Chapter 1

Introduction

1.1 Preliminary Background

Many scientific and engineering problems from technological discoveries of the industrial revolution as well as in the most modern technological applications can be well modelled by ordinary differential equations (ODEs), FDEs and coupled differential equations. The phenomena modelled by ODEs, FDEs, etc., can be described by abstract differential equation/inclusion in finite or infinite dimensional spaces. Most of the systems arising from practical situations are nonlinear in nature. The nonlinear systems are more complex than linear systems, so the first step in dealing with nonlinear system is to linearize it, if possible. The better approximation to the nonlinear system is the semilinear system, that is, a system consists of both linear and nonlinear parts and can be derived from general nonlinear system by making local approximation about some nominal trajectory. This provides motivation to the study of qualitative properties of semilinear systems in abstract form, so throughout this thesis semilinear systems are taken into account and studied their qualitative properties, namely existence of mild solutions, controllability results and existence of optimal control.

In the study of linear differential equations some conventional methods such as Laplace transform method, power series method etc., are used to solve the differential equations
analytically. In the case of nonlinear differential equations, analytical solutions cannot be found in general. In fact, due to recent advances in computer technology some numerical tools and powerful softwares are available to analyse and visualize the approximate solutions of nonlinear differential equations. However, the real life phenomena in science and technology require answer to the following basic questions related to the so called qualitative properties, namely, (i) whether the system has at least one solution? (ii) whether the system has at most one solution? (iii) can certain behaviour of the system be controlled or stabilized? etc. In the study of qualitative properties, these kinds of questions can be answered without solving the differential equations, especially when analytical solutions are unavailable.

The fractional calculus insists new dimensions to understand or describe basic nature of the real life phenomena arising in science and engineering in a better way. Probably several applications of this 300 year old subject could be experienced in near future, since integer order differentiation and integration are only the special cases in the world of fractional calculus (see [70, 106]). During the last decades, the theory and applications of FDEs are undergoing rapid development with more and more real world applications, namely in the fields of viscoelasticity, fractals, signal processing, system modeling and identification also in other areas of science and technology (see [9]).

In the theory of abstract differential equations of integer order, $C_0$-semigroup, also known as a strongly continuous one-parameter family of semigroup, is a generalization of the exponential function, which provides solutions of integer order differential equations in Banach spaces, such as delay differential equations and partial differential equations etc. However, in the case of FDEs, the Riemann-Liouville and Caputo fractional operators do possess neither semigroup nor commutative properties, which are inherent to the derivatives of integer order [10]. Various kinds of approaches are delivered, towards the ultimate aim of finding the unified technique for the concept of solution representation of FDEs. The methods employing solution operators, resolvent operators, $(a, k)$-regularized resolvent, $\alpha$-resolvent operator functions and fractional resolvents were proposed so far in the literature to solve the corresponding FDEs (see [79, 80, 103]).
On the other hand, stochastic differential equations (SDEs) are used in the modeling of real life phenomena, where there is a need for an aspect of randomness. In such cases, SDEs have been applied successfully to several fields of science and engineering. It is of great significant to study FDEs with stochastic effects. While studying FSDEs, the naturally arising question is “what is its solution?” In the study of FSDEs, one of the main objectives is to establish a general procedure to derive mild solution for wide classes of FSDEs, since in-order to study the qualitative properties; one need exact solution representation of the corresponding system. The techniques used in the solution concept of FDEs could be adapted to the stochastic settings, unfortunately there is no unified approach for the solution concept of FSDEs so far, since the obtained techniques are best fit only for some class of FSDEs.

Further, the notion of multimaps arises in many branches of mathematics, namely mathematical economics, theory of games, convex analysis etc. The multimaps play a significant role in the description of process in control theory, since the presence of control provides an intrinsic multivalence in the evolution of the system [69]. Differential inclusions have wide applications in economics, engineering, and so on. The theory of differential inclusions has been developed accordingly in the past three decades (see [8, 13, 14, 36, 64]). The extension of the study of differential inclusions to the world of fractional calculus is essential based on the wide spread applications of multivalued analysis in science and engineering. The study of fractional differential inclusions (FDIs) was initiated by El-Sayed and Ibrahim [41] and much interest has been given along this line (see [20, 58, 134]). However, the study of FSDEs/FSDIs is yet in the initial stage and needs to be analysed in the aspect of qualitative behaviours.

### 1.2 Fractional Calculus

Fractional calculus was born from the question “what if \( n \) is fractional?”, which was raised by the French mathematician L’Hopital, while Leibniz brings in the notation \( \frac{d^n f(x)}{dx^n} \) for the \( n \)-th order derivative of the function \( f(x) \) in 1695. Fractional calculus has undergone the tremendous development by the works of the eminent mathematicians Leibniz, Euler,
Lacroix, Laplace, Fourier, Abel, Liouville, Riemann etc.

The first expression for the fractional derivative of the function \( f(x) = x^m \) was given by S.F. Lacroix in 1819 by using Euler’s gamma function as follows

\[
\frac{d^n x^m}{dx^n} = \frac{\Gamma(m + 1)}{\Gamma(m - n + 1)} x^{m-n}.
\]

For \( n = \frac{1}{2} \) and \( m = 1 \), one can obtain the derivative of order \( \frac{1}{2} \) of the function \( f(x) = x \) as below

\[
\frac{d^{1/2} x}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \sqrt{x}.
\]

The first application of the fractional calculus was given by N.H. Abel in his tautochrone problem of determining the shape of the frictionless wire in a vertical plane such that the time for a bead to slide to the lowest point of the wire is independent of the start point, while solving this problem Abel encountered the fractional integral equation.

There are so many definitions are given by various mathematicians for the fractional differentiation and integration. However, universally accepted and widely used definitions are

- Riemann-Liouville fractional integral and derivative
- Caputo fractional derivative
- Grunwald-Letnikov fractional derivative.

### 1.2.1 Riemann-Liouville Fractional Integral

The Riemann-Liouville fractional integral is a direct generalization of Cauchy’s formula for an \( n \)-fold integral

\[
\int_a^t dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n) dt_n = \frac{1}{(n-1)!} \int_a^t (t - s)^{n-1} f(s) ds.
\]

to an arbitrary order \( \alpha \in \mathbb{R}^+ \). The Riemann-Liouville fractional integral of order \( \alpha \) is defined as follows

\[
J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \; t > 0, \; \alpha \in \mathbb{R}^+.
\]
The Riemann-Liouville fractional integral satisfies the semigroup property, that is
\[ J_t^{\alpha + \beta} = J_t^\alpha J_t^\beta = J_t^\beta J_t^\alpha, \quad \alpha, \beta > 0. \] (1.1)

The Laplace transform of Riemann-Liouville fractional integral is given by
\[ L[J_t^\alpha f(t)] = \frac{1}{\lambda^\alpha} \hat{f}(\lambda) \]
where
\[ \hat{f}(\lambda) = \int_0^\infty e^{\lambda t} f(t) dt, \quad \text{Re}(\lambda) > w. \]

### 1.2.2 Riemann-Liouville Fractional Derivative

The Riemann-Liouville fractional derivative of order \( n - 1 < \alpha \leq n \) is defined by
\[ aD_t^\alpha = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{ds^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds. \]

The Riemann-Liouville fractional differentiation is the left inverse of the Riemann-Liouville fractional integral of order \( \alpha \), that is
\[ D_t^\alpha J_t^\alpha = I, \quad \alpha > 0. \] (1.2)

If \( n \) denotes the positive integer such that \( n - 1 < \alpha \leq n \), it could be recognized from (1.1) and (1.2) that
\[ D_t^\alpha f(t) = D_t^n J_t^{n-\alpha} f(t). \]

Some limitations of Riemann-Liouville’s definition are

- Initial value problems in terms of the Riemann-Liouville fractional derivatives need initial conditions expressed in terms of fractional derivatives of the unknown function.
- Riemann-Liouville fractional derivative of constant is not zero; it leads to some unacceptable result for a dynamic process even if it is at a steady state.
1.2.3 Caputo Fractional Derivative

Caputo fractional derivative overcomes the caveats of Riemann-Liouville fractional derivative. The Caputo fractional derivative of the function of order $n - 1 < \alpha < n$ is given by

$$c_a D^\alpha_t f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^n(s)}{(t - s)^{\alpha - n + 1}} ds.$$ 

Some advantages of Caputo’s definition compared with Riemann-Liouville’s definition are

- The initial conditions can be expressed in terms of integer order derivatives for the initial value problems with Caputo fractional derivative.

- Caputo derivative of a constant is equal to zero.

Throughout this thesis, Caputo fractional derivative is employed in-order to acquire physically accepted results.

1.3 Qualitative Properties

In the following subsections, some basic qualitative properties of differential equations such as existence of mild solution, controllability and existence of optimal control are described.

1.3.1 Existence of Solution

In the theory of abstract Cauchy problems, the problem of existence of solutions is basis for the system validation and further undertaking study of the corresponding dynamic processes [23].

**Definition 1.1.** A one parameter family $T(t)$, $0 \leq t \leq b$ of bounded liner operators from a Banach space $X$ into itself is called a strongly continuous semigroup or $C_0$-semigroup on $X$ if
(i) $T(0) = I$, where $I$ is the identity on $X$.

(ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$, the semigroup property.

(iii) $\|T(t)x - x\|_X \to 0$ as $t \to 0^+$ for all $x \in X$, the strongly continuous property.

If $T(t)$ is a semigroup of class $C_0$, then it follows that there exist constants $w \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{wt}, \forall t \geq 0.$$

**Definition 1.2.** The infinitesimal generator of a $C_0$-semigroup $T(t)$, $0 \leq t \leq b$ on a Banach space $X$ is the operator $A$ defined by

$$Ax = \lim_{t \to 0^+} \frac{1}{t}(T(t)x - x)$$

is exists. The domain of $A$, $D(A)$ is the set of all points $x \in X$ for which the above limit exists.

**Remark 1.1.** [102] If $A$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$, then the domain of $A$, $D(A)$ is dense in $X$ and $A$ is a closed linear operator.

If $A$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$, then the homogeneous initial value problem (the Cauchy problem) given by

$$\frac{dx(t)}{dt} = Ax(t), \ t \geq 0$$

$$x(0) = x_0$$

has a unique solution $x \in C^1([0, \infty); X)$ for every $x_0 \in D(A)$ and is given by

$$x(t) = T(t)x_0.$$

If the initial time is $t_0$, then the solution is given by

$$x(t) = T(t - t_0)x_0.$$

Now, consider the nonhomogeneous Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + f(t), \ t \in [0, b] \\ x(0) = x_0 \end{cases} \quad (1.3)$$
where \( f \in L^1([0, b]; X) \) and \( x_0 \in X \).

The continuous function \( x : [0, b] \to X \) defined by the variation of constants formula

\[
x(t) = T(t)x_0 + \int_0^t T(t - s)f(s)ds
\]

is called the mild solution of the nonhomogeneous Cauchy problem (1.3).

**Definition 1.3.** A function \( x : [0, b] \to X \) is said to be a classical solution of the Cauchy problem (1.3), if it is continuously differentiable on \([0, b]\), \( x(t) \in D(A) \) for all \( t \in [0, b] \) and it satisfies equation (1.3) on \([0, b]\).

**Theorem 1.1** (Hille-Yosida Theorem). [102] A necessary and sufficient condition for a closed and densely defined linear operator \( A \) on a Banach space \( X \) to be the infinitesimal generator of a \( C_0 \)-semigroup \( T(t) \), \( 0 \leq t \leq b \) on \( X \) is that there exist real numbers \( M \) and \( w \), such that for every \( \lambda > w \) belongs to the resolvent set \( \rho(A) \) of \( A \) and for all positive integer \( r \),

\[
\| R(\lambda, A)^r \| \leq \frac{M}{(\lambda - w)^r}
\]

where \( R(\lambda, A) = (\lambda I - A)^{-1} \) is the resolvent operator of \( A \).

If \( A \) generates a \( C_0 \)-semigroup \( T(t) \), \( t \geq 0 \) on \( X \), then the following properties are hold (see [28, 29, 102])

(i) \( \| T(t) \| \) is bounded on every finite subinterval of \([0, \infty)\),

(ii) for all \( x \in X \), \( \lim_{t \to 0^+} \frac{1}{t} \int_0^t T(s)xds = 0 \),

(iii) if \( w_0 = \inf_{t \geq 0} \left( \frac{1}{t} \log(\| T(t) \|) \right) \), then \( w_0 = \inf_{t \to \infty} \left( \frac{1}{t} \log(\| T(t) \|) \right) < \infty \) (\( w_0 \) is called the growth bound of the semigroup),

(iv) for all \( w > w_0 \), there exists a constant \( M \) such that \( \| T(t) \| \leq Me^{wt} \) for all \( t \geq 0 \),

(v) for \( x \in D(A) \), \( T(t)x \in D(A) \) for all \( t \geq 0 \),

(vi) \( \frac{d}{dt}(T(t)x) = AT(t)x = T(t)Ax \) for all \( x \in D(A) \), \( t \geq 0 \),
(vii) \( T(t)x - x = \int_0^t T(s)Axds \) for all \( x \in D(A) \),

(viii) if \( w_0 \) is the growth bound of \( T(t) \) and \( \Re(\lambda) > w_0 \), then \( \lambda \in \rho(A) \) and for all \( x \in X \), \( \int_0^\infty e^{-\lambda t}T(t)xdt = (\lambda I - A)^{-1}x \), that is, the resolvent operator is the Laplace transformation of \( T(t) \),

(ix) \( \lim_{\lambda \to \infty} \lambda(\lambda I - A)^{-1}x = x \) for all \( x \in X \) and \( \lambda \in \mathbb{R} \).

Consider the fractional order system of the form

\[
\begin{align*}
^C D_t^\alpha x(t) &= Ax(t), \quad t \in [0, b] \\
x(0) &= x_0
\end{align*}
\]  

(1.4)

where \(^C D_t^\alpha \) is the Caputo fractional derivative of order \( \alpha \), \( 0 < \alpha < 1 \). The state \( x(t) \) takes values in a Banach space \( X \), \( A : D(A) \subseteq X \to X \) is a closed linear operator with dense domain \( D(A) \) and generates a \( C_0 \)-semigroup \( T(t) \).

The Cauchy problem (1.4) can be rewritten in the following form

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}Ax(s)ds.
\]

The function \( x(t) \) is solution of the Cauchy problem (1.4) means that

(i) \( x \) is continuous on \([0, b]\) and \( x(t) \in D(A) \) for each \( t \in [0, b] \),

(ii) \(^C D_t^\alpha x(t) \) exists and is continuous on \([0, b]\),

(iii) \( x \) satisfies the equation (1.4) on \([0, b]\) and the initial condition \( x(0) = x_0 \).

In recent years, the study of existence of mild solution of FDEs has grown dramatically. Zhou and Jiao [133] discussed the nonlocal Cauchy problem for the fractional evolution equations in Banach space and obtained various criteria on the existence and uniqueness of mild solutions. Li et al. [81] established the existence results for nonlocal semilinear FDEs by using convex-power condensing operator, Mainardi’s function and fixed point theorem. Hernandez et al. [62] studied the existence result for a general class of abstract FDEs by using the well-developed theory of resolvent operators for integral equations. The functional equation associated with general \((a, k)\) regularized resolvent families, which can replace the
property of semigroups, has been studied by Lizama and Poble [85]. Moreover, there has been tremendous development in existence, controllability and other qualitative properties of FDEs (for details, see [78, 82, 133]).

1.3.2 Controllability

Control theory is certainly one of the most significant interdisciplinary areas of research nowadays. The concept of controllability plays a crucial role in analysis and design of control system. The main objective of the control action is to drive the system from one state to another state in an optimal fashion. In infinite dimensional dynamical systems, there are two concept of controllability, namely exact and approximate controllability. This is strongly motivated by the fact that in infinite dimensional spaces there exist linear spaces, which are not closed. The exact controllability of the dynamical system means that it is possible to steer the system from an arbitrary initial state to an arbitrary final state using the set of admissible controls, while approximate controllability means that the system can be steered from the arbitrary initial state to arbitrary small neighborhood of the final state. It should be pointed out that in the case of finite dimensional systems, the concepts of exact and approximate controllability coincide.

Kalman [66, 67, 68] introduced the concept of controllability for finite dimensional linear control systems and proposed the controllability results under some rank condition of the controllability matrix. Hermes [59] investigated the controllability of a class of nonlinear systems by using fixed point theorem. Tarnove [120] suggested the concept of controllability of nonlinear systems by investigating the existence of a fixed point of a certain set valued map. Fattorini [46, 47] considered a more general model of the system and studied the controllability for the case, when \( A \) is infinitesimal generator of a strongly continuous semigroup \( T(t), \ t \geq 0 \), where \( A \) is closed, linear with dense domain.

The classical theory of controllability and observability in finite dimensional spaces have been extended to linear abstract systems defined on infinite dimensional Banach spaces by Triggiani [123] under the basic assumption that the operator acting on the state be bounded.
Triggiani [124] shown the lack of exact controllability for mild solutions in Banach spaces using locally $L_1$-controls, if the operator through which the control acts on the system is compact. Further, in [125] Triggiani proved that if $A$ generates a compact $C_0$-semigroup $T(t)$, then linear system could never be exact controllable in an infinite dimensional space.

The linear deterministic control system in an infinite dimensional space can be written as

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad t \in [t_0, b]$$

(1.5)

where $A : D(A) \subset X \to X$ is a closed, densely defined, but not necessarily bounded linear operator, the state $x(t)$ takes values in a Banach space $X$, the control function $u(t)$ takes its values in another Banach space $U$ and $B : U \to X$ is a bounded linear operator.

For $x_0 \in X$, the function $x \in C([t_0, b], X)$ given by

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)Bu(s)ds$$

is called the mild solution of the system (1.5).

**Definition 1.4.** The linear system (1.5) is said to be approximately controllable in the finite time interval $[t_0, b]$, if for given $\epsilon > 0$, there exists an admissible control $u(t)$ on $[t_0, b]$ steering the initial point $x_0$ along a trajectory (mild solution) $x(t)$ of (1.5) to an $\epsilon$-neighborhood of the final state $x_b$ such that

$$\|x(b) - x_b\| \leq \epsilon.$$

**Remark 1.2.** For $\epsilon = 0$, the above definition will be a definition of exact controllability of the system (1.5).

The controllability map $L : L_2([t_0, b], U) \to X$ and the controllable Grammian $\Gamma_{t_0}^b : X \to X$ are defined respectively for the system (1.5) as follows

$$Lu = \int_{t_0}^{b} T(b - s)Bu(s)ds$$

and

$$\Gamma_{t_0}^b = \int_{t_0}^{b} T(b - s)BB^*T^*(b - s)ds.$$
Remark 1.3. The following properties are satisfied

(i) $L : L_2([t_0, b], U) \rightarrow X$ is a bounded linear map.

(ii) The adjoint operator $L^*$ of $L$ is defined as $(L^*x)(s) = B^*T^*(b - s)x$ on $[t_0, b]$.

(iii) $\Gamma_{t_0}^b = LL^*$ is a bounded linear operator on $X$.

Consider the deterministic semilinear control system given by

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in [t_0, b]$$

(1.6)

where $f : [t_0, b] \times X \rightarrow X$ is a nonlinear function.

Definition 1.5. The set of all possible final states

$$\mathcal{R}_b(f) = \{ x(b) \in X : x(\cdot) \text{ is the mild solution of (1.6) for } u(\cdot) \in L_2([0, b], U) \}$$

is called the reachable set of the semilinear system (1.6).

The definition of approximate controllability and exact controllability can be given in terms of reachable set as follows

Definition 1.6. The semilinear control system (1.6) is said to be approximately controllable if and only if

$$\overline{\mathcal{R}_b(f)} = X,$$

where $\overline{\mathcal{R}_b(f)}$ represents the closure of $\mathcal{R}_b(f)$ and the control system (1.6) is said to be exact controllable if and only if

$$\mathcal{R}_b(f) = X.$$

Theorem 1.2. [98] The semilinear control system is approximately controllable under the following conditions

(i) the $C_0$-semigroup $T(t)$ is compact,

(ii) the nonlinear function $f(t, x(t))$ is Lipschitz continuous,
(iii) \[ \| f(t, x(t)) \| \leq M, \text{ where } M \text{ is a positive constant}, \]

(iv) for every \( \hat{p} \in L_2([0, b], X) \), there exists a \( \hat{q} \in \overline{R(B)} \) such that \( \hat{L}_\hat{p} = \hat{L}_\hat{q} \), where \( R(B) \) denotes the range of the operator \( B \) and \( \hat{L} \) is a bounded linear operator from \( L_2([0, b], X) \) to \( X \) defined as

\[ \hat{L}_\hat{p} = \int_0^b T(b - s)\hat{p}(s)ds. \]

By condition (iv) of the above theorem, the corresponding linear system of (1.6) is approximately controllable.

Stochastic control theory is considered to be a stochastic generalization of classical control theory. Controllability of stochastic systems is a well-known problem and frequently discussed in the literature (see [88, 91, 101]). Some major contributions concerning controllability of semilinear SDEs are the following. Mahmudov [87] extended the classical theory of controllability of deterministic systems to linear stochastic systems defined on infinite-dimensional Hilbert spaces. Dauer and Mahmudov [34] established sufficient conditions for the approximate and exact controllability of semilinear stochastic functional differential equations in Hilbert space by using the Banach fixed point theorem. Mahmudov [89] obtained various sufficient conditions for approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces by using some new properties of symmetric operators, compact semigroups, the Schauder fixed point theorem and the contraction mapping principle under the natural assumption that the corresponding linear control system is approximately controllable. Balasubramaniam and Dauer [12] studied the controllability of semilinear stochastic delay systems represented by evolution equations with unbounded linear operators in Hilbert space. Subalakshmi et al. [119] investigated the approximate and exact controllability for semilinear stochastic functional integrodifferential systems by using Banach fixed point theorem.

The criteria for the controllability in infinite-dimensional systems are obtained by considering the operator \( L_0^b(u) = \int_0^b T(b - t)B(t)dt \) and \( \Gamma_0^b = L_0^b(L_0^b)^* \). The stochastic analogue of \( L_0^b, (L_0^b)^* \) are defined in-order to study the concept of stochastic controllability.
The relationship between the controllability operator $\Gamma^b_s$ and its stochastic analogue $\Pi^b_s$ can be found in [87]. The stochastic analogue $\Pi^b_s$ of

$$\Gamma^b_s = \int^b_s T(b - s)BB^*T^*(b - s)dt$$

is defined as follows

$$\Pi^b_s \{ \cdot \} = \int^b_s T(b - s)BB^*T^*(b - s)E\{ \cdot | \mathcal{F}_t \}dt.$$ 

It is straightforward that $L^b_0$, $\Pi^b_s$, $\Gamma^b_s$ are bounded linear operators, and the stochastic analogue of the adjoint operator $(L^b_0)^*$ of $L^b_0$ is defined by

$$(L^b_0)^*(\cdot) = B^*T^*(b - s)E\{ \cdot | \mathcal{F}_t \}$$

and

$$\Pi^b_0 = L^b_0(L^b_0)^*.$$ 

The natural extension of the concept of controllability to the fractional order dynamical systems is required. The concept of approximate controllability of the following fractional order linear control system of order $0 < \alpha \leq 1$

$$^C D^\alpha_t x(t) = Ax(t) + Bu(t), \ t \in [0, b]$$

$$x(0) = x_0$$

is considered to be a natural generalization of the approximate controllability of the first order linear control system.

The fractional order semilinear control system

$$^C D^\alpha_t x(t) = Ax(t) + Bu(t) + f(t, x(t)), \ t \in [0, b]$$

$$x(0) = x_0$$

is approximately controllable if the following conditions are satisfied [114]

(i) the $C_0$-semigroup $T(t)$ generated by $A$ is compact,
(ii) for each $t \in [0, b]$, the function $f(t, \cdot) : X \to X$ is continuous and for each $x \in C([0, b], X)$, the function $f(\cdot, x) : [0, b] \to X$ is strongly measurable,

(iii) there exists a constant $q_1 \in [0, \alpha]$ and $m \in L^\infty_{q_1}([0, b], \mathbb{R}^+)$, such that $|f(t, x)| \leq m(t)$ for all $x \in X$ and almost all $t \in [0, b]$,

(iv) the function $f : [0, b] \times X \to X$ is continuous and uniformly bounded, that is, there exists $N > 0$ such that $\|f(t, x)\| \leq N$ for all $(t, x) \in [0, b] \times X$.

### 1.3.3 Optimal Control

The theory of optimal control is considered to be the most important field of modern mathematics and it has many practical applications in science and technology. Variational calculus is one of the techniques, which is based on the Euler-Lagrange theory, to obtain analytical solutions of optimization or optimal control problems. The first formulation of the variational principle was made by Fermat for a physical problem. After the Fermat’s discovery, the variational formulations became well known in many science and engineering fields, namely mechanics, electrodynamics, quantum mechanics, quantum field theory, etc. Euler published a manuscript under the title “A method of finding curves possessing the properties of a maximum or a minimum or the solution of the isoperimetric problem taken in the broadest sense” in 1744, followed by his first work on variational calculus in 1732. The work elaborated new methods of calculus of variations was published by Lagrange in 1759 based on the main idea to consider a variation of the curve, which is assumed to be an extremal. The Lagrange method became generally accepted one by other eminent mathematicians.

Consider the dynamical system described by integer order differential equations

$$\dot{x}(t) = f(t, x(t), u(t))$$

$$x(0) = x_0$$

where $x(t)$ represent the state variable, which characterize the behaviour of the system at any time instant $t$, $u(t) \in U$ represents the control variable at time instant $t$. 
The cost functional or performance criterion is specified for evaluating the performance of a system quantitatively. By analogy to the problems of the calculus of variations, the cost functional $J$ is defined in the so-called Lagrange form,

$$J(u) := \int_0^b L(t, x(t), u(t)) dt.$$ 

The stochastic optimal control problem is extension of optimal control problem of dynamical systems subject to stochastic disturbances. In the case of stochastic optimal control problems, the cost function could be defined as

$$J(u) := \mathbb{E}\left\{ \int_0^b L(t, x(t), u(t)) dt \right\}.$$ 

One can state the optimal control problem as follows, finding an admissible control, which satisfies the given constraints in such a way that the cost functional $J$ has a minimum value. In other words, finding the control $u_0 \in U_{ad}$ such that

$$J(u_0) \leq J(u), \text{ for all } u \in U_{ad}$$

where $U_{ad}$ is the set of all admissible controls. The optimal control problems are well known for the dynamical systems of integer order, the optimal control problems of dynamical systems of fractional order are the subject of current strong research (see [2, 3, 50]). The fractional calculus of variations is one of the recent essential field of research in various disciplines, due to its salient features in science, engineering, and applied mathematics (see [1, 40, 49]).

It is absolutely essential to assure the existence of optimal control, while studying fractional optimal control problems arisen from science and technology. In this regard, the solvability and existence of optimal controls of different classes of FDEs are studied in fewer works. Wang et al. [129] investigated the solvability and optimal controls of a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. Wang and Zhou [127] established the existence of mild solutions for semilinear fractional evolution equations and also analysed the existence of optimal controls in $\alpha$-norm by means of fractional calculus, singular version Gronwall inequality and Leray-Schauder fixed point.
theorem. Fan and Mophou [44] derived the existence of an optimal state-control pair solution to the associated Lagrange optimal control problem governed by a semilinear composite fractional relaxation equation in Banach space. However, in the case of fractional stochastic optimal control problems, no result could be found in the literature. Recognizing the importance of fractional stochastic optimal control problems, this thesis successfully deals with the existence of optimal control problems for some class of FSDEs.

1.4 Some Basic Definitions, Lemmas and Fixed Point Theorems

Definition 1.7. [95] Let $\Omega$ be a given set, then a $\sigma$-algebra $\mathcal{F}$ on $\Omega$ is a family $\mathcal{F}$ of subsets of $\Omega$ with the following properties:

(i) $\emptyset \in \mathcal{F}$, where $\emptyset$ denotes the empty set,

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, where $A^c = \Omega - A$ is the complement of $A$ in $\Omega$,

(iii) $A_1, A_2, \cdots \subset \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. The subset $A \subset \Omega$, which belongs to $\mathcal{F}$ is called $\mathcal{F}$-measurable set.

Definition 1.8. [95] A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

- $\mathbb{P}(\Omega) = 1$,

- for any disjoint sequence $\{A_i\}_{i \geq 1} \subset \mathcal{F}$,

$$
\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).
$$

Definition 1.9. [95] The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, set

$$
\bar{\mathcal{F}} = \{ A \subset \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C) \}.
$$
Then, $\bar{\mathcal{F}}$ is a $\sigma$-algebra and is called the completion of $\mathcal{F}$. If $\bar{\mathcal{F}} = \mathcal{F}$, then the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete.

**Definition 1.10.** [95] A random variable $X$ is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, if $X$ is a real valued random variable and is integrable with respect to the probability measure $\mathbb{P}$, then the number

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega)$$

is called the expectation of $\mathbb{P}$.

**Definition 1.11.** [95] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $\{\mathcal{F}_t\}_{t \in \mathbb{J}}$ of $\sigma$-algebras of $\mathcal{F}$ with $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leq t < s < \infty$ is called a filtration of $\mathcal{F}$.

**Definition 1.12.** [95] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of random variables $\{X_t\}_{t \in \mathbb{J}}$ is called stochastic process with index set $\mathbb{J}$.

**Definition 1.13.** [95] The stochastic process $\{X_t\}_{t \in \mathbb{J}}$ is said to be $\{\mathcal{F}_t\}$-adapted, if for every $t$, $X_t$ is $\mathcal{F}_t$-measurable.

**Definition 1.14.** [95] The stochastic process $\{X_t\}_{t \geq 0}$ is said to be right (left) continuous, if for almost all $\omega \in \Omega$, the function $X_t(\omega)$ is right (left) continuous on $t \geq 0$. It is said to be c.d.l.g., if it is right continuous and for almost all $\omega \in \Omega$, the left limit $\lim_{s \to t^-} X_s(\omega)$ exists and is finite for all $t > 0$.

**Definition 1.15.** [95] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. A standard one-dimensional Brownian motion (or Wiener process) is a real valued continuous $\mathcal{F}_t$-adapted process $\{B_t\}_{t \geq 0}$ with the following properties:

(i) $B_0 = 0$ almost surely (a.s.);

(ii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;

(iii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$. 19
In the following, basic definitions, lemmas and some fixed point theorems in functional analysis, which are related to this thesis are discussed.

**Definition 1.16.** [28] Let $X$ and $Y$ be two Banach spaces, then $f : X \to Y$ is said to be Lipschitz continuous if there exists a constant $l > 0$ such that

$$
\|f(x) - f(y)\|_Y \leq l\|x - y\|_X, \text{ for all } x, y \in X.
$$

**Definition 1.17.** [28] Let $X$ be a Banach space and $T : X \to X$ be a nonlinear operator then each solution of the equation $T(x) = x, x \in X$ is called fixed point of the operator $T$.

**Definition 1.18.** [28] Let $X$ and $Y$ be real Banach spaces.

(i) An operator $T : X \to Y$ is said to be compact, if it maps every bounded subset of $X$ into a relatively compact subset of $Y$.

(ii) $T : X \to Y$ is said to be completely continuous, if for any sequence $\{x_n\}$ converging weakly to $x_0$, the sequence $\{Tx_n\}$ converges to $Tx_0$.

**Definition 1.19.** [28] Let $X$ be a Banach space.

(i) A family of functions $\hat{F} = \{f_n(x)\}$ defined on $X$ is said to be uniformly bounded, if there exists a number $M > 0$ such that

$$
|f_n(x)| < M, \forall x \in X, f_n \in \hat{F}.
$$

(ii) A family of functions is said to be equicontinuous, if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$
|f_n(x) - f_n(y)| < \epsilon
$$

for all $x, y \in X$ such that $|x - y| < \delta$ and for all $f_n$ in the given family.

**Theorem 1.3.** [28](Arzela-Ascoli theorem) Let $E$ be a compact metric space and $C(E)$ be the Banach space of real or complex valued continuous functions defined on $E$. If $A = \{f_n\}$ be a sequence in $C(E)$ such that
(i) $f_n$ is uniformly bounded

(ii) $f_n$ is equicontinuous

then the closure of $A$ is compact.

**Theorem 1.4.** [65](Banach’s contraction principle) Let $f$ be a contraction mapping of a complete metric space $X$ into itself with contraction constant $0 \leq k < 1$, then $f$ has a unique fixed point in $X$.

**Theorem 1.5.** [132](Schauder fixed point theorem) Let $S$ be a non-empty closed bounded convex subset of a Banach space $X$ and suppose that $T : S \to S$ is compact then $T$ has a fixed point.

**Theorem 1.6.** [51](Leray-Schauder alternative fixed point theorem) Let $X$ be a Banach space, and $T : X \to X$ be a completely continuous operator, then either

- $T$ has a fixed point or
- the set $\{x \in X : x = \lambda T(x), \ 0 < \lambda < 1\}$ is unbounded.

In some other books this theorem is also referred as Schaefer’s fixed point theorem.

**Lemma 1.1.** (Nussbaum fixed point theorem) Let $S$ be a closed, bounded and convex subset of a Banach space $X$. Let $T_1, T_2$ be continuous mappings from $S$ into $X$ such that

(i) $(T_1 + T_2)S \subseteq S$,

(ii) $\|T_1x_1 - T_1x_2\| \leq k \|x_1 - x_2\|$ for all $x_1, x_2 \in S$, where $k$ is a constant and $0 \leq k < 1$,

(iii) $\overline{T_2(S)}$ is compact,

then the operator $T_1 + T_2$ has a fixed point in $S$.

**Lemma 1.2.** (Bohnenblust-Karlin’s fixed point theorem) Let $H$ be a Hilbert space, $S$ be a nonempty subset of $H$, which is bounded, closed and convex. Suppose $F : S \to 2^H \setminus \{\emptyset\}$ is upper semi continuous (u.s.c) with closed, convex values such that $F(S) \subseteq S$ and $F(S)$ is compact, then $F$ has a fixed point.
1.5 Thesis Outline and Contributions Overview

This thesis examines the qualitative properties such as existence of mild solution, controllability and existence of optimal control for various classes of FSDEs and FSDIs by using fixed point techniques. The objective of Chapter 2 is to examine the local and global existence of mild solutions for impulsive FSDEs with nonlocal condition in Hilbert space. The objective of Chapter 3 is to propose some new kind mild solution for the fractional neutral stochastic integrodifferential equations with infinite delay by using Mainardi’s function and to study its existence result. The objectives of Chapters 4, 5 are to investigate the approximate controllability of fractional neutral stochastic integrodifferential inclusions with infinite delay and FSDEs driven by mixed fBm respectively using Mainardi’s function and resolvent operators. The objective of Chapter 6 is to address the solvability and existence of optimal controls for various classes of FSDEs namely impulsive fractional stochastic integrodifferential equations and FSDEs driven by Poisson jumps via resolvent operators.

In Chapter 2, local and global existence of mild solutions are studied for the impulsive fractional semilinear stochastic differential equation with nonlocal condition in Hilbert space. The local existence of mild solution is obtained by using both the Banach contraction mapping principle and Schauder fixed point theorem. Then, the global existence of mild solution is examined. Finally, an illustrative example is provided to validate the obtained theoretical results.

Chapter 3 proposes the different concept of mild solution for the considered FSDEs in stochastic settings. Further, the existence of mild solution is analysed for fractional neutral stochastic integrodifferential equations with infinite delay in Hilbert space by employing Nussbaum fixed point theorem, Mainardi’s function and fractional calculus. Finally, efficiency of the derived theoretical results is verified by an example.

Chapter 4 is concerned with the approximate controllability of fractional neutral stochastic integrodifferential inclusions with infinite delay in Hilbert space. The new set of sufficient conditions are established for approximate controllability of the considered
system by utilizing Bohnenblust-Karlin’s fixed point theorem, Mainardi’s function, fractional calculus and operator semigroups under the assumption that corresponding linear system is approximately controllable. Finally, the proposed theoretical results are validated through an illustrative example.

Chapter 5 considers the class of FSDEs driven by mixed fractional Brownian motion with Hurst parameter $\mathcal{H} \in \left(\frac{1}{2}, 1\right)$. The approximate controllability of the considered system is investigated by employing fractional calculus, analytic resolvent operators and with the help of Schaefer’s fixed point theorem under the assumption that corresponding linear system is approximately controllable. An illustrative example is provided to verify the obtained theoretical results.

Chapter 6 is devoted to examine the solvability and optimal controls for impulsive fractional stochastic integrodifferential equations and FSDEs driven by Poisson jumps in Hilbert space by using Leray-Schauder fixed point theorem and classical Banach contraction mapping principle respectively. Firstly, the sufficient conditions are obtained for the existence of mild solution of the considered systems and then the existence of optimal controls is investigated for the corresponding Lagrange optimal control problems. The effectiveness of the acquired theoretical results are examined with respective examples.