. On the other hand, in [82], triple and quadruple integrals are introduced to give better results than in [84, 88, 92]. Unlike [82, 84, 88, 92], in this section, the integral term is used in $V_5(t, x(t), i)$ in order to derive the less conservative passivity results.

3.3 Dissipativity and passivity analysis of Markovian jump stochastic neural networks with two delay components

3.3.1 Problem description

Consider the Markovian jump stochastic neural networks with two additive time-varying delay components as follows:

$$\frac{dx(t)}{dt} = [-A_i x(t) + B_i g(x(t)) + C_i g(x(t - d_1(t) - d_2(t))) + u(t)]dt$$
$$+ \delta(t, x(t), x(t - d_1(t) - d_2(t), i)d\omega(t),$$
$$y(t) = g(x(t)).$$

(3.32)

Assume that $\delta : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^{n \times m}$ is Borel measurable with $\delta(0, 0, 0, i) \equiv 0$; $\omega(t) = [\omega_1(t), \cdots, \omega_m(t)]^T \in \mathbb{R}^m$ is an $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ satisfying $\mathcal{E}\{d\omega(t)\} = 0$ and $\mathcal{E}\{(d\omega(t))^2\} = dt$. Moreover, assume that the Brownian motion $\{\omega(t), t \geq 0\}$ is independent from the Markov chain $\{r(t), t \geq 0\}$.

The time-varying delays $d_1(t)$ and $d_2(t)$ satisfy $0 < d_1(t) \leq d_1 < \infty$, $0 < d_2(t) \leq d_2 < \infty$, $\dot{d}_1(t) \leq \bar{\mu}_1 < \infty$, $\dot{d}_2(t) \leq \bar{\mu}_2 < \infty$, where $d_1, d_2, \bar{\mu}_1$ and $\bar{\mu}_2$ are positive constants.
For the sake of convenience, the following abbreviations are adopted in the sequel:

\[
\varpi(t) \triangleq -A_xt(t) + B_ig(x(t)) + C_i(g(x(t - d_1(t) - d_2(t)))) + u(t),
\]

\[
\varpi(t) \triangleq \theta(t, x(t), x(t - d_1(t) - d_2(t)), i). \tag{3.33}
\]

Then, Markovian jump stochastic neural networks with two additive time-varying delay components (3.32) can be rewritten as

\[
dx(t) = \varpi(t)dt + \varpi(t)d\varpi(t),
\]

\[
y(t) = g(x(t)).
\tag{3.34}
\]

**Assumption 3.2.** The stochastic noise \( \varpi(t) \) is locally Lipschitz and satisfies the following condition:

\[
\text{Trace} |\varpi^T(t) \varpi(t)| \leq x^T(t)T_{11}x(t) + x^T(t - d_1(t) - d_2(t))T_{22}x(t - d_1(t) - d_2(t)), \tag{3.35}
\]

where \( T_{11} \) and \( T_{22} (i \in \mathbb{S}) \) are positive definite matrices.

Let \( C([-\bar{d}, 0]; \mathbb{R}^n) \) denote the family of continuously differentiable function \( \varphi \) from \([-\bar{d}, 0] \) to \( \mathbb{R}^n \) with the norm \( \| \varphi \| = \sup_{-\bar{d} \leq \theta \leq 0} |\varphi(\theta)| \), where \( | \cdot | \) is the Euclidean norm in \( \mathbb{R}^n \) and \( \bar{d} = \max\{d_1, d_2\} \). Denote by \( C^2_{\mathcal{F}_0}([[-\bar{d}, 0]; \mathbb{R}^n) \) the family of all bounded \( \mathcal{F}_0 \)-measurable \( C([-\bar{d}, 0]; \mathbb{R}^n) \)-valued stochastic variables \( \xi = \{ \xi(\theta) : -\bar{d} \leq \theta \leq 0 \} \) such that

\[
\int_{-\bar{d}}^0 \mathbb{E}[|\xi(s)|^2]ds < \infty.
\]

Let \( x(t, \phi) \) denote the state trajectory from the initial data \( x(t) = \phi(t) \) on \([-\bar{d}, t] \leq 0 \), where \( \phi \in C^2_{\mathcal{F}_0}([-\bar{d}, 0]; \mathbb{R}^n) \). Let \( C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R}) \) denote the family of all nonnegative functions \( V(t, x(t), i) \) on \( \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{S} \), which are continuously twice differentiable in \( x \) and once differentiable in \( t \). If \( V \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R}) \) then along trajectory of system (3.34), define an operator \( \mathbb{L}V \) from \( \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{S} \) to \( \mathbb{R} \) by

\[
\mathbb{L}V(t, x(t), i) = V_i(t, x(t), i) + V_x(t, x(t), i)[-A_x x(t) + B_x g(x(t)) + C_i g(x(t - d_1(t) - d_2(t)))]
+ u(t)] + \frac{1}{2}\text{Trace}[\varpi^T(t)V_{xx}(t, x(t), i)\varpi(t)] + \sum_{j=1}^{N} q_{ij} V(t, x(t), j) \tag{3.36}
\]

where \( V_i(t, x(t), i) = \frac{\partial V(t, x(t), i)}{\partial t}, \quad V_x(t, x(t), i) = \begin{pmatrix} \frac{\partial V(t, x(t), i)}{\partial x_1} & \cdots & \frac{\partial V(t, x(t), i)}{\partial x_n} \end{pmatrix}, \)

\[
V_{xx}(t, x(t), i) = \begin{pmatrix} \frac{\partial^2 V(t, x(t), i)}{\partial x_j \partial x_k} \end{pmatrix}_{n \times n}.
\]
3.3.2 Dissipativity analysis for Markovian jump Stochastic neural networks

The notion of dissipativity is closely related to the intuitive phenomena of loss or dissipation of energy of a physical system. Based on this, in this section, the dissipativity criteria for stochastic neural networks with Markovian jumping parameters and two additive time-varying delays is considered. By employing the idea of GFL as in [25], an appropriate LKF candidate for stochastic neural networks model (3.34) is introduced. Based on this LKF candidate, the dissipativity criterion for stochastic neural networks (3.34) with two additive time-varying delays is analyzed in the following theorem.

**Theorem 3.2.** Under Assumptions 3.1 and 3.2, for given scalars \( d_1, d_2, \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \), the system (3.34) is \((Q, S, R)\)-dissipative, if there exist positive scalars \( \lambda_i (i \in \mathcal{S}) \), positive definite matrices \( P_i (i \in \mathcal{S}), Q_s, M_s (s = 1, 2, \cdots, 6), R_1, R_2, S_1, S_2 \), positive diagonal matrices \( U, W \) and scalar \( \gamma > 0 \), such that the following LMI holds for all \( i \in \mathcal{S} \):

\[
\Omega_i = (\Omega_{i,m,i})_{17 \times 17} < 0, \tag{3.37}
\]

and

\[
P_i \leq \lambda_i I, \tag{3.38}
\]

where

\[
\begin{align*}
\Omega_{1,1,i} &= \sum_{j=1}^{N} q_{ij} P_j + \lambda_i T_{1i} + Q_1 + Q_3 + Q_4 + Q_5 + Q_6 + d_1^2 M_1 + d_2^2 M_2 + \frac{d_1^4}{4} R_1 + \frac{d_2^4}{4} R_2 \\
&\quad + \frac{d_3^4}{6} S_1 + \frac{d_3^4}{6} S_2 - A_i^T P_i - P_i A_i + d_1^2 A_i^T M_5 A_i + d_2^2 A_i^T M_6 A_i - Q_5 - Q_6, \\
\Omega_{1,2,i} &= P_i B_i - d_1^2 A_i^T M_5 B_i - d_2^2 A_i^T M_6 B_i + U L, \; \Omega_{1,6,i} = P_i C_i - d_1^2 A_i^T M_5 C_i - d_2^2 A_i^T M_6 C_i, \\
\Omega_{1,7,i} &= P_i - d_1^2 A_i^T M_5 - d_2^2 A_i^T M_6, \; \Omega_{1,16,i} = -P_i + d_1^2 A_i^T M_5 + d_2^2 A_i^T M_6, \\
\Omega_{2,2,i} &= d_1^2 B_i^T M_5 B_i + d_2^2 B_i^T M_6 B_i + Q_2 + d_1^2 M_3 + d_2^2 M_4 - 2U - Q, \; \Omega_{2,6,i} = d_1^2 B_i^T M_5 C_i \\
&\quad + d_2^2 B_i^T M_6 C_i, \; \Omega_{2,7,i} = d_1^2 B_i^T M_5 + d_2^2 B_i^T M_6 - S, \; \Omega_{2,16,i} = -d_1^2 B_i^T M_5 - d_2^2 B_i^T M_6 \\
&\quad + S, \; \Omega_{3,3,i} = -Q_3 (1 - \tilde{\mu}_1), \; \Omega_{3,17,i} = -Q_3 (1 - \tilde{\mu}_1), \; \Omega_{4,4,i} = -Q_4 (1 - \tilde{\mu}_2),
\end{align*}
\]
\[ \Omega_{5,5,i} = -Q_1(1 - \hat{\mu}_1 - \hat{\mu}_2) + \lambda_i T_{2i}, \quad \Omega_{5,6,i} = WL, \quad \Omega_{6,6,i} = d_1^2 C_i^T M_5 C_i + d_2^2 C_i^T M_6 C_i \\
- Q_2(1 - \hat{\mu}_1 - \hat{\mu}_2) - 2W, \quad \Omega_{6,7,i} = d_1^2 C_i^T M_5 + d_2^2 C_i^T M_6, \quad \Omega_{6,16,i} = -d_1^2 C_i^T M_5 \\
- d_2^2 C_i^T M_6, \quad \Omega_{7,7,i} = d_1^2 M_5 + d_2^2 M_6 - \hat{R} + \gamma I, \quad \Omega_{7,16,i} = -d_1^2 M_5 - d_2^2 M_6 \\
- \hat{R} + \gamma I, \quad \Omega_{8,8,i} = -M_1, \quad \Omega_{9,9,i} = -M_2, \quad \Omega_{10,10,i} = -M_3, \quad \Omega_{11,11,i} = -M_4, \\
\Omega_{12,12,i} = -R_1, \quad \Omega_{13,13,i} = -R_2, \quad \Omega_{13,17,i} = -R_2, \quad \Omega_{14,14,i} = -\frac{6}{d_1^2} \tilde{S}_1, \quad \Omega_{15,15,i} = -\frac{6}{d_2^2} \tilde{S}_2, \\
\Omega_{16,16,i} = d_1^2 M_5 + d_2^2 M_6 + \hat{R} - \gamma I, \quad \Omega_{17,17,i} = -Q_3(1 - \hat{\mu}_1) - R_2, \]

and the remaining coefficients are zero.

**Proof.** To establish the dissipativity conditions, introduce the following LKF candidate as follows:

\[ V(t, x(t), i) = \sum_{i=1}^{5} V_i(t, x(t), i), \quad (3.39) \]

where

\[ V_1(t, x(t), i) = x^T(t) P_i x(t), \]

\[ V_2(t, x(t), i) = \int_{t-d_1(t)-d_2(t)}^{t} x^T(s) Q_1 x(s) ds + \int_{t-d_1(t)-d_2(t)}^{t} g^T(x(s)) Q_2 g(x(s)) ds \]

\[ + \int_{t-d_1(t)-d_2(t)}^{t} x^T(s) Q_3 x(s) ds + \int_{t-d_2(t)}^{t} x^T(s) Q_4 x(s) ds \]

\[ + \int_{t-d_1(t)-d_2(t)}^{t} x^T(s) Q_5 x(s) ds + \int_{t-d_2(t)}^{t} x^T(s) Q_6 x(s) ds, \]

\[ V_3(t, x(t), i) = d_1 \int_{t-d_1(t)}^{t} x^T(s) M_1 x(s) ds d\theta + d_2 \int_{t-d_2(t)}^{t} x^T(s) M_2 x(s) ds d\theta \]

\[ + d_1 \int_{t-d_1(t)}^{t} g^T(x(s)) M_3 g(x(s)) ds d\theta \]

\[ + d_2 \int_{t-d_2(t)}^{t} g^T(x(s)) M_4 g(x(s)) ds d\theta \]

\[ + d_1 \int_{t-d_1(t)}^{t} \omega^T(s) M_5 \omega(s) ds d\theta + d_2 \int_{t-d_2(t)}^{t} \omega^T(s) M_6 \omega(s) ds d\theta, \]

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Let \( L \) be the weak infinitesimal operator. Then, along the trajectory of (3.34) one has

\[
V_4(t, x(t), i) = \frac{d_1^2}{2} \int_{-d_1}^0 \int_0^1 \int_{t+\lambda}^t x^T(s) R_1 x(s) ds d\lambda d\theta \\
+ \frac{d_2^2}{2} \int_{-d_2}^0 \int_0^1 \int_{t+\lambda}^t x^T(s) R_2 x(s) ds d\lambda d\theta,
\]

\[
V_5(t, x(t), i) = \int_{-d_1}^0 \int_0^1 \int_{t+\lambda}^t x^T(s) S_1 x(s) ds d\lambda d\theta d\alpha \\
+ \int_{-d_2}^0 \int_0^1 \int_{t+\lambda}^t x^T(s) S_2 x(s) ds d\lambda d\theta d\alpha.
\]

Let \( \mathbb{L} \) be the weak infinitesimal operator. Then, along the trajectory of (3.34) one has

\[
\mathbb{L} V(t, x(t), i) = \sum_{i=1}^5 \mathbb{L} V_i(t, x(t), i),
\]

where

\[
\mathbb{L} V_1(t, x(t), i) = 2x^T(t) P_1 \bar{\omega}(t) + \sum_{j=1}^N q_{ij} x^T(t) P_j x(t) + \text{Trace}\{\bar{\omega}^T(t) P_i \bar{\omega}(t)\}, \quad \text{(using (3.36))}
\]

\[
\mathbb{L} V_2(t, x(t), i) \leq x^T(t) Q_1 x(t) - x^T(t - d_1(t) - d_2(t)) Q_1 x(t - d_1(t) - d_2(t))(1 - \tilde{\mu}_1 - \tilde{\mu}_2) \\
+ f^T(x(t)) Q_2 f(x(t)) + x^T(t) Q_3 x(t) + x^T(t) Q_4 x(t) \\
- f^T(x(t-d_1(t)-d_2(t))) Q_2 f(x(t-d_1(t)-d_2(t)))(1 - \tilde{\mu}_1 - \tilde{\mu}_2) \\
- x^T(t-d_1(t)) Q_3 x(t-d_1(t))(1 - \tilde{\mu}_1) \\
- x^T(t-d_2(t)) Q_4 x(t-d_2(t))(1 - \tilde{\mu}_2) \\
+ x^T(t) Q_5 x(t) - x^T(t-d_1(t)) Q_5 x(t-d_1(t)) + x^T(t) Q_6 x(t) \\
- x^T(t-d_2(t)) Q_6 x(t-d_2(t)),
\]

\[
\mathbb{L} V_3(t, x(t), i) \leq x^T(t) [d_1^2 M_1 + d_2^2 M_2] x(t) - \int_{t-d_1}^t x^T(s) dM_1 \int_{t-d_1}^t x(s) ds \\
- \int_{t-d_2}^t x^T(s) dM_2 \int_{t-d_2}^t x(s) ds + g^T(x(t)) [d_1^2 M_3 + d_2^2 M_4] g(x(t)) \\
- \int_{t-d_1}^t g^T(x(s)) dM_3 \int_{t-d_1}^t g(x(s)) ds \\
- \int_{t-d_2}^t g^T(x(s)) dM_4 \int_{t-d_2}^t g(x(s)) ds
\]
the following inequalities hold based on Assumption 3.1:

\[ E(3.48), \text{it is easy to get} \]

Noting \( E \)

Assumption 3.2 combined with condition (3.38) yields

\[
\begin{align*}
\mathbb{L}V_4(t, x(t), i) & \leq x^T(t) \left( \frac{d_1^4}{4} R_1 + \frac{d_2^2}{4} R_2 \right) x(t) - \int_{t-d_1}^{t} x^T(s) d \theta R_1 \int_{t-d_1}^{t} x(s) d \theta \\
& - \int_{t-d_2}^{t} x^T(s) d \theta R_2 \int_{t-d_2}^{t} x(s) d \theta,
\end{align*}
\]

(3.44)

\[
\begin{align*}
\mathbb{L}V_5(t, x(t), i) & \leq x^T(t) \left( \frac{d_1^3}{6} S_1 + \frac{d_2^3}{6} S_2 \right) x(t) - \frac{6}{d_1^3} \int_{t-d_1}^{t} \int_{t-d_2}^{t} x^T(s) d \theta d \alpha S_1 \\
& \times \int_{t-d_1}^{t} \int_{t-d_2}^{t} x(s) d \theta d \alpha - \frac{6}{d_2^3} \int_{t-d_1}^{t} \int_{t-d_2}^{t} x^T(s) d \theta d \alpha S_2 \\
& \times \int_{t-d_1}^{t} \int_{t-d_2}^{t} x(s) d \theta d \alpha.
\end{align*}
\]

(3.45)

Let \( L = \text{diag}\{l_1, l_2, \ldots, l_n\}. \) Then there exist positive diagonal matrices \( U \) and \( W \) such that the following inequalities hold based on Assumption 3.1:

\[
\begin{align*}
-2g^T(x(t))Ug(x(t)) + 2x^T(t)ULg(x(t)) & \geq 0, \\
-2g^T(x(t-d_1(t) - d_2(t)))Wg(x(t-d_1(t) - d_2(t))) \\
+ 2x^T(t-d_1(t) - d_2(t))WLg(x(t-d_1(t) - d_2(t))) & \geq 0.
\end{align*}
\]

(3.47)

(3.48)

Noting \( \mathbb{E}\{dw(t)\} = 0 \) and substituting (3.41)-(3.46) in (3.40) and then adding (3.47) and (3.48), it is easy to get

\[
\begin{align*}
\mathbb{E}\{\mathbb{L}V(t, x(t), i) - y^T(t)Qy(t) - 2y^T(t)Su(t) - u^T(t)(R - \gamma I)u(t)\} & \leq \mathbb{E}\{\Psi^T(t)\Phi_i \Psi(t)\},
\end{align*}
\]

(3.49)
where

\[
\Psi^T(t) = \left[ x^T(t) \quad x^T(t - d_1(t)) \quad x^T(t - d_2(t)) \quad x^T(t - d_1(t) - d_2(t)) \quad g^T(x(t)) \right] \\
g^T(x(t - d_1(t) - d_2(t))) \quad x^T(t - d_1) \quad x^T(t - d_2) \quad \int_{t-d_1}^t x^T(s)ds \\
\int_{t-d_2}^t x^T(s)ds \quad \int_{t-d_1}^t g^T(x(s))ds \quad \int_{t-d_2}^t g^T(x(s))ds \quad \int_{t-d_1}^t x^T(s)ds \quad \int_{t-d_2}^t x^T(s)ds \quad \int_{t-d_1}^t x^T(s)ds \quad \int_{t-d_2}^t x^T(s)ds \quad \int_{t-d_1}^t x^T(s)ds \\
u^T(t) \quad \int_{t-d_1}^t \varpi^T(s)ds \quad \int_{t-d_2}^t \varpi^T(s)ds \quad \int_{t-d_2}^t \varpi^T(s)ds \quad \int_{t-d_1}^t \varpi^T(s)ds \\
\right]
\]

and

\[
\Phi_i = (\Phi_{l,m,i})_{20 \times 20} \quad (3.50)
\]

with

\[
\Phi_{1,1,i} = \sum_{j=1}^{N} q_{ij} P_j + \lambda_i T_{1i} + Q_1 + Q_3 + Q_4 + Q_5 + Q_6 + d_1^2 M_1 + d_2^2 M_2 + \frac{d_1^4}{4} R_1 + \frac{d_2^4}{4} R_2 \\
+ \frac{d_1^6}{6} S_1 + \frac{d_2^6}{6} S_2, \quad \Phi_{1,2,i} = P_i, \quad \Phi_{1,6,i} = UL, \quad \Phi_{2,2,i} = d_1^2 M_5 + d_2^2 M_6, \\
\Phi_{3,3,i} = -Q_3(1 - \bar{\mu}_1), \quad \Phi_{4,4,i} = -Q_4(1 - \bar{\mu}_2), \quad \Phi_{5,5,i} = -Q_1(1 - \bar{\mu}_1 - \bar{\mu}_2) + \lambda_i T_{2i}, \\
\Phi_{5,7,i} = W L, \quad \Phi_{6,6,i} = Q_2 + d_1^2 M_3 + d_2^2 M_4 - 2U - Q, \quad \Phi_{6,18,i} = -\bar{S}, \\
\Phi_{7,7,i} = -Q_2(1 - \bar{\mu}_1 - \bar{\mu}_2) - 2W, \quad \Phi_{8,8,i} = -Q_5, \quad \Phi_{9,9,i} = -Q_6, \quad \Phi_{10,10,i} = -M_1, \\
\Phi_{11,11,i} = -M_2, \quad \Phi_{12,12,i} = -M_3, \quad \Phi_{13,13,i} = -M_4, \quad \Phi_{14,14,i} = -R_1, \quad \Phi_{15,15,i} = -R_2, \\
\Phi_{16,16,i} = -\frac{6}{d_1^2} S_1, \quad \Phi_{17,17,i} = -\frac{6}{d_2^2} S_2, \quad \Phi_{18,18,i} = -(\bar{R} - \gamma I), \quad \Phi_{19,19,i} = -M_5, \\
\Phi_{20,20,i} = -M_6 \quad \text{and the remaining coefficients are zero.}
\]

Taking mathematical expectation on both sides of (3.34) and using \( E\{d\omega(t)\} = 0 \) and then integrating from \( t - d_1 \) to \( t \), it follows that

\[
E\left\{ x(t) - x(t - d_1) - \int_{t-d_1}^t \varpi(s)ds \right\} = 0. \quad (3.51)
\]
Similarly, taking mathematical expectation on both sides of (3.34) and using $\mathbb{E}\{d\omega(t)\} = 0$ and then integrating from $t - d_2$ to $t$, one can easily obtain

$$
\mathbb{E}\left\{ x(t) - x(t - d_2) - \int_{t-d_2}^{t} \mathcal{w}(s) ds \right\} = 0.
$$

(3.52)

Combining (3.34) together with (3.51) and (3.52) yields

$$
\mathbb{E}\{\Sigma_i \Psi(t)\} = 0,
$$

(3.53)

where

$$
\Sigma_i = \begin{bmatrix}
-A_i & -I & 0_{n,3m} & B_i & C_i & 0_{n,10m} & I & 0_{n,2m} \\
-I & 0_{n,6m} & I & 0_{n,10m} & I & 0_{n,m} \\
-I & 0_{n,7m} & I & 0_{n,10m} & I & 0_{n,m}
\end{bmatrix}.
$$

According to Lemma 1.5, $\mathbb{E}\{\Psi^T(t) \Psi(t)\} < 0$ if and only if $\Omega_i = \Sigma_i^T \Phi_i \Sigma_i^T < 0$ for the right orthogonal complement of $\Sigma_i$ given by

$$
\Sigma_i^T = \begin{bmatrix}
I & -A_i & 0_{n,5m} & I & I & 0_{n,11m} \\
0_{n,m} & B_i & 0_{n,3m} & I & 0_{n,14m} \\
0_{n,2m} & I & 0_{n,17m} \\
0_{n,3m} & I & 0_{n,16m} \\
0_{n,4m} & I & 0_{n,15m} \\
0_{n,m} & C_i & 0_{n,4m} & I & 0_{n,13m} \\
0_{n,m} & I & 0_{n,14m} & I & 0_{n,3m} \\
0_{n,9m} & I & 0_{n,10m} \\
0_{n,10m} & I & 0_{n,9m} \\
0_{n,11m} & I & 0_{n,8m} \\
0_{n,12m} & I & 0_{n,7m} \\
0_{n,13m} & I & 0_{n,6m} \\
0_{n,14m} & I & 0_{n,5m} \\
0_{n,15m} & I & 0_{n,4m} \\
0_{n,16m} & I & 0_{n,3m} \\
0_{n,2m} & I & 0_{n,11m} & -I & 0_{n,5m}
\end{bmatrix}^T.
$$

Hence, if $\Omega_i < 0$ holds, then (3.49) implies that

$$
\mathbb{E}\{y^T(t)Qy(t) + 2y^T(t)Su(t) + u^T(t)Ru(t)\} \geq \mathbb{E}\{\mathbb{L}V(t, x(t), i) + \gamma u^T(t)u(t)\}.
$$

(3.54)

Integrating (3.54) from 0 to $t_p$, under zero initial condition one can obtain

$$
\mathbb{E}\{G(y, u, t_p)\} \geq \mathbb{E}\{\gamma \langle u, u \rangle_{t_p} + V(t, t_p, i) - V(t, 0, i)\} \geq \mathbb{E}\{\gamma \langle u, u \rangle_{t_p}\}
$$

for all $t_p \geq 0$. Therefore, the stochastic neural network (3.34) is strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R}) - \gamma$ dissipative according to Definition 1.3. This completes the proof. \(\square\)
Remark 3.11. To reduce the computational burden of the theoretical results in Theorem 3.2, the Lemma 1.6 is used to deal with the derivative of the terms in $V_4(t, x(t), i)$ and $V_5(t, x(t), i)$ i.e., the triple integral terms in $\mathbb{L}V_4(t, x(t), i)$ and quadruple integral terms in $\mathbb{L}V_5(t, x(t), i)$ are bounded by double and triple integral terms, respectively. These double and triple integral terms in the augmented matrix use more information on state variables in Theorem 3.2.

Remark 3.12. By GFL, the inequalities $\mathbb{E}\{\Psi^T(t)\Phi_i \Psi(t)\} < 0$ including vectors in the stochastic scope are equivalently rewritten as deterministic matrix inequalities $\Omega_i = \Sigma_i^{-1} \Phi_i \Sigma_i^{-1} < 0$. Thus, it is easy to check the dissipativity of system (3.34) by solving the LMIs (3.37) and (3.38). It should be pointed out that the use of generalized Finsler lemma in the proof of Theorem 3.2 will lead to the less conservative results by minimizing the dimensions of the LMIs.

Remark 3.13. In Assumption 3.2, the noise intensity function $\varpi(t)$ is locally Lipschitz continuous and satisfies the linear growth condition (3.35) as well. The stochastic perturbation $\varpi(t)$ to be considered in this section is of nonlinear form. If the LKF is chosen as $V(t) = x^T(t) P_i x(t)$, then by virtue of Ito’s differential formula, there exists a nonlinear term $\text{Trace}\{\varpi^T(t) P_i \varpi(t)\}$. In order to express the dissipativity criteria for the neural network (3.34) in terms of strict LMIs, the restrictive condition $P_i \leq \lambda_i I$ is required. Under the assumptions 3.2 and 3.35, and according to the work in Chapter 5 of [66], it is easy to see that the stochastic neural networks (3.34) expressed by delayed stochastic differential equation has a unique continuous solution.

Remark 3.14. Theorem 3.2 provides a dissipativity analysis of Markovian jump stochastic neural network (3.34). Now, Theorem 3.2 is followed to obtain the passivity analysis of neural network (3.34) in the absence of stochastic term. In such case, the system (3.34) reduces to

$$
\dot{x}(t) = -A_i x(t) + B_i g(x(t)) + C_i g(x(t - d_1(t) - d_2(t))) + u(t),
$$

$$
y(t) = g(x(t)).
$$
By choosing $Q = 0, S = I$ and $R = 2\gamma I$, the following corollary is obtained from Theorem 3.2.

**Corollary 3.4.** Under Assumptions 3.1 and 3.2, for given scalars $d_1, d_2, \tilde{\mu}_1$ and $\tilde{\mu}_2$, the system (3.55) is passive, if there exist positive definite matrices $P_i, Q_i(t = 1, 2, \cdots, 6), M_v(v = 1, 2, \cdots, 4), R_1, R_2, S_1, S_2$, any matrix $V$ and positive diagonal matrices $U, W$ and scalar $\gamma > 0$ such that the following LMI holds for all $i \in \mathcal{S}$:

$$
\Xi_i = (\Xi_{l,m,i})_{17 \times 17} < 0,
$$

(3.56)

where

$$
\Xi_{1,1,i} = \sum_{j=1}^{N} q_{ij} P_j + Q_1 + Q_3 + Q_4 + Q_5 + Q_6 + d_1^2 M_1 + d_2^2 M_2 + \frac{d_1^4}{4} R_1 + \frac{d_2^4}{4} R_2
$$

$$
+ \frac{d_1^2}{6} S_1 + \frac{d_2^2}{6} S_2 - A_i^T \gamma A_i, \quad \Xi_{1,2,i} = -U L + A_i - A_i^T \gamma B_i, \quad \Xi_{1,3,i} = -A_i^T \gamma C_i,
$$

$$
\Xi_{1,17,i} = -A_i \gamma, \quad \Xi_{2,2,i} = Q_2 + d_1^2 M_3 + d_2^2 M_4 - 2U + B_i + B_i^T - B_i^T \gamma B_i,
$$

$$
\Xi_{2,3,i} = C_i - B_i^T \gamma C_i, \quad \Xi_{2,17,i} = I - B_i \gamma, \quad \Xi_{3,3,i} = -2W - Q_2(1 - \tilde{\mu}_1 - \tilde{\mu}_2) - C_i^T C_i,
$$

$$
\Xi_{3,4,i} = -W L, \quad \Xi_{3,17,i} = -C_i \gamma, \quad \Xi_{4,4,i} = -Q_1(1 - \tilde{\mu}_1 - \tilde{\mu}_2), \quad \Xi_{5,5,i} = -Q_3(1 - \tilde{\mu}_1),
$$

$$
\Xi_{6,6,i} = -Q_4(1 - \tilde{\mu}_2), \quad \Xi_{7,7,i} = -Q_5, \quad \Xi_{8,8,i} = -\frac{6}{d_1^4} S_1, \quad \Xi_{9,9,i} = -Q_6, \quad \Xi_{10,10,i} = -M_1,
$$

$$
\Xi_{11,11,i} = -M_2, \quad \Xi_{12,12,i} = -M_3, \quad \Xi_{13,13,i} = -M_4, \quad \Xi_{14,14,i} = -R_1, \quad \Xi_{15,15,i} = -R_2,
$$

$$
\Xi_{16,16,i} = -\frac{6}{d_2^4} S_2, \quad \Xi_{17,17,i} = -\gamma, \quad L = \text{diag}\{l_1, l_2, \cdots, l_n\}
$$

and the remaining coefficients are zero.

**Proof.** For the system (3.55), the following equation is true for any matrix $V$ with appropriate dimension:

$$
\mathbb{E}\{2\dot{x}^T(t)V[-\dot{x}(t) - A_i x(t) + B_i g(x(t)) + C_i g(x(t - d_1(t) - d_2(t))) + u(t)]\} = 0.
$$

Considering the same LKF as in Theorem 3.2 excluding stochastic term and following the similar argument of Theorem 3.2 and adding the above equation with (3.49), the following can be obtained:

$$
\mathbb{E}\{LV(t, x(t), i) - 2y^T(t)u(t) - u^T(t)\gamma u(t)\} \leq \mathbb{E}\{\zeta^T(t)\Phi_i \zeta(t)\},
$$

(3.57)
Therefore, without stochastic term, Lemma 1.5 yield 
\[\zeta^T(t) = \left[x^T(t) \ g^T(x(t)) \ g^T(x(t) - d_1(t) - d_2(t)) \ x^T(t - d_1(t) - d_2(t)) \ x^T(t - d_1(t)) \right]
\[x^T(t - d_2(t)) \ x^T(t - d_1(t)) \ u^T(t) \ x^T(t - d_2) \ \int_{t-d_1}^t x^T(s) ds \ \int_{t-d_2}^t x^T(s) ds
\int_{t-d_1}^t g^T(x(s)) ds \ \int_{t-d_2}^t g^T(x(s)) ds \ \int_{-d_1}^{t-d_1} \int_{t+\theta}^{t+\theta} x^T(s) ds d\theta \ \int_{-d_2}^{t-d_2} \int_{t+\theta}^{t+\theta} x^T(s) ds d\theta
\int_{-d_1}^{t} \int_{t+\theta}^{t+\theta} x^T(s) ds d\theta d\alpha \ \int_{-d_2}^{t} \int_{t+\theta}^{t+\theta} x^T(s) ds d\theta d\alpha \ \dot{x}^T(t)\]

and
\[\Phi_i = (\Phi_{i,m,i})_{18 \times 18} \quad (3.58)\]

with
\[\Phi_{1,1,i} = \sum_{j=1}^{N} q_j P_j + Q_1 + Q_3 + Q_4 + Q_5 + Q_6 + d_1^2 M_1 + d_2^2 M_2 + \frac{d_1^4}{4} R_1 + \frac{d_2^4}{4} R_2 + \frac{d_1^3}{6} S_1
\[+ \frac{d_2^3}{6} S_2, \ \Phi_{1,2,i} = U L, \ \Phi_{1,18,i} = -A_t^T V + P_i, \ \Phi_{2,2,i} = Q_2 + d_1^2 M_3 + d_2^2 M_4 - 2U,
\Phi_{2,8,i} = -I, \ \Phi_{2,18,i} = B_t^T V, \ \Phi_{3,3,i} = -2W - Q_2(1 - \bar{\mu}_1 - \bar{\mu}_2), \ \Phi_{3,4,i} = W L,
\Phi_{3,18,i} = C_t^T V, \ \Phi_{4,4,i} = -Q_1(1 - \bar{\mu}_1 - \bar{\mu}_2), \ \Phi_{5,5,i} = -Q_3(1 - \bar{\mu}_1), \ \Phi_{6,6,i} = -Q_4(1 - \bar{\mu}_2),
\Phi_{7,7,i} = -Q_5, \ \Phi_{8,8,i} = -\gamma, \ \Phi_{8,18,i} = V, \ \Phi_{9,9,i} = -Q_6, \ \Phi_{10,10,i} = -M_1, \ \Phi_{11,11,i} = -M_2,
\Phi_{12,12,i} = -M_3, \ \Phi_{13,13,i} = -M_4, \ \Phi_{14,14,i} = -R_1, \ \Phi_{15,15,i} = -R_2, \ \Phi_{16,16,i} = -\frac{6}{d_1^2} S_1,
\Phi_{17,17,i} = -\frac{6}{d_2^2} S_2, \ \Phi_{18,18,i} = -V - V^T\]

and the remaining coefficients are zero.

For \(\Gamma_i = [-A_i \ B_i \ C_i \ 0_{n,4m} \ I \ 0_{n,9m} \ -I]\), we have \(\Gamma_i \zeta(t) = 0\).

Therefore, without stochastic term, Lemma 1.5 yield \(\zeta^T(t) \Phi_i \zeta(t) < 0\) if and only if \(\Xi_i = \).
\[ \Gamma_i^T \Phi_i \Gamma_i \preceq 0 \] for the right orthogonal complement of \( \Gamma_i \) given by

\[
\Gamma_i^\perp = \begin{bmatrix}
I & 0_{n,6m} & A_i & 0_{n,10m} \\
0_{n,m} & -I & 0_{n,5m} & B_i & 0_{n,10m} \\
0_{n,2m} & -I & 0_{n,4m} & C_i & 0_{n,10m} \\
0_{n,3m} & I & 0_{n,14m} & & \\
0_{n,4m} & I & 0_{n,13m} & & \\
0_{n,5m} & I & 0_{n,12m} & & \\
0_{n,6m} & I & 0_{n,11m} & & \\
0_{n,15m} & I & 0_{n,2m} & & \\
0_{n,8m} & I & 0_{n,9m} & & \\
0_{n,9m} & I & 0_{n,8m} & & \\
0_{n,10m} & I & 0_{n,7m} & & \\
0_{n,11m} & I & 0_{n,6m} & & \\
0_{n,12m} & I & 0_{n,5m} & & \\
0_{n,13m} & I & 0_{n,4m} & & \\
0_{n,14m} & I & 0_{n,3m} & & \\
0_{n,16m} & I & 0_{n,m} & & \\
0_{n,7m} & I & 0_{n,9m} & & I
\end{bmatrix}^T.
\]

Hence, if \( \Theta_i < 0 \) holds, then (3.57) implies that

\[
\mathbb{E}\{LV(t, x(t), i) - 2y^T(t)u(t) - u^T(t)\gamma u(t)\} \leq 0. \tag{3.59}
\]

Integrating (3.59) from 0 to \( t_p \), under zero initial condition, one can obtain

\[
2 \int_0^{t_p} \mathbb{E}\{y^T(s)u(s)\} ds \geq \mathbb{E}\left\{ V(t, x(t), i) - V(t, x(0), i) - \gamma \int_0^{t_p} u^T(s)u(s) ds \right\} \\
\geq - \gamma \int_0^{t_p} \mathbb{E}\{u^T(s)u(s)\} ds, \tag{3.60}
\]

for all \( t_p \geq 0 \). Therefore, the neural network (3.34) is passive according to Definition 1.4. This completes the proof. \( \blacklozenge \)

**Remark 3.15.** When \( d_2(t) = 0 \) and \( d_1(t) = d(t) \), system (3.55) reduces to the following form with the single delay \( d(t) \):

\[
\dot{x}(t) = - A_i x(t) + B_i g(x(t)) + C_i g(x(t - d(t))) + u(t), \tag{3.61}
\]

\[
y(t) = g(x(t)),
\]

where \( d(t) \) satisfies \( 0 < d(t) \leq d \) and \( \dot{d}(t) \leq \dot{\mu} \). By letting \( Q_b = 0 (b = 1, 2, 4, 6) \), \( M_c = 0 \ (c = 2, 4, 5, 6) \), \( R_2 = 0 \) and \( S_2 = 0 \) in Lyapunov functional (3.39), along a similar line as in the derivation of Corollary 3.4, a passivity criterion for delayed neural network (3.61) can be obtained as in the following corollary.
Corollary 3.5. Under Assumptions 3.1 and 3.2, for given scalars $d$ and $\tilde{\mu}$, the system (3.61) is passive, if there exist positive definite matrices $P_i, Q_3, Q_5, M_1, M_3, R_1, S_1$ and positive diagonal matrices $U, W$ and scalar $\gamma > 0$, such that the following LMI holds:

$$\Lambda_i = (\Lambda_{i,m,i})_{10 \times 10} < 0,$$  \hspace{1cm} (3.62)

where

$$\begin{align*}
\Lambda_{1,1,i} &= \sum_{j=1}^{N} q_{ij} P_j + Q_3 + Q_5 + d^2 M_1 + \frac{d^4}{4} R_1 + \frac{d^6}{6} S_1 - A_i^T \gamma A_i, \quad \Lambda_{1,2,i} = -LU + A_i \\
- A_i^T \gamma B_i, \quad \Lambda_{1,3,i} &= -A_i^T \gamma C_i, \quad \Lambda_{1,10,i} = -A_i \gamma + P_i, \quad \Lambda_{2,2,i} = d^2 M_3 - 2U + B_i \\
+ B_i^T - B_i^T \gamma B_i, \quad \Lambda_{2,3,i} &= -B_i^T \gamma C_i, \quad \Lambda_{2,10,i} = I - B_i^T \gamma, \quad \Lambda_{3,3,i} = -2W - C_i^T \gamma C_i, \\
\Lambda_{3,4,i} &= -LW, \quad \Lambda_{3,10,i} = -C_i^T \gamma, \quad \Lambda_{4,4,i} = -Q_3(1 - \tilde{\mu}), \quad \Lambda_{5,5,i} = -Q_5, \quad \Lambda_{6,6,i} = -M_1, \\
\Lambda_{7,7,i} &= -R_1, \quad \Lambda_{8,8,i} = -\frac{6}{d^3} S_1, \quad \Lambda_{9,9,i} = -M_3, \quad \Lambda_{10,10,i} = -\gamma, \quad L = \text{diag}\{l_1, l_2, \ldots, l_n\}
\end{align*}$$

and the remaining coefficients are zero.

Proof. The proof is same as that of Corollary 3.4 and hence it is omitted. \qed

Remark 3.16. The system (3.61) without Markovian jump reduces to the following form

$$\begin{align*}
\dot{x}(t) &= -Ax(t) + Bg(x(t)) + Cg(x(t - d(t))) + u(t), \quad (3.63) \\
y(t) &= g(x(t)).
\end{align*}$$

A passivity criterion for delayed neural network (3.63) can be derived as follows from Corollary 3.4.

Corollary 3.6. Under Assumptions 3.1 and 3.2, for given scalars $d$ and $\tilde{\mu}$, the system (3.63) is passive, if there exist positive definite matrices $P, Q_3, Q_5, M_1, M_3, R_1, S_1$ and positive diagonal matrices $U, W$ and scalar $\gamma > 0$, such that the following LMI holds:

$$\Theta = (\Theta_{i,m})_{10 \times 10} < 0,$$  \hspace{1cm} (3.64)
where

\[ \Theta_{1,1} = Q_3 + Q_5 + d^2 M_1 + \frac{d^4}{4} R_1 + \frac{d^6}{6} S_1, \quad -A^T \gamma A, \quad \Theta_{1,2} = -LU + A - A^T \gamma B, \]

\[ \Theta_{1,3} = -A^T \gamma C, \quad \Theta_{1,10} = -A \gamma + P, \quad \Theta_{2,2} = d^2 M_3 - 2U + B + B^T - B^T \gamma B, \]

\[ \Theta_{2,3} = -B^T \gamma C, \quad \Theta_{2,10} = I - B^T \gamma, \quad \Theta_{3,3} = -2W - C^T \gamma C, \quad \Theta_{3,4} = -LW, \]

\[ \Theta_{3,10} = -C^T \gamma, \quad \Theta_{4,4} = -Q_3 (1 - \hat{\mu}), \quad \Theta_{5,5} = -Q_5, \quad \Theta_{6,6} = -M_1, \quad \Theta_{7,7} = -R_1, \]

\[ \Theta_{8,8} = -\frac{6}{d^3} S_1, \quad \Theta_{9,9} = -M_3, \quad \Theta_{10,10} = -\gamma, \quad L = \text{diag} \{ l_1, l_2, \ldots, l_n \} \]

and the remaining coefficients are zero.

**Proof.** The proof is same as that of Corollary 3.5 and hence it is omitted. \(\square\)

### 3.3.3 Numerical examples

In this section, four illustrative examples are given to show the effectiveness of the proposed criterion.

**Example 3.4.** Consider the Markovian jump stochastic neural networks (3.34) with the following parameters:

\[ A_1 = \begin{bmatrix} 2.3 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.9 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 & 0.5 \\ -0.2 & 0.1 \end{bmatrix}, \]

\[ C_1 = \begin{bmatrix} 0.2 & -0.3 \\ 0.4 & 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.5 \end{bmatrix}, \quad T_{11} = 0.24 I, \quad T_{12} = 0.20 I, \quad T_{21} = 0.19 I, \quad T_{22} = 0.10 I. \]

The density matrix \( \Gamma = \{ q_{ij} \} \) is assumed to be \( \Gamma = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \). The purpose of this example is to find the maximal allowable time-delay \( d_2 \) such that the system is strictly \((Q, S, R)\) dissipative. For this, the following matrices are chosen:

\[ Q = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \quad R = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}, \quad S = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}. \]

In this example, the activation functions are assumed to be \( g_1(x) = 0.4 \tanh(x) \) and \( g_2(x) = 0.8 \tanh(x) \). It is easy to check that these activation functions satisfy the Assumption 3.1
Figure 3.7: The state trajectories of system (3.34) with additive time-varying delays $d_1(t) = 0.6 + 0.2 \sin^2(t)$ and $d_2(t) = 3.0462 + 0.25 \sin^2(t)$ for Example 3.4.

Figure 3.8: The state trajectories of system (3.34) with additive time-varying delays $d_1(t) = 0.7 + 0.15 \sin^2(t)$ and $d_2(t) = 2.9753 + 0.2 \sin^2(t)$ for Example 3.4.

Figure 3.9: Unstable behavior of system (3.34) with time-varying delay $d_1(t) = 2.6 + 0.2 \sin^2(t)$ for Example 3.4.
Figure 3.10: Unstable behavior of system (3.34) with time-varying delay $d_1(t) = 2.7 + 0.15 \sin^2(t)$ for Example 3.4.

Figure 3.11: The Markovian switching signal $r(t)$ for Example 3.4.
Table 3.6: Maximum allowable upper bounds of $d_2$ for various $d_1$ for Example 3.4.

<table>
<thead>
<tr>
<th>Theorem 3.2</th>
<th>$\tilde{\mu}_1 = 0.4$, $\tilde{\mu}_2 = 0.5$</th>
<th>$\tilde{\mu}_1 = 0.3$, $\tilde{\mu}_2 = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 = 0.5$</td>
<td>3.7321</td>
<td>3.5432</td>
</tr>
<tr>
<td>$d_1 = 1$</td>
<td>3.5462</td>
<td>3.3753</td>
</tr>
<tr>
<td>$d_1 = 1.5$</td>
<td>3.3981</td>
<td>3.1907</td>
</tr>
<tr>
<td>$d_1 = 2$</td>
<td>3.1542</td>
<td>2.9321</td>
</tr>
<tr>
<td>$d_1 = 2.5$</td>
<td>2.9584</td>
<td>2.7652</td>
</tr>
<tr>
<td>$d_1 = 3$</td>
<td>Infeasible</td>
<td>Infeasible</td>
</tr>
</tbody>
</table>

with $L = \text{diag}\{0.4, 0.8\}$. By using MATLAB LMI toolbox and by taking $d_1 = 0.5$, $d_2 = 3.7321$, $\tilde{\mu}_1 = 0.4$ and $\tilde{\mu}_2 = 0.5$, the LMIs in Theorem 3.2 are solvable and the corresponding feasible solutions are obtained:

$$P_1 = \begin{bmatrix} 1.9016 & 0.0002 \\ 0.0002 & 2.0253 \end{bmatrix}, P_2 = \begin{bmatrix} 0.1840 & -0.0031 \\ -0.0031 & 0.0892 \end{bmatrix},$$

$$\lambda_1 = 2.1387, \lambda_2 = 2.1562, \text{and } \gamma = 8.1139.$$

By taking $\tilde{\mu}_1 = 0.4$ and $\tilde{\mu}_2 = 0.5$, and then $\tilde{\mu}_1 = 0.3$ and $\tilde{\mu}_2 = 0.4$, the maximum allowable upper bounds $d_2$ are obtained, respectively, for various $d_1$ as listed in Table 3.6. Furthermore, Figure 3.7 shows the state trajectories of variable $x(t)$ with the initial condition $[0.4, 0.8]^T$ for the additive delays $d_1(t) = 0.6 + 0.2\sin^2(t)$ and $d_2(t) = 3.0462 + 0.25\sin^2(t)$. Further, the state trajectories of variable $x(t)$ for $d_1(t) = 0.7 + 0.15\sin^2(t)$ and $d_2(t) = 2.9753 + 0.2\sin^2(t)$ are depicted in Figure 3.8. For $d_1(t) = 2.6 + 0.2\sin^2(t)$, the system (3.34) becomes unstable which can be seen through Figure 3.9 where as Figure 3.10 depicts the unstable behavior of neural network (3.34) for the delay $d_1(t) = 2.7 + 0.15\sin^2(t)$. Further, the possible jumping between modes for the above parameters is given in Figure 3.11. The results in Table 3.6 indicate that Theorem 3.2 endorsed the dissipativity criterion of Markovian jump stochastic neural networks with two additive time-varying delays.

**Example 3.5.** Consider the Markovian jump neural network (3.61) as described in [6] and
The density matrix $\Gamma = \{q_{ij}\}$ is chosen in such a way that $\Gamma = \begin{bmatrix} -7 & 7 \\ 6 & -6 \end{bmatrix}$. In this example, the activation functions are assumed to be $g_1(x) = g_2(x) = \tanh(x)$. It is easy to see that these activation functions must satisfy the Assumption 3.1 with $L = \text{diag}\{1, 1\}$. By using Corollary 3.5 of this section, the upper bounds of the delay are computed and they
Table 3.7: Maximum allowable upper bounds of $d$ for various $\tilde{\mu}$ for Example 3.5.

<table>
<thead>
<tr>
<th>$\tilde{\mu}$</th>
<th>0</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>No. of Decision Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[6] Large</td>
<td>0.2320</td>
<td>0.2185</td>
<td>0.2031</td>
<td>0.2024</td>
<td></td>
<td>$12n^2 + 11n + 7$</td>
</tr>
<tr>
<td>[13] Large</td>
<td>1.2045</td>
<td>0.9628</td>
<td>0.9042</td>
<td>0.8635</td>
<td></td>
<td>$22n^2 + 18n + 1$</td>
</tr>
<tr>
<td>Corollary 3.5</td>
<td>2.2389</td>
<td>1.9432</td>
<td>1.7328</td>
<td>1.6329</td>
<td></td>
<td>$3.5n^2 + 5.5n + 1$</td>
</tr>
</tbody>
</table>

Table 3.8: Maximum allowable upper bounds of $d$ for various $\tilde{\mu}$ for Example 3.6.

<table>
<thead>
<tr>
<th>$\tilde{\mu}$</th>
<th>0</th>
<th>0.5</th>
<th>0.9</th>
<th>No. of Decision Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[23]</td>
<td>4.4119</td>
<td>3.9100</td>
<td>0.5223</td>
<td>$4n^2 + 4n + 2$</td>
</tr>
<tr>
<td>[79]</td>
<td>11.0075</td>
<td>10.4567</td>
<td>9.6032</td>
<td>$13.5n^2 + 9.5n + 1$</td>
</tr>
<tr>
<td>Corollary 3.6</td>
<td>13.5642</td>
<td>12.8312</td>
<td>10.5213</td>
<td>$3.5n^2 + 5.5n + 1$</td>
</tr>
</tbody>
</table>

are compared with the previous existing results established in [6] and [13] in Table 3.7. From Table 3.7, it can be easily seen that the method proposed in this section is much less conservative than the corresponding method in [6] and [13]. When $u(t) = 0$, one can obtain the state trajectories of the state $x(t)$ for the delay $d(t) = 1.9389 + 0.3\sin(t)$ which is shown in Figure 3.12. Furthermore, the random jumping mode $r(t)$ is predicted in Figure 3.13.

**Example 3.6.** Consider the neural network (3.63) as described in [23] and [79]:

$$A = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1.05 & -1.1 \\ -0.4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.3 \end{bmatrix}. $$

The activation functions are same as in Example 3.5. Solving the LMI in Corollary 3.6, the maximum allowable upper bounds of the time-varying delay $d(t)$ is obtained as in Table 3.8 for different $\tilde{\mu}$'s. Obviously, the comparison results in Table 3.8 describe the significance of the method proposed in this section over the methods in [23] and [79].

**Example 3.7.** Consider the following neural network (3.63) as described in [40], [94], and...
Figure 3.14: The state trajectories of system (3.61) with time-varying delay $d(t) = 1.9389 + 0.3\sin(t)$ for Example 3.7.

\[ A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \]

\[ B = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}, \]

\[ C = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}. \]

The activation functions are assumed to be $g_1(x) = 0.1137\tanh(x)$, $g_2(x) = 0.1279\tanh(x)$, $g_3(x) = 0.7994\tanh(x)$ and $g_4(x) = 0.2368\tanh(x)$ to satisfy the Assumption 3.1 with $L = \text{diag}\{0.1137, 0.1279, 0.7994, 0.2368\}$. Further, the maximum allowable time delay values of $d(t)$ are concluded by the feasibility of the LMIs in Corollary 3.6 under different $\tilde{\mu}$ so that the system is passive and the computed values are listed in Table 3.9. From Table 3.9, it can be easily seen that the method proposed in this section is much less conservative than the corresponding methods in [40], [94], and [63]. When $u(t) = 0$, the behavior of the state trajectories of the state $x(t)$ for the delay $d(t) = 6.1287 + 0.1\sin^2(t)$ is predicted in Figure 3.14.
Table 3.9: Admissible upper bounds of $d$ for various $\tilde{\mu}$ for Example 3.7.

<table>
<thead>
<tr>
<th>$\tilde{\mu}$</th>
<th>0</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
<th>No. of Decision Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[40]</td>
<td>3.5841</td>
<td>3.3125</td>
<td>2.7500</td>
<td>2.6468</td>
<td>2.6361</td>
<td>$10.5n^2 + 10.5n$</td>
</tr>
<tr>
<td>[94]</td>
<td>3.8363</td>
<td>3.5290</td>
<td>2.8606</td>
<td>2.7273</td>
<td>2.7098</td>
<td>$15.5n^2 + 10.5n$</td>
</tr>
<tr>
<td>[63]</td>
<td>3.9428</td>
<td>3.7512</td>
<td>2.9514</td>
<td>2.7743</td>
<td>2.9712</td>
<td>$9n^2 + 7n$</td>
</tr>
<tr>
<td>[63]</td>
<td>4.3259</td>
<td>4.0980</td>
<td>3.3912</td>
<td>2.8335</td>
<td>2.7821</td>
<td>$10n^2 + 12n$</td>
</tr>
<tr>
<td>Corollary 3.6</td>
<td>6.9756</td>
<td>6.3287</td>
<td>5.9184</td>
<td>5.6523</td>
<td>4.2865</td>
<td>$3.5n^2 + 5.5n + 1$</td>
</tr>
</tbody>
</table>

**Remark 3.17.** The obtained results and comparison to most recent results in the literature are given in Tables 3.7, 3.8 and 3.9. From these tables, it is obvious that, technique proposed by using Lemmas 1.5 and 1.6 delivers significantly better results with less number of decision variables comparable to other methods such as the Jensen inequality lemma [7] and delay decomposition approach [13].

### 3.4 Conclusions and future directions

In this chapter, the problem of dissipativity and passivity analysis has been described for Markovian jump stochastic neural networks with two additive time-varying delays. By introducing triple and quadruple integral terms with an augmented LKF and utilizing stochastic analysis, novel dissipativity and passivity criteria are proposed. The approach is based on GFL lemma and second order reciprocal convex combination combined with LMI techniques. Moreover, in this chapter the derivation has avoided some redundant free weighting matrices due to the usage of GFL; thus, compared with existing relevant results, the GFL approach reduces the computational complexity and hence leads to less conservatism. Finally, illustrative examples are provided to show the effectiveness of the proposed method. It is clear that the methodology proposed in this section has inherent flexibility and is potentially applicable to other problems with time-varying delays, such as Memristive neural networks, state estimation for complex-valued neural networks and Memristive bidirectional associative memory neural networks using GFL approach. The corresponding results will appear in the near future.