Chapter 2

Local and Global Existence of Mild Solution for Impulsive FSDEs

2.1 Introduction

An ODE coupled with impulsive effects is considered as impulsive differential equations and it was introduced by Milman and Myshkis in the year 1960. The dynamics of process in which sudden discontinuous jumps occur in the real world problems can be described by the impulsive differential equations, namely perturbations such as earthquake, harvesting, shock etc., can be well-approximated as instantaneous change of state or impulses. Such processes are naturally seen in biology, physics, engineering, etc. Moreover, a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solution and noncontinuability of solutions, hence it has been developed tremendously (see [73]).

In the study of nonlinear evolution equations, on certain occasions, the solution exists locally in some short interval but not globally in time, which is caused by the phenomena called “blow-up”. For instance, the solution \( x(t) = \frac{1}{c-t} \) of the Ricatti equation \( \dot{x}(t) = x^2(t) \) blow-up at \( t \to c \) and losses the global existence of solution. The question of global existence and uniqueness of solutions are not resolved, even for most important nonlinear
evolution equations arising in classical physics, such as Navier-Stokes, Euler equations in fluid mechanics and Einstein field equations in general relativity etc. It strongly prompts the study of local and global existence of solution in the study of abstract differential equations.

In [100], Ouahab studied the local and global existence results for first order impulsive functional differential equations with multiple delays by means of the nonlinear alternative of Leray-Schauder fixed point theorem. The local and global existence of mild solution for a class of impulsive fractional semilinear integrodifferential equations have been studied by Rashid and Al-Omari [108]. Recently Chauhan and Dabas [25] discussed the local and global existence of mild solution for impulsive fractional functional integrodifferential equations with nonlocal condition.

The nonlocal condition has been considered to be more useful than the standard initial condition to describe some real life phenomena. For instance, the Cauchy problem describing the diffusion phenomenon of a small amount of gas in a transparent tube with nonlocal conditions gives better results, than using the usual classical initial conditions [37]. Byszewski [22] introduced nonlocal initial conditions into the initial-value problems and indicated that the corresponding systems more accurately describe the phenomena, since more information has been taken into account at the onset of the experiment, thereby reducing the ill effects incurred by a single (possibly erroneous) initial measurement. For more details on nonlocal problems, one can refer [17, 22, 83] and the references therein.

On the other hand, the abstract fractional Cauchy problems with nonlocal conditions have been considered to be the field of increasing interest. N’Guerekata [53, 54] obtained the existence and uniqueness of solutions for some fractional abstract differential equation with nonlocal conditions with the aid of contraction mapping principle and Krasnoselskii fixed point theorem. Debbouche [35] established the existence and uniqueness of local mild solution and then local classical solution of a class of nonlinear fractional evolution integrodifferential systems with nonlocal conditions by using theory of resolvent operators, fractional powers of operators, fixed point technique and the Gelfand-Shilov principle. Wang et al. [128] studied the nonlocal Cauchy problem for semilinear fractional order evolution equations with nonlocal conditions.
Motivated by the above observations, the research in this area is evidently very important on both theoretic and practical points of view. Hence, this Chapter concerned with the local and global existence of mild solution for impulsive FSDE with nonlocal conditions.

2.2 Local and Global Existence of Mild Solution for impulsive fractional semilinear SDEs with nonlocal condition

2.2.1 Problem Description

Consider the following form of impulsive fractional semilinear SDEs with nonlocal condition

\[
\begin{aligned}
C D^\alpha_t x(t) &= -Ax(t) + f(t, x_t) + \int_0^t \sigma(t, s, x_s)dw(s), \quad t \in J := [0, b], \ t \neq t_k \\
\Delta x(t_k) &= I_k(x(t_k^-)), \ k = 1, 2, \cdots m \\
h(x) &= \phi_0 \text{ on } [-r, 0], 
\end{aligned}
\]  

(2.1)

where \(0 < \alpha < 1\), \(C D^\alpha_t\) denotes the Caputo fractional derivative, \(-A\) is sectorial operator, \(x(\cdot)\) takes values in a Hilbert space \(H\), the nonlinear maps \(f : J \times PC_0 \to H\) and \(\sigma : J \times J \times PC_0 \to L(K, H)\) are continuous, where \(PC_0 = PC([-r, 0], H)\). For any \(x \in PC_b = PC([-r, b], H)\), and \(t \in J\), define the element \(x_t\) of \(PC_0\) by \(x_t(\theta) = x(t + \theta), \ \theta \in [-r, 0]\). The function \(\phi_0 \in PC_b\), and the map \(h\) is defined from \(PC_b\) into \(PC_b\).

2.2.2 Basic Preliminaries and Hypotheses

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space furnished with a complete family of right continuous increasing sub-\(\sigma\)-algebras \(\{\mathcal{F}_t : \ t \in J\}\) satisfying \(\mathcal{F}_t \subset \mathcal{F}\). Let \(x(t) : \Omega \to H\) be a continuous \(\mathcal{F}_t\)-adapted, \(H\)-valued stochastic process. Let \(\{\zeta_n\}_{n=1}^\infty\) be a complete orthonormal basis of \(K\). Suppose that \(w(t), \ t \geq 0\) is a cylindrical \(K\)-valued Wiener process with finite trace nuclear covariance operator \(Q \geq 0\), denote \(Tr(Q) = \sum_{n=1}^\infty \lambda_n < \infty\), which satisfies that \(Q \zeta_n = \lambda_n \zeta_n\). Indeed \(w(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} \zeta_n \zeta_n^t\), where \(\{w_n(t)\}_{n=1}^\infty\) are mutually
independent one dimensional standard Wiener processes. Let \( \varphi \in L(K, H) \), and define
\[
\|\varphi\|_{Q}^{2} = \text{Tr}(\varphi Q \varphi^{*}) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \varphi \zeta_n\|^{2}.
\]
If \( \|\varphi\|_{Q} < \infty \), then \( \varphi \) is called a \( Q \)-Hilbert-Schmidt operator. The completion \( L_{Q}(K, H) \) of \( L(K, H) \) with respect to the topology induced by the norm \( \cdot \|_{Q} \), where \( \|\varphi\|_{Q}^{2} = \langle \varphi, \varphi \rangle \) is a Hilbert space with the above norm topology.

**Definition 2.1.** [106]. The two parameter function of the Mittag-Leffler type is defined as
\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{C} \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - z} d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},
\]
where \( C \) is a contour, which starts and ends at \( -\infty \) and encircles the disc \( |\mu| \leq |z|^{1/2} \) counter clockwise.

The Laplace transform of the Mittag-Leffler function is given by
\[
\int_{0}^{\infty} e^{-\lambda t} t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(\pm a t^{\alpha}) dt = \frac{k! \lambda^{\alpha-\beta}}{(\lambda^{\alpha} \pm a)^{k+1}}, \quad \text{Re}(\lambda) > |a|^{1/\alpha}.
\]

**Definition 2.2.** [55]. A closed and linear operator \( A \) is said to be sectorial if there are constants \( \omega \in \mathbb{R}, \theta \in \left[\frac{\pi}{2}, \pi\right], \tilde{M} > 0 \) such that the following conditions are satisfied
\begin{enumerate}
  \item \( \rho(A) \subset \sum_{(\theta, \omega)} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\text{arg}(\lambda - \omega)| < \theta\}, \)
  \item \( \|R(\lambda, A)\| \leq \frac{\tilde{M}}{|\lambda - \omega|}, \lambda \in \sum_{(\theta, \omega)} \).
\end{enumerate}

**Definition 2.3.** [115]. Let \( A \) be a closed and linear operator with the domain \( D(A) \) defined in a Hilbert space \( H \). Let \( \rho(A) \) be the resolvent set of \( A \). One can say that \( A \) is the generator of an \( \alpha \)-resolvent family, if there exists \( \omega \geq 0 \) and a strongly continuous function \( \hat{S}_{\alpha} : \mathbb{R}_{+} \to L(H) \), where \( L(H) \) is a Banach space of all bounded linear operator from \( H \) into \( H \) and the corresponding norm is denoted by \( \|\cdot\| \), such that \( \{\lambda^{\alpha} : \text{Re}(\lambda) > \omega\} \subset \rho(A) \) and
\[
(\lambda^{\alpha} I - A)^{-1} x = \int_{0}^{\infty} e^{-\lambda t} \hat{S}_{\alpha}(t) x dt, \quad \text{Re}(\lambda) > \omega, \quad x \in H,
\]
where \( \hat{S}_{\alpha}(t) \) is called the \( \alpha \)-resolvent family generated by \( A \).
Definition 2.4. [32]. Let $A$ be a closed linear operator with the domain $D(A)$ defined in a Hilbert space $H$ and $\alpha > 0$. One can say that $A$ is the generator of a solution operator, if there exists $\omega \geq 0$ and a strongly continuous function $\hat{T}_\alpha : \mathbb{R}_+ \to L(H)$ such that

$$\{\lambda^\alpha : \text{Re}(\lambda) > \omega\} \subset \rho(A)$$

and

$$\lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}x = \int_0^\infty e^{-\lambda t}\hat{T}_\alpha(t)x \, dt, \quad \text{Re}(\lambda) > \omega, \quad x \in H,$$

where $\hat{T}_\alpha(t)$ is called the solution operator generated by $A$.

Definition 2.5. An $\mathcal{F}_t$ adapted stochastic process $x : [-r, b] \to H$ is called mild solution of the system (2.1), if $x(t) = \psi(t)$ on $[-r, 0]$ where $\psi \in PC_b$ such that $h(\psi) = \varphi_0$ on $[-r, 0]$, and satisfies the following conditions

(i) $x(t)$ is $PC_b$ valued and the restrictions of $x(\cdot)$ to $(t_k, t_{k+1}]$, $k = 1, 2, \cdots m$ is continuous.

(ii) For each $t \in J$, $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \hat{T}_\alpha(t)\psi(0) + \int_0^t \hat{S}_\alpha(t - s) \left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) \, dw(\tau)\right] \, ds, & t \in [0, t_1], \\ \hat{T}_\alpha(t)\psi(0) + \sum_{i=1}^m \int_0^t \hat{T}_\alpha(t - t_i)I_i(x(t_i^-)) \\ + \int_0^t \hat{S}_\alpha(t - s) \left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) \, dw(\tau)\right] \, ds, & t \in (t_k, t_{k+1}], \quad k = 1, 2, \cdots m \end{cases}$$

where

$$\hat{T}_\alpha(t) = E_{\alpha,1}(-At^\alpha) = \frac{1}{2\pi i} \int_{\hat{B}_r} e^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^{\alpha}I + A} \, d\lambda,$$

$$\hat{S}_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha) = \frac{1}{2\pi i} \int_{\hat{B}_r} e^{\lambda t} \frac{1}{\lambda^{\alpha}I + A} \, d\lambda,$$

here $\hat{B}_r$ denotes the Bromwich path, $\hat{S}_\alpha(t)$ is $\alpha$-resolvent family and $\hat{T}_\alpha(t)$ is the solution operator, both are generated by $A$.

The following hypotheses are assumed to establish the main results.

(H2.1) The function $f : J \times H \to H$ is continuous and there exists a constant $N_1$ such that

$$\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq N_1\mathbb{E}\|x - y\|^2$$

for all $x, y \in H$.  

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(H_{2.2}) The function $\sigma : D \times H \to L(K, H)$ is continuous and there exists a constant $N_2$ such that
\[ \int_0^t E \left\| \sigma(t, s, x) - \sigma(t, s, y) \right\|^2_Q \, ds \leq N_2 E \| x - y \|^2 \]
for all $x, y \in H$, where $D = J \times J = \{(t, s), t, s \in J\}$.

(H_{2.3}) The nonlinear map $h : PC_b \to PC_b$ is such that for any $x_1$ and $x_2$ in $PC_b$ with $x_1 = x_2$
on $[\rho, 0], h(x_1) = h(x_2)$ on $[\rho, 0]$.

(H_{2.4}) The functions $I_k : H \to H$ are continuous and there exists a constant $\mu > 0$ such that
\[ E \| I_k(x) - I_k(y) \|^2 \leq \mu E \| x - y \|^2 \]
for all $x, y \in H$, $k = 1, 2, \ldots, m$.

(H_{2.5}) The functions $I_k : H \to H$ are continuous and there exists a constant $\rho > 0$ such that
\[ E \| I_k(x) \|^2 \leq \rho E \| x \|^2 \]
for all $x \in H$, $k = 1, 2, \ldots, m$.

(H_{2.6}) The functions $f : J \times PC_0 \to H$, $\sigma : D \times PC_0 \to L(K, H)$ and $I_k : H \to H$, $k = 1, 2, \ldots, m$ are completely continuous.

(H_{2.7}) The operator family $\{\hat{T}_\alpha(t)\}_{t \geq 0}$ and $\{\hat{S}_\alpha(t)\}_{t \geq 0}$ are compact, where $\hat{S}_\alpha(t) = t^{1-\alpha} \hat{S}_\alpha(t)$.
If $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then
\[ \| \hat{T}_\alpha(t) \|_{L(H)} \leq M e^{\omega t} \]
and
\[ \| \hat{S}_\alpha(t) \|_{L(H)} \leq C e^{\omega t} (1 + t^{\alpha-1}) \].
\[ M_T = \sup_{0 \leq t \leq b} \| \hat{T}_\alpha(t) \|_{L(H)}, \]
\[ M_S = \sup_{0 \leq t \leq b} C e^{\omega t} (1 + t^{1-\alpha}) \].
\[ \| \hat{S}_\alpha(t) \|_{L(H)} \leq t^{\alpha-1} M_S \] (see [118]).

2.2.3 Local Existence of Mild Solution of impulsive fractional semilinear SDEs

**Theorem 2.1.** If the hypotheses (H_{2.1}) – (H_{2.5}) are satisfied, and there exists $x_0 \in PC_b$ such
that $h(x_0) = \phi_0$ on $[\rho, 0]$. Then for every $\phi_0 \in PC_b$ there exists a $\tau_0 = \tau_0(\phi_0), 0 < \tau_0 < b$
such that the initial value problem (2.1) has a unique mild solution $x \in PC([\rho, \tau_0], H)$.

**Proof.** Since, only the local solution is considered here, one may assume that $b < \infty$. Let $t' > 0, R > 0$ be such that $B_R(x_0) = \{x : E \| x - x_0 \|^2_\nu \leq R\}$, $E \| f(t, x) \|^2_H \leq N_1$, $\int_0^t E \| \sigma(t, s, x) \|^2_Q \, ds \leq N_2$ for $0 \leq t \leq t'$ and $x \in B_R(x_0)$. Choose $t'' > 0$ such that...
\[ \mathbb{E} \left\| \hat{T}_\alpha(t)x_0(0) - x_0(0) \right\|_H^2 \leq \frac{R}{15} \text{ for } 0 \leq t \leq t'' \text{ and } \mathbb{E} \left\| x_0(t) - x_0(0) \right\|_H^2 \leq \frac{R}{15} \text{ for } 0 \leq t \leq t'' \]

and choose
\[
\tau_0 = \min \left\{ b, t', t'', \left[ \frac{R}{15} - M_7 \rho m}{\alpha^2(N_1 + N_2 Tr(Q))} \right] \right\}
\]

Set \( Y = PC_{\tau_0} = PC([-r, \tau_0], H) \) and \( Y_0 = \{ x : x \in Y, x = x_0 \text{ on } [-r, 0], x(t) \in B_R(x_0) \text{ for } 0 \leq t \leq \tau_0 \} \). It is clear that \( Y_0 \) is a bounded closed convex subset of \( Y \).

Define a mapping \( \Phi : Y_0 \to Y \) by
\[
(\Phi x)(t) = \begin{cases} 
  x_0(t), & t \in [-r, 0] \\
  \hat{T}_\alpha(t)x_0(0) + \sum_{0 < t_i < t} \hat{T}_\alpha(t - t_i)I_i(x(t_i^-)) \\
  +\int_0^t \hat{S}_\alpha(t - s) \left[ f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau)dw(\tau) \right] ds, & t \in [0, \tau_0].
\end{cases}
\]

For \( x \in Y_0, t \in [0, \tau_0] \), one have
\[
\mathbb{E} \left\| (\Phi x)(t) - x_0(t) \right\|_H^2 \leq 5 \left\{ \mathbb{E} \left\| \hat{T}_\alpha(t)x_0(0) - x_0(0) \right\|_H^2 + \mathbb{E} \left\| x_0(t) - x_0(0) \right\|_H^2 \right.
\]
\[
+ \mathbb{E} \left\| \sum_{0 < t_i < t} \hat{T}_\alpha(t - t_i)I_i(x(t_i^-)) \right\|_H^2 \\
+ M_7^2 \int_0^t (t - s)^{\alpha - 1} ds \int_0^t (t - s)^{\alpha - 1} \mathbb{E} \left\| f(s, x_s) \right\|_H^2 ds
\]
\[
+ M_7^2 \int_0^t (t - s)^{\alpha - 1} ds \int_0^t (t - s)^{\alpha - 1} Tr(Q) \int_0^s \mathbb{E} \left\| \sigma(s, \tau, x_\tau) \right\|_Q^2 d\tau ds \right\}
\]
\[
\leq 5 \left\{ \frac{R}{15} + \frac{R}{15} + \frac{M_7^2 \tau_0^2 \alpha^2}{\alpha^2} (N_1 + N_2 Tr(Q)) + M_7^2 \rho m \right\} \leq R.
\]

Thus \( \Phi : Y_0 \to Y_0 \), if one choose \( \tau_0 > 0 \) such that
\[
3 \left( \frac{M_7^2 \rho m}{\alpha^2} + \frac{M_7^2 \tau_0^2 \alpha^2}{\alpha^2} (N_1 + N_2 Tr(Q)) \right) < 1. \tag{2.2}
\]

Now, let \( x, y \in Y_0 \), then
\[
\mathbb{E} \left\| (\Phi x)(t) - (\Phi y)(t) \right\|_H^2 \leq 3 \left\{ \sum_{0 < t_i < t} \mathbb{E} \left\| \hat{T}_\alpha(t - t_i)I_i(x(t_i^-)) - \hat{T}_\alpha(t - t_i)I_i(y(t_i^-)) \right\|_H^2 \\
+ M_7^2 \int_0^t (t - s)^{\alpha - 1} ds \int_0^t (t - s)^{\alpha - 1} \mathbb{E} \left\| f(s, x_s) - f(s, y_s) \right\|_H^2 ds
\]
\[
+ M_7^2 \int_0^t (t - s)^{\alpha - 1} ds \int_0^t (t - s)^{\alpha - 1} Tr(Q) \int_0^s \mathbb{E} \left\| \sigma(s, \tau, x_\tau) - \sigma(s, \tau, y_\tau) \right\|_Q^2 d\tau ds \right\}
\]

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which implies
\[
\mathbb{E} \| (\Phi x)(t) - (\Phi y)(t) \|_H^2 \leq 3 \left[ M_I^2 \mu_m + \frac{M_S^2 \tau_0^{2\alpha}}{\alpha^2} (N_1 + N_2 Tr(Q)) \right] \mathbb{E} \| x - y \|^2.
\]

It follows from (2.2) and Banach contraction mapping principle that there exists a unique \( x \in Y_0 \) such that \( x \) is a mild solution of the system (2.1) on \([-r, \tau_0]\). This completes the proof. \( \square \)

**Theorem 2.2.** If the hypotheses \((H_{2.5}) - (H_{2.7})\) are satisfied and there exists \( x_0 \in PC_b \) such that \( h(x_0) = \phi_0 \) on \([-r, 0]\), then for every \( \phi_0 \in PC_b \) there exists a \( \tau_0 = \tau_0(\phi_0), \ 0 < \tau_0 < b \) such that the initial value problem (2.1) has a mild solution \( x \in PC([-r, \tau_0], H) \).

**Proof.** The proof of this theorem is based on the Schauder’s fixed point theorem. Let \( \Phi : Y_0 \to Y_0 \) be defined as in Theorem 2.1.

**Step 1:** To show that, \( \Phi \) is continuous from \( Y_0 \) into \( Y_0 \). Let \( \{x^n\} \) be a sequence in \( Y_0 \), such that \( x^n \to x \) in \( Y_0 \). Then \( f(t, x^n) \to f(t, x) \) and \( \sigma(t, s, x^n_s) \to \sigma(t, s, x_s) \) as \( n \to \infty \), because the functions \( f \) and \( \sigma \) are continuous on \( J \times PC_0 \) and \( D \times PC_0 \) respectively. Now, for every \( t \in [0, \tau_0] \), one can estimate
\[
\mathbb{E} \| (\Phi x^n)(t) - (\Phi x)(t) \|_H^2 \leq 3 \left\{ M_I^2 \mathbb{E} \| I_i(x^n(t^-_i)) - I_i(x(t^-_i)) \|_H^2 + M_I^2 b \int_0^t (t-s)^{2(\alpha-1)} \mathbb{E} \| f(s, x^n_s) - f(s, x_s) \|_H^2 \, ds + M_S^2 b \int_0^t (t-s)^{2(\alpha-1)} \times Tr(Q) \left( \int_0^s \mathbb{E} \| \sigma(s, \tau, x^n_{\tau^+}) - \sigma(s, \tau, x_{\tau^+}) \|_Q^2 \, d\tau \right) \, ds \right\}.
\]

Now, using the fact that,
\[
(t - s)^{2(\alpha-1)} \mathbb{E} \| f(s, x^n_s) - f(s, x_s) \|_H^2 \leq 2N_1(t - s)^{2(\alpha-1)} \in L^1(J, \mathbb{R}^+)
\]
\[
(t - s)^{2(\alpha-1)} \int_0^s \mathbb{E} \| \sigma(s, \tau, x^n_{\tau^+}) - \sigma(s, \tau, x_{\tau^+}) \|_Q^2 \, d\tau \, ds \leq 2N_2(t - s)^{2(\alpha-1)} \in L^1(J, \mathbb{R}^+)
\]

and by means of Lebesgue dominated convergence theorem, one can obtain
\[
\int_0^t (t - s)^{2(\alpha-1)} \mathbb{E} \| f(s, x^n_s) - f(s, x_s) \|_H^2 \, ds \to 0,
\]
\[
\int_0^t (t - s)^{2(\alpha-1)} \int_0^s \mathbb{E} \| \sigma(s, \tau, x^n_{\tau^+}) - \sigma(s, \tau, x_{\tau^+}) \|_Q^2 \, d\tau \, ds \to 0.
\]
Hence \( \lim_{n \to \infty} E \| \Phi x^n - \Phi x \|_{\tau_0}^2 = 0 \). Since, the functions \( I_k, \ k = 1, 2, \ldots m \) are continuous. This means that \( \Phi \) is continuous.

**Step 2:** To show that \( \Phi(Y_0) = \{ \Phi x : x \in Y_0 \} \) be an equicontinuous family of functions. For \( \tau_0 > \tau_2 > \tau_1 > 0 \), one have

\[
E \| (\Phi x)(\tau_2) - (\Phi x)(\tau_1) \|^2_H \\
\leq 4 \left\{ E \left\| \hat{T}_\alpha(\tau_2)x_0(0) - \hat{T}_\alpha(\tau_1)x_0(0) \right\|^2_H \\
+ E \left\| \sum_{0 < t_k < \tau_2} \hat{T}_\alpha(\tau_2 - t_k)I_k(x(t_k^{-})) - \sum_{0 < t_k < \tau_1} \hat{T}_\alpha(\tau_1 - t_k)I_k(x(t_k^{-})) \right\|^2_H \\
+ E \left\| \int_0^{\tau_2} \hat{S}_\alpha(\tau_2 - s)f(s, x_s)ds - \int_0^{\tau_1} \hat{S}_\alpha(\tau_1 - s)f(s, x_s)ds \right\|^2_H \\
+ E \left\| \int_0^{\tau_2} \hat{S}_\alpha(\tau_2 - s) \left( \int_0^s \sigma(s, \tau, x_\tau)dw(\tau) \right) ds \\
- \int_0^{\tau_1} \hat{S}_\alpha(\tau_1 - s) \left( \int_0^s \sigma(s, \tau, x_\tau)dw(\tau) \right) ds \right\|^2_H \right\} \\
\leq 6 \left\{ E \| \hat{T}_\alpha(\tau_2)x_0(0) - \hat{T}_\alpha(\tau_1)x_0(0) \|^2_H \\
+ \sum_{i=1}^k E \| (\hat{T}_\alpha(\tau_2 - t_i) - \hat{T}_\alpha(\tau_1 - t_i))I_i(x(t_i^{-})) \|^2_H \\
+ \int_0^{\tau_1} \| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \|_{L(H)}^2 ds \int_0^{\tau_1} E \| f(s, x_s) \|^2 ds \\
+ \int_0^{\tau_2} \| \hat{S}_\alpha(\tau_2 - s) \|_{L(H)} ds \int_0^{\tau_2} \| \hat{S}_\alpha(\tau_2 - s) \|_{L(H)} E \| f(s, x_s) \|^2_H ds \\
+ \int_0^{\tau_1} \| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \|_{L(H)}^2 ds \int_0^{\tau_1} Tr(Q) \left( \int_0^s E \| \sigma(s, \tau, x_\tau) \|^2_Q d\tau \right) ds \\
+ \int_0^{\tau_2} \| \hat{S}_\alpha(\tau_2 - s) \|_{L(H)} ds \int_0^{\tau_2} \| \hat{S}_\alpha(\tau_2 - s) \|_{L(H)} Tr(Q) \left( \int_0^s E \| \sigma(s, \tau, x_\tau) \|^2_Q d\tau \right) ds \right\} \\
\leq 6 \sum_{j=1}^6 J_{2,j} \tag{2.3}
\]

where

\[
J_{2,3} = \int_0^{\tau_1} \| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \|_{L(H)}^2 ds \int_0^{\tau_1} E \| f(s, x_s) \|^2 ds \\
J_{2,3} \leq \tau_1 N_1 \int_0^{\tau_1} \| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \|_{L(H)}^2 ds
\]
and

\[ J_{2.5} = \int_0^{\tau_1} \left\| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \right\|^2_{L(H)} ds \int_0^{\tau_1} Tr(Q) \left( \int_0^s E \| \sigma(s, \tau, x_\tau) \|^2_Q d\tau \right) ds \]
\[ \leq Tr(Q)\tau_1 N_2 \int_0^{\tau_1} \left\| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \right\|^2_{L(H)} ds. \]

Since \( \left\| \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \right\|^2_{L(H)} \leq 2M_2^2(\tau_0 - s)^{2(\alpha - 1)} \in L^1(J, \mathbb{R}^+) \) for \( s \in [0, \tau_0] \) and \( \hat{S}_\alpha(\tau_2 - s) - \hat{S}_\alpha(\tau_1 - s) \to 0 \) as \( \tau_1 \to \tau_2 \), because \( \hat{S}_\alpha(\cdot) \) is strongly continuous. This implies that \( \lim_{\tau_1 \to \tau_2} J_{2.3} = \lim_{\tau_1 \to \tau_2} J_{2.5} = 0 \). Also

\[ J_{2.4} = \int_{\tau_1}^{\tau_2} \left\| \hat{S}_\alpha(\tau_2 - s) \right\|_{L(H)} ds \int_{\tau_1}^{\tau_2} \left\| \hat{S}_\alpha(\tau_2 - s) \right\|_{L(H)} E \| f(s, x_s) \|^2_{H} ds \leq \frac{M_2^2 N_1(\tau_2 - \tau_1)^{2\alpha}}{\alpha^2}, \]

and

\[ J_{2.6} = \int_{\tau_1}^{\tau_2} \left\| \hat{S}_\alpha(\tau_2 - s) \right\|_{L(H)} ds \int_{\tau_1}^{\tau_2} \left\| \hat{S}_\alpha(\tau_2 - s) \right\|_{L(H)} Tr(Q) \left( \int_0^s E \| \sigma(s, \tau, x_\tau) \|^2_Q d\tau \right) ds \]
\[ \leq \frac{M_2^2 N_2 Tr(Q)(\tau_2 - \tau_1)^{2\alpha}}{\alpha^2}. \]

Hence, \( \lim_{\tau_1 \to \tau_2} J_{2.4} = 0 \) and \( \lim_{\tau_1 \to \tau_2} J_{2.6} = 0 \). Since, \( \hat{T}_\alpha(\cdot) \) is strongly continuous and the continuity of \( t \mapsto \left\| \hat{T}_\alpha(t) \right\|_{L(H)} \) allows us to conclude that the right hand side of (2.3) is tends to zero as \( \tau_1 \to \tau_2 \), which implies that \( \Phi(Y_0) \) is equicontinuous.

**Step 3:** To prove \( \Phi_1 \) is completely continuous operator on \( H \). Decompose \( \Phi \) by \( \Phi = \Phi_1 + \Phi_2 \), where

\[
(\Phi_1x)(t) = \begin{cases} 0, & t \in [-r, 0] \\ \int_0^t \hat{S}_\alpha(t-s) \left[ f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) d\tau \right] ds, & t \in [0, \tau_0]. \end{cases}
\]

\[
(\Phi_2x)(t) = \begin{cases} x_0(t), & t \in [-r, 0] \\ \hat{T}_\alpha(t)x_0(0) + \sum_{0 < t_i < t} \hat{T}_\alpha(t - t_i)I_i(x(t_i^-)), & t \in [0, \tau_0]. \end{cases}
\]

From the compactness of \( \overline{S}_\alpha(\cdot) \) and \( (H_{2.6}) \), one can conclude that the set

\[
\left\{ \overline{S}_\alpha(t-s) \left[ f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) d\tau \right], t, s \in [0, \tau_0], x \in Y_0 \right\}
\]

is relatively compact in \( H \). Furthermore, by using the mean value theorem for Bochner integral, one can conclude that \( (\Phi_1x)(t) \) belongs to the set

\[
\frac{\alpha^{\alpha+1}}{\alpha} \text{conv} \left\{ \overline{S}_\alpha(t-s) \left[ f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) d\tau \right], t, s \in [0, \tau_0], x \in Y_0 \right\}
\]

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for all $t \in [0, \tau_0]$, $x \in H$, where $\text{conv}(\cdot)$ denotes the convex hull. Accordingly, the set 
\[ \{ \Phi_1 x(t) : x \in Y_0 \} \] is relatively compact.

Now, one need to show that, 
\[ \{ \Phi_2 x(t) : t \in [-r,0], x \in Y_0 \} \] is compact in $H$. For all $t \in [-r,0]$, $(\Phi_2 x)(t) = x_0(t)$, since $x_0(t)$ is a fixed function, it follows that 
\[ \{ \Phi_2 x(t) : t \in [-r,0], x \in Y_0 \} \] is a compact subset of $H$. But then, for $t \in [0, \tau_0]$ and $x \in Y_0$, 
\[ \Phi_2 x(t) = \hat{T}_\alpha(t)x_0(0) + \sum_{0<t_i<t} \hat{T}_\alpha(t-t_i)I_i(x(t^-_i)). \]

Since, $\hat{T}_\alpha(t)$ is compact for all $t \in [0, \tau_0]$, it follows that the set 
\[ \{ (\Phi_2 x)(t) : t \in [-r,0], x \in Y_0 \} \] is precompact in $H$, which implies $\Phi_2$ is also compact. Therefore $\Phi = \Phi_1 + \Phi_2$ is compact. Furthermore, the set 
\[ \hat{E} = \{ x \in Y_0 : x = \lambda \Phi x \text{ for some } 0 < \lambda < 1 \} \] is bounded, since $\hat{E} \subset Y_0$ and $Y_0$ is closed bounded convex set. By Schauder’s fixed point theorem, it can be concluded that $\Phi$ has a fixed point in $Y_0$ and any fixed point of $\Phi$ is a mild solution of the system (2.1) on $[-r, \tau_0]$.

### 2.2.4 Global Existence of Mild Solution of impulsive fractional semilinear SDEs

This subsection considers the global existence of mild solution for the system (2.1).

**Theorem 2.3.** Assume the hypotheses of Theorem 2.2. Let $f : [-r,b) \times PC_0 \to H$ and $\sigma : [-r,b) \times [-r,b) \times PC_0 \to H$, $0 < b \leq \infty$ are continuous, and map bounded sets in $[-r,b) \times PC_0$ and $[-r,b) \times PC_0$ into bounded sets in $H$, then for every $\Phi_0 \in PC_b$ the initial value problem (2.1) has a mild solution $x$ on a maximal interval of existence $[-r, t_{\text{max}})$. If $t_{\text{max}} < \infty$, then $\lim t \uparrow t_{\text{max}} \mathbb{E} \| x(t) \|_H = \infty$.

**Proof.** By defining $x(t + \tau_0) = V(t)$, the problem (2.1) can be translated into the following form
\[
\begin{align*}
C D^\alpha_t V(t) + AV(t) &= F(t, V(t)) + \int_0^t G(t,s,V_s)dw(s), \ t \in [0,b-\tau_0], \ t \neq t_k \\
\Delta V(t_k) &= I_k(V(t_k)), \ k = 1,2, \cdots m \\
h(V(t)) &= \phi_0(t), \ t \in [-r-\tau_0,0],
\end{align*}
\]
(2.4)
where

\[ F(t, V_t) = f(t + \tau_0, V_t), \quad t \in [0, b - \tau_0], \]
\[ G(t, s, V_s) = \sigma(t + \tau_0, s, V_s), \quad t \in [0, b - \tau_0], \]
\[ \Delta V(\tilde{t}_k) = I_k(V(\tilde{t}_k)), \quad k = 1, 2, \ldots, m, \]
\[ \tilde{\phi}(t) = x(t + \tau_0), \]

and \( \tilde{t}_k = t_k - \tau_0. \) Since, the functions \( F, G \) are bounded functions, by Theorem 2.2 there exists a function \( V \in PC([-r - \tau_0, b - \tau_0], H) \) such that \( V \) is a mild solution of (2.4) on \([-r - \tau_0, \tau_1]\) for some \( 0 < \tau_1 \leq b - \tau_0 \) and given by

\[ V(t) = \begin{cases} 
\hat{T}_\alpha(t)V(0) + \sum_{0 < \tilde{t}_k < t} \hat{T}_\alpha(t - \tilde{t}_k)I_k(V(\tilde{t}_k)) \\
+ \int_{\tilde{t}_k}^t \hat{S}_\alpha(t - s) \left[ F(s, V_s) + \int_c^s G(s, \tau, V_\tau) d\tau \right] ds, \quad t \in [0, \tau_1] 
\end{cases} \]

\( \tilde{h}(V(t)) = \tilde{\phi}(t), \quad t \in [-r - \tau_0, 0]. \)

Then

\[ \tilde{x}(t) = \begin{cases} 
x(t), \quad t \in [-r, \tau_0] \\
V(t - \tau_0), \quad t \in [\tau_0, \tau_0 + \tau_1], 
\end{cases} \]

is a mild solution of the system (2.1) on \([-r, \tau_0 + \tau_1]. \) Since \( x(t + \tau_0) = V(t), \) thus for \( t \in [\tau_0, \tau_0 + \tau_1], \) one have

\[ V(t - \tau_0) := x(t) = \begin{cases} 
\hat{T}_\alpha(t - \tau_0)x(\tau_0) + \sum_{\tau_0 < \tilde{t}_k < t} \hat{T}_\alpha(t - \tilde{t}_k)I_k(x(\tilde{t}_k)) \\
+ \int_{\tau_0}^t \hat{S}_\alpha(t - s) \left[ f(s, x_s) + \int_c^s \sigma(s, \tau, x_\tau) d\tau \right] ds. 
\end{cases} \]

The solution of the system (2.1) can be extended to the maximal interval \([-r, t_{\text{max}}]\) by continuing in this way. Now, one can prove that, if \( t_{\text{max}} < \infty, \) then \( \mathbb{E} \| x(t) \|^2_H \to \infty \) as \( t \to t_{\text{max}}. \) To do this, one shall prove that \( t \to t_{\text{max}} \) implies \( \lim_{t \to t_{\text{max}}} \mathbb{E} \| x(t) \|^2_H = \infty. \) Indeed, if \( t \uparrow t_{\text{max}} \) and \( \lim_{t \uparrow t_{\text{max}}} \mathbb{E} \| x(t) \|^2_H < \infty, \) one can assume that \( \| \hat{T}_\alpha(t) \|_{L(H)} \leq M_T \) and \( \mathbb{E} \| x(t) \|^2_H \leq k_1 \) for \( 0 \leq t < t_{\text{max}}, \) where \( M_T \) and \( k_1 \) are constants. Now, if \( 0 < R < t < t' < t_{\text{max}}, \) then

\[ \mathbb{E} \| x(t') - x(t) \|^2_H \leq 7 \left\{ \mathbb{E} \left\| \hat{T}_\alpha(t')x_0(0) - \hat{T}_\alpha(t)x_0(0) \right\|^2_H + \sum_{t < t_i < t'} \mathbb{E} \left\| \hat{T}_\alpha(t' - t_i)I_i(x(t_i^-)) \right\|^2_H \\
+ \sum_{0 < t_i < t} \left\| \hat{T}_\alpha(t' - t_i) - \hat{T}_\alpha(t - t_i) \right\|^2_{L(H)} \mathbb{E} \left\| I_i(x(t_i^-)) \right\|^2_H \right\}. \]

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\[
\begin{align*}
&\int_0^t \left\| \hat{S}_\alpha(t' - s) - \hat{S}_\alpha(t - s) \right\|_{L(H)}^2 \, ds \int_0^t \mathbb{E} \left\| f(s, x_s) \right\|_H^2 \, ds \\
&+ \int_t^{t'} \left\| \hat{S}_\alpha(t' - s) \right\|_{L(H)} \, ds \int_t^{t'} \left\| \hat{S}_\alpha(t' - s) \right\|_{L(H)} \mathbb{E} \left\| f(s, x_s) \right\|_H^2 \, ds \\
&+ \int_0^t \left\| \hat{S}_\alpha(t' - s) - \hat{S}_\alpha(t - s) \right\|_{L(H)}^2 \, ds \int_0^t \int_t^{t'} \mathbb{E} \left\| \sigma(s, \tau, x_\tau) \right\|_Q^2 \, d\tau \, ds \\
&+ \int_t^{t'} \left\| \hat{S}_\alpha(t' - s) \right\|_{L(H)} \, ds \int_t^{t'} \left\| \hat{S}_\alpha(t' - s) \right\|_{L(H)} \mathbb{E} \left\| \sigma(s, \tau, x_\tau) \right\|_Q^2 \, d\tau \, ds \\
&\leq 7 \left\{ \mathbb{E} \left\| \hat{T}_\alpha(t')x_0(0) - \hat{T}_\alpha(t)x_0(0) \right\|_H^2 + \rho \sum_{0 < t_i < t} \left\| \hat{T}_\alpha(t' - t_i) - \hat{T}_\alpha(t - t_i) \right\|_{L(H)}^2 \\
&+ \rho \sum_{t < t_i < t'} \left\| \hat{T}_\alpha(t' - t_i) \right\|_{L(H)}^2 + N_1 t_{\max} \int_0^t \left\| \hat{S}_\alpha(t' - s) - \hat{S}_\alpha(t - s) \right\|_{L(H)}^2 \, ds \\
&+ \frac{N_1 (t' - t)^{2\alpha} M_2^2}{\alpha^2} + N_2 T \hat{R}(Q)t_{\max} \int_0^t \left\| \hat{S}_\alpha(t' - s) - \hat{S}_\alpha(t - s) \right\|_{L(H)}^2 \, ds \\
&+ \frac{T \hat{R}(Q)N_2 M_2^2 (t' - t)^{2\alpha}}{\alpha^2} \right\}.
\end{align*}
\]

Since \( t > R > 0 \) is arbitrary, and \( \hat{T}_\alpha(t), \hat{S}_\alpha(t) \) are continuous in the uniform operator topology for \( t \geq R > 0 \), which implies that the right hand side of (2.5) tends to zero as \( t \) and \( t' \) tend to \( t_{\max} \). Therefore, \( \lim_{t \uparrow \max} x(t) = x(t_{\max}) \) exists and the solution \( x \) can be extended beyond \( t_{\max} \), contradicting the maximality of \( t_{\max} \). Therefore, it follows from the assumption \( t_{\max} < \infty \) that \( \lim_{t \uparrow \max} \mathbb{E} \left\| x(t) \right\|_H^2 = \infty \). Now, the proof of the theorem can be concluded by showing \( \lim_{t \uparrow \max} \mathbb{E} \left\| x(t) \right\|_H^2 = \infty \). If it is not true, then there is a sequence \( \tau_n \uparrow \max \) and a constant \( k_1 \) such that \( \mathbb{E} \left\| x(\tau_n) \right\|_H^2 \leq k_1 \) for all \( n \).

Let

\[
\begin{align*}
\beta_1 &= \sup \left\{ \mathbb{E} \left\| f(t, x_t) \right\|_H^2 : 0 \leq t \leq t_{\max}, \mathbb{E} \left\| x(t) \right\|_H^2 \leq M_2^2 (k_1 + 1) \right\} \\
\beta_2 &= \sup \left\{ \int_0^s \mathbb{E} \left\| \sigma(s, \tau, x_\tau) \right\|_Q^2 \, d\tau : 0 \leq t \leq t_{\max}, \mathbb{E} \left\| x(t) \right\|_H^2 \leq M_2^2 (k_1 + 1) \right\}
\end{align*}
\]

and choose \( \rho_1 \) such that \( \rho_1 < \frac{1 - 6k_1}{8k_1} \).

Since, \( t \rightarrow \mathbb{E} \left\| x(t) \right\|_H^2 \) is continuous and \( \lim_{t \uparrow \max} \mathbb{E} \left\| x(t) \right\|_H^2 = \infty \), one can find a sequence \( \{ \tilde{\lambda}_n \} \) with the following properties: \( \tilde{\lambda}_n \rightarrow 0 \) as \( n \rightarrow \infty \), \( \mathbb{E} \left\| x(t) \right\|_H^2 \leq M_2^2 (k_1 + 1) \) for
\[ \tau_n \leq t \leq \tau_n + \lambda_n \text{ and } \mathbb{E} \| x(\tau_n + \lambda_n) \|_H^2 = M_T^2(k_1 + 1). \] On the other hand, one have

\[ M_T^2(k_1 + 1) = \mathbb{E} \| x(\tau_n + \lambda_n) \|_H^2 \]
\[ \leq 4 \left\{ \mathbb{E} \left\| \hat{T}_\alpha(\lambda_n):x(\tau_n) \right\|_H^2 \right. \]
\[ + \sum_{\tau_n < t_k < \tau_n + \lambda_n} \mathbb{E} \left\| \hat{T}_\alpha(\lambda_n - t_k) \right\|_{L(H)}^2 \mathbb{E} \left\| I_k(x(t_k^-)) \right\|_H^2 \]
\[ + M_S^2 \int_{\tau_n}^{\tau_n + \lambda_n} (\tau_n + \lambda_n - s)^{\alpha-1} ds \int_{\tau_n}^{\tau_n + \lambda_n} (\tau_n + \lambda_n - s)^{\alpha-1} \mathbb{E} \| f(s, x_s) \|_H^2 ds \]
\[ + M_S^2 \int_{\tau_n}^{\tau_n + \lambda_n} (\tau_n + \lambda_n - s)^{\alpha-1} ds \int_{\tau_n}^{\tau_n + \lambda_n} (\tau_n + \lambda_n - s)^{\alpha-1} \]
\[ \times \left( \mathbb{E} \int_0^s \| \sigma(s, \tau, x_\tau) \|_Q^2 d\tau \right) ds \right\} \]
\[ \leq 4 \left\{ M_S^2 \mathbb{E} \| x \|_H^2 + M_T^2 m \rho \mathbb{E} \| x \|_H^2 + M_S^2 \frac{\lambda_{2n}}{\alpha^2} \beta_1 + M_S^2 \frac{\lambda_{2n}}{\alpha^2} Tr(Q) \beta_2 \right\} \]
\[ \leq 4 M_T^2 k_1 \left( 1 + m \rho_1 \right) + \frac{4 M_T^2 \lambda_{2n}}{\alpha^2} \left\{ \beta_1 + Tr(Q) \beta_2 \right\} \]
\[ \leq 4 M_T^2 k_1 \left[ 1 + m \left( \frac{1 - 6k_1}{8k_1} \right) \right] \text{ as } \lambda_n \to 0 \]
\[ \leq M_T^2 (k_1 + \frac{1}{2}) \]

which is absurd as \( \lambda_n \to 0 \). Therefore, \( \lim_{t \uparrow \tau_{\max}} \mathbb{E} \| x(t) \|_H = \infty \). This completes the proof of the theorem. \( \square \)

### 2.2.5 Example

Consider the following impulsive semilinear FSDE

\[ \frac{\partial^2 u(t, x)}{\partial t^2} + \frac{\partial^2 u(t, x)}{\partial x^2} = \int_0^t e^{-\frac{u(s, x)}{2}} \frac{\| u(s, x) \|}{25 + \| u(s, x) \|} ds + \int_0^t \frac{e^{\frac{t-s}{4}}}{4 + \| u(s, x) \|} dw(s), \quad t \in [0, 1], \ x \in [0, \pi], \ t \neq \frac{1}{2}, \] \[ u(t, 0) = u(t, \pi) = 0, \ t \geq 0, \] \[ \Delta u|_{t=1/2} = \sin \left( \frac{1}{2} \left\| u \left( \frac{1}{2}, x \right) \right\| \right), \] \[ \frac{1}{\tau} \int_{-\tau}^{0} e^{2s} u(s, x) ds = u_0(x), \ 0 \leq x \leq \pi. \]
Let $H = L_2[0, \pi]$, $w(t)$ is standard cylindrical Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $A : D(A) \subset H \to H$ be defined by $Az = z''$ with the domain $D(A) = \{z \in H : z, z' \text{ are absolutely continuous}, z'' \in H, z(0) = z(\pi) = 0\}$. Then

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, z_n \rangle z_n, \quad z \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \in \mathbb{N}$ is the orthonormal set of eigenvectors of $A$. It is well known that $A$ generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2t} \langle z, z_n \rangle z_n, \quad z \in H, \text{ for every } t > 0.$$

It follows from the above expression that $\{T(t)\}_{t \geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda^\alpha, A) = (\lambda^\alpha I - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$. In-order to define the operator $Q : H \to H$, one can choose a sequence $\{\xi_n\}$ and set $Qz = \xi_n z_n$ with

$$\text{Tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\xi_n} < 0.0718.$$

By putting $t - s = -\theta$ in the first and second terms of the right hand side of (2.6) and taking $u(t, x) = u(t)x$, one can get

$$\int_0^t e^{\frac{u(s)-u(t)}{25} + \|u(s, x)\|} ds = \int_{-t}^0 e^{\frac{u(\theta)}{25} + \|u(\theta)(x)\|} d\theta$$

and

$$\int_0^t \frac{e^{t-s}}{4 + \|u(s, x)\|} dw(s) = \int_{-t}^0 \frac{e^{-\theta}}{4 + \|u(\theta)x\|} dw(\theta),$$

which implies (2.6) takes the following abstract form

$$D_t^{\frac{1}{2}} u(t)x + Au(t)x = f(t, u_t)(x) + \int_0^t \sigma(t, s, u_s)(x) dw(s)$$

where $f : [0, 1] \times PC_0 \to H$ and $\sigma : [0, 1] \times [0, 1] \times PC_0 \to L_2(K, H)$ given by

$$f(t, \phi)(x) = \int_{-t}^0 e^{\frac{-\theta}{25} + \|\phi(\theta)(x)\|} d\theta,$$

$$\sigma(t, s, \phi)(x) = \int_{-t}^0 \frac{e^{-\theta}}{4 + \|\phi(\theta)x\|} dw(\theta).$$
\( I_k(u) = \sin(\frac{1}{k} \|u\|), \ k = 1, \ h(u)(\theta) = \sigma(u) \) for \( u \in PC_1, \ \theta \in [-\tau, 0] \), \( \phi(\theta) \equiv u_0 \) for \( \theta \in [-\tau, 0] \),

where \( \sigma : PC_1 \to L_2([0, \pi]) \) is such that

\[
\sigma(u) = \frac{1}{\tau} \int_{-\tau}^{0} e^{2s}u(s, x)ds.
\]

Then, the system (2.6)-(2.9) can be written in the abstract form of the system (2.1). For \((t, \phi), (s, \psi) \in [0, 1] \times PC_0, \) one have

\[
\mathbb{E} \|f(t, \phi) - f(s, \psi)\|_{H}^2 \leq 2 \int_{-t}^{-s} \pi e^{-\theta} \left\| \frac{\phi(\theta)(x)}{25 + \psi(\theta)(x)} \right\|_{H}^2 d\theta
\]

\[
+ 2 \int_{-s}^{0} \pi e^{-\theta} \left\| \frac{\phi(\theta)(\cdot)}{25 + \psi(\theta)(\cdot)} - \frac{\psi(\theta)(\cdot)}{25 + \psi(\theta)(\cdot)} \right\|_{H}^2 d\theta
\]

\[
< 2\pi \left( 1 + e^1 \right) \left| t - s \right|^2
\]

\[
+ \frac{2\pi(e^1 - 1)}{625} \mathbb{E} \|\phi - \psi\|_{PC_0}^2
\]

Similarly,

\[
\mathbb{E} \|\sigma(t, \tau, \phi) - \sigma(s, \tau, \psi)\|_{Q}^2 \leq \frac{\pi e^{2} Tr(Q) \left| t - s \right|^2}{8} + \frac{\pi e^{2} Tr(Q) \mathbb{E} \|\phi - \psi\|_{PC_0}^2}{8}
\]

and

\[
\mathbb{E} \|I_k(u(t)) - I_k(v(t))\|_{H}^2 = \mathbb{E} \left\| \sin\left( \frac{1}{t} u(t) \right) - \sin\left( \frac{1}{t} v(t) \right) \right\|_{H}^2
\]

\[
\leq \frac{1}{49} \mathbb{E} \|u(t) - v(t)\|_{PC_1}^2, \ u, v \in PC_1
\]

Furthermore, for a defined \( h \), one can find \( \eta(t) = \frac{u_0}{k^*} \in PC_1 \) on \([0, \tau]\) with \( k^* = \frac{1}{\tau} \int_{0}^{\tau} e^{-2s}ds \neq 0 \) such that

\[
h(\eta)(\theta) \equiv \sigma(\eta) = \frac{1}{\tau} \int_{-\tau}^{0} e^{2s} \left( \frac{1}{k^*} u_0 \right) ds = u_0 \equiv \phi(\theta).
\]

That is \( h(\eta) = \phi \). Hence, the hypotheses \((H_{2.1}) - (H_{2.7})\) are satisfied with \( N_1 = \frac{2\pi(1+e)}{625}, \ N_2 = \frac{2\pi e^2}{16}, \ \mu = \frac{1}{49}, \ m = 1, \ M_T = 1, \ M_S = \frac{1}{\Gamma\left(\frac{1}{2}\right)}, \) and \( \alpha = \frac{1}{2} \). Further,

\[
3 \left[ M_T^2 \mu m + \left( \frac{N_1}{\alpha^2} + \frac{N_2 Tr(Q)}{\alpha^2} \right) M_S^2 \tau_0 \right] = 3 \left[ \frac{1}{49} + \frac{2\pi(1+e)4}{625\Gamma\left(\frac{1}{2}\right)^2} + \frac{2\pi e^2(4) Tr(Q)}{16\Gamma\left(\frac{1}{2}\right)^2} \right] < 1.
\]

Therefore by Theorem 2.1, the problem (2.6)-(2.9) has a unique mild solution on \([0, 1]\).
2.3 Conclusions and Future Directions

In this Chapter, the local and global existences of mild solutions have been studied for impulsive fractional semilinear SDEs with nonlocal condition in a Hilbert space. At first, the local existence of mild solution has been proved respectively by using the Banach contraction principle and Schauder fixed point theorem, then obtained result has been extended to global existence. The fixed point technique and solution operator are employed to obtain the main theoretical results, which are valid for all values of $\alpha \in (0, 1)$. Finally, the derived theoretical results have been validated through an example.

The FSDE are very efficient to describe the real life phenomena with noises, the real life phenomena from science and technology require different kinds of dynamical systems with various natures such as jumps, neutral, fractional Brownian motion (fBm), mixed fBm etc. Hence it would be great significant to discuss the local and global existence of mild solutions for FSDEs with Poisson jumps, neutral FSDEs with impulses, FSDEs driven by fBm etc.