Chapter 6

The Solvability and Optimal Controls for FSDEs via Resolvent Operators

6.1 Introduction

Optimal control theory is an outcome of the calculus of variations, it has broad history of more than 360 years. The optimal control problems are widely studied in the case of integer order control system. But in the case of fractional optimal control problems, only limited numbers of contributions are available in the literature (see [3, 4, 44, 127]). Agrawal [3] introduced the general formulation and solution scheme for fractional optimal control problems by using the fractional variational principle and Lagrange multiplier technique, where fractional derivatives have been taken in the sense of Riemann-Liouville. Wang et al. [130] obtained the sufficient conditions for the existence, uniqueness and continuous dependence of mild solutions also proved the existence of fractional optimal controls for the considered Lagrange problem by using fractional calculus, singular version of the Gronwall inequality and Banach contraction mapping principle. Wang et al. [129] studied the solvability and optimal controls for a class of fractional integrodifferential evolution systems with infinite delay. Liu et al. [77] established the solvability and optimal controls for some fractional evolution equation with impulse effects by using fractional calculus, Gronwall inequality, and Leray-Schauder’s fixed point theorem. It is natural to raise the question of existence of stochastic optimal controls for a large class of fractional dynamical systems with stochastic
perturbations. In this regard, nowadays, it could be a thrust area of research.

The stochastic processes with jumps are being more commonly used, for instance in the classical financial stock price model, the stock prices are represented by a process that has jumps. In particular, Poisson jumps are widely used in the modelling of real life phenomena arising in economics, finance, physics, biology and medicine etc. In literature, the qualitative properties of abstract SDEs with Poisson jumps have been studied in great details (see [7, 109, 110]). Ren and Chen [109] obtained existence and uniqueness of the solution for neutral stochastic functional differential equation with infinite delay and Poisson jumps in some abstract space with non-Lipschitz conditions. Ren et al. [111] established the existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay under non-Lipschitz and Lipschitz conditions. Luo and Taniguchi [86] studied the existence and uniqueness of mild solutions to non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jump processes in Hilbert space. The better descriptions of the dynamical behaviour of the real life phenomena have been provided by FSDEs, it is noteworthy to examine FSDEs driven by Poisson jumps. However, in-order to consider FSDEs with Poisson jumps, one needs some intellectual to rewrite fractional derivative, otherwise there will be a flaw, due to the fact that the fractional derivative conflicts with Poisson jump term.

Strongly inspired by the above discussion, this chapter addresses fractional optimal control problems with stochastic effects for various classes of FSDEs. Section 6.2 is concerned with solvability and optimal controls for impulsive fractional stochastic integrodifferential equations in Hilbert space. Sufficient conditions are derived for the existence of mild solution of the considered system by using analytic resolvent operators, the uniform continuity of the resolvent and Leray-Schauder fixed point theorem. Then, the existence of optimal control for the corresponding Lagrange optimal control problem is investigated. Further, Section 6.3 investigates the solvability and optimal controls for FSDEs driven by Poisson jumps in Hilbert space. The sufficient conditions are derived to prove that the system has a unique mild solution by using classical Banach contraction mapping principle and then existence of optimal control is proved for the corresponding Lagrange optimal control problem. The applications of derived theoretical results are verified by illustrative examples in both sections.
6.2 The Solvability and Optimal Controls for Impulsive Fractional Stochastic Integrodifferential Equations via Resolvent Operators

6.2.1 Problem Description

Consider the impulsive fractional stochastic integrodifferential equations of the following form

\[
\begin{align*}
C D_t^\alpha x(t) &= Ax(t) + J_t^{1-\alpha} [B(t) u(t) + f(t, x(t), x(a_1(t)), x(a_2(t)), \ldots, x(a_m(t))] \\
\Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \ldots, \hat{q}, \quad x(0) = x_0
\end{align*}
\]

where \( J = [0, b], \) \( 0 < \alpha < 1, \) \( A : D(A) \subset H \to H \) is the infinitesimal generator of a resolvent \( S(t), t \geq 0 \) and \( x(\cdot) \) takes values in the separable Hilbert space \( H. \) \( u \) is a given control function and it takes values from another separable reflexive Hilbert space \( U. \) \( B \) is a linear operator from \( U \) into \( H, \) \( \Delta x(t_k) = x(t_k^+) - x(t_k) \) represents the jump in the state \( x \) at \( t_k, \) \( k = 1, 2, \ldots, \hat{q}. \) The functions \( f : J \times H^{m+1} \to H \) and \( \sigma : J \times H^{n+1} \to L(K, H) \) are the appropriate functions.

6.2.2 Basic Definitions, Lemmas and Hypotheses

Let \( C(J, L_2(\Omega, H)) \) be the Banach space of all continuous maps \( f \) into \( L_2(\Omega, H) \) satisfying

\[
\sup_{t \in J} \mathbb{E} \| x(t) \|^2 < \infty.
\]

Let \( PC(J, L_2(\Omega, H)) = \{ \psi : J \to L_2(\Omega, H) / \psi \in C((t_k, t_{k+1}], H), \ k = 1, 2, \ldots, \hat{q}, \ \psi(t_k^+), \ \psi(t_k^-) \text{ exist and } \psi(t_k^-) = \psi(t_k) \} \) with the norm \( \| \psi \|_{PC} = \sup_{t \in J} (\| \psi(t) \|^2_{L_2})^{\frac{1}{2}}, \) then \( (PC(J, L_2(\Omega, H)), \| \cdot \|_{PC}) \) is a Banach space.

**Definition 6.1.** [107] A family \( \{S(t)\}_{t \geq 0} \subset L(H) \) of bounded linear operators in \( H \) is called a resolvent (or solution operator) generated by \( A \) if and only if the following conditions are satisfied

(i) \( S(t) \) is strongly continuous on \( \mathbb{R}_+ \) and \( S(0) = I, \)

(ii) \( S(t) D(A) \subseteq D(A) \) and \( AS(t)x = S(t)Ax \) for all \( x \in D(A) \) and \( t \geq 0, \)

(iii) the resolvent equation holds for all \( x \in D(A) \) and \( t \geq 0, \)

\[
S(t)x = x + \int_0^t g_\alpha(t - s) A S(s) x ds,
\]

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where \( g_\alpha(t) = \frac{t^{\alpha-1}}{(\alpha)} \), \( t > 0 \). \( A \) is a closed and densely defined linear operator on \( H \), which implies, it is easy to show that the resolvent equation holds for all \( x \in H \) (see [107]). For \( \hat{w}, \theta \in \mathbb{R} \), let 
\[ \sum(\hat{w}, \theta) := \{ \hat{\lambda} \in \mathbb{C} : |\text{arg}(\hat{\lambda} - \hat{w})| < \theta \} \].

**Definition 6.2.** [107] A resolvent \( S(t) \) is called analytic, if and only if the function \( S(\cdot) : \mathbb{R}_+ \to L(H) \) admits analytic extension to a sector \( \sum(0, \theta_0) \) for some \( 0 < \theta_0 \leq \frac{\pi}{2} \). An analytic resolvent \( S(t) \) is said to be of analyticity type \((\hat{w}_0, \theta_0)\), if for each \( \theta < \theta_0 \) and \( \hat{w} > \hat{w}_0 \), there is \( M_1 = M_1(\hat{w}, \theta) \) such that for \( z \in \sum(0, \theta) \), \( \|S(z)\| \leq M_1e^{\hat{w}\text{Re}(z)} \), where \( \text{Re}(z) \) denotes the real part of \( z \).

**Lemma 6.1.** [45] Suppose that \( S(t) \) is a compact analytic resolvent of analyticity type \((\hat{w}_0, \theta_0)\), then 
\[
(i) \lim_{\hat{h} \to 0} \|S(t + \hat{h}) - S(t)\| = 0 \text{ for } t > 0,
(ii) \lim_{\hat{h} \to 0^+} \|S(t + \hat{h}) - \hat{h}S(t) - \hat{h}S(t)\| = 0 \text{ for } t > 0,
(iii) \lim_{\hat{h} \to 0^+} \|S(t) - \hat{h}S(t - \hat{h})\| = 0 \text{ for } t > 0.
\]

**Definition 6.3.** [42] A resolvent \( S(t) \) is called compact for \( t > 0 \) if and only if for every \( t > 0 \), \( S(t) \) is a compact operator.

Consider the following form of FDEs 
\[
\begin{align*}
CD_0^\alpha x(t) &= Ax(t) + J_t^{1-\alpha}[f(t) + B(t)u(t)], \ 0 < t \leq b \\
x(0) &= x_0. 
\end{align*}
\]

(6.2) Suppose that \( x_0 \in H, f \in L^1(J, H) \) and \( x \) is an integral solution of (6.2), then the variation of constant formula is given by 
\[
x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)B(s)u(s)ds, \ 0 \leq t \leq b.
\]

In fact, if \( x \) satisfies (6.2), then for \( 0 \leq t \leq b \), one can have 
\[
x(t) = x_0 + A(g_\alpha \ast x)(t) + \int_0^t f(s)ds + \int_0^t B(s)u(s)ds.
\]

By the definition of a resolvent and the definition of an integral solution, it follows that 
\[
1 \ast x = (S - Ag_\alpha \ast S) \ast x \\
= S \ast [x - (Ag_\alpha \ast x)] \\
= S \ast [x - (x - x_0 - 1 \ast f - 1 \ast Bu)] \\
= S \ast (x_0 + 1 \ast f + 1 \ast Bu) \\
= S \ast x_0 + 1 \ast S \ast f + 1 \ast S \ast Bu.
\]

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which implies that
\[ x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)B(s)u(s)ds, \quad 0 \leq t \leq b \]
is the variation of constant formula satisfying (6.2). Now, one can define the mild solution of the system (6.1) as follows.

**Definition 6.4.** A stochastic process \( x(t) \in PC(J, L_2(\Omega, H)) \) is called a mild solution for (6.1), if

(i) \( x(t) \) is \( \mathcal{F}_t \)-adapted,

(ii) \( x(t) \in H \) has a cadlag path on \( t \in [0, b] \) a.s

(iii) for arbitrary \( t \in [0, b] \),

\[
 x(t) = S(t)x_0 + \sum_{k=1}^{\hat{q}} S(t-t_k)I_k(x(t_k)) + \int_0^t S(t-s)f(s, x(s), x(a_1(s)), x(a_2(s)), \ldots, x(a_m(s)))ds \\
+ \int_0^t S(t-s)\left( \int_0^s \sigma(\tau, x(\tau), x(b_1(\tau)), x(b_2(\tau)), \ldots, x(b_n(\tau)))dw(\tau) \right)ds.
\]

Assume the following hypotheses,

\((H_{6.1})\) \( S(t) \) is a compact analytic resolvent of analyticity type \((\hat{w}_0, \theta_0)\) and

\[
\hat{M} = \sup_{t \in [0, b]} \|S(t)\| < +\infty.
\]

\((H_{6.2})\) The function \( f : J \times H^{m+1} \to H \) is a continuous function, there exist the constants \( M_f > 0 \), \( \bar{M}_f > 0 \) such that

\[
\|f(s_1, x_0, x_1, \ldots, x_m) - f(s_2, y_0, y_1, \ldots, y_m)\|^2 \leq M_f(\|s_1 - s_2\|^2 + \max_{i=0,1,\ldots,m} \|x_i - y_i\|^2)
\]

for \( 0 \leq s_1, s_2 \leq b, x_i, y_i \in H, \; i = 0, 1, 2, \ldots, m \) and the inequality

\[
\|f(t, x_0, x_1, \ldots, x_m)\|^2 \leq \bar{M}_f \left( \max_{i=0,1,\ldots,m} \|x_i\|^2 + 1 \right)
\]

holds for \( (t, x_0, x_1, \ldots, x_m) \in J \times H^{m+1} \).

\((H_{6.3})\) The function \( \sigma : J \times H^{m+1} \to L(K, H) \) satisfies the following conditions
(i) for each \( t \in J \), the function \( \sigma(t, \cdot) : H^{n+1} \to L(K, H) \) is continuous, and the function
\[
\sigma(\cdot, x_0, x_1, \cdots, x_n) : J \to L(K, H)
\]
is \( \mathcal{F}_t \)-measurable for each \((x_0, x_1, \cdots, x_n) \in H^{n+1}\).

(ii) for each positive \( r \in \mathbb{N} \), there exists a positive function \( h_r(\cdot) \in L^1(J) \) such that
\[
\sup_{\|x_0\|^2, \|x_1\|^2, \cdots, \|x_n\|^2 \leq r} \int_0^t \mathbb{E}\left\|\sigma(s, x_0, x_1, \cdots, x_n)\right\|_Q^2 ds \leq h_r(t)
\]
and \( \liminf_{r \to \infty} \frac{1}{r} \int_0^b h_r(s) ds = \Upsilon < \infty \),

(iii) the function \( \sigma : J \times H^{n+1} \to L(K, H) \) satisfies \( (H_{6.3}) \) (i) and there exists a constant \( M_\sigma > 0 \) such that
\[
\mathbb{E}\left\|\sigma(s_1, x_0, x_1, \cdots, x_m) - \sigma(s_2, y_0, y_1, \cdots, y_n)\right\|_Q^2 \leq M_\sigma (\|s_1 - s_2\|^2 + \max_{i=0,1,\cdots,n} \mathbb{E}\|x_i - y_i\|^2)
\]
for \( 0 \leq s_1, s_2 \leq b \), \( x_i, y_i \in H \), \( i = 0, 1, 2, \cdots, n \).

\( (H_{6.4}) \) \( I_k : H \to H \), \( k = 1, 2, \cdots, \hat{q} \) and there exist \( M_k \geq 0 \), \( \tilde{M}_k \geq 0 \) such that
\[
\mathbb{E}\|I_k(x) - I_k(y)\|^2 \leq M_k \mathbb{E}\|x - y\|^2 \quad \text{and} \quad \mathbb{E}\|I_k(x)\|^2 \leq \tilde{M}_k \mathbb{E}\|x\|^2
\]
for any \( x, y \in H \).

\( (H_{6.5}) \) Let \( u \in U \) be the control function and the operator \( B \in L_\infty(J, L(U, H)) \). \( \|B\| \) stands for the norm of the operator \( B \).

\( (H_{6.6}) \) The multivalued map \( A : J \to \mathcal{P}(U) \) (where \( \mathcal{P}(U) \) is a class of nonempty closed, convex subsets of \( U \)) are measurable and \( A(\cdot) \subseteq \Pi \), where \( \Pi \) is a bounded set of \( U \).

Let the admissible set be \( \mathcal{A}_{ad} = \left\{ v : J \times \Omega \to H / v \text{ is } \mathcal{F}_t \text{- adapted stochastic process and } \mathbb{E}\int_0^b \|v(t)\|^2 dt < \infty \right\} \).

### 6.2.3 Existence of Mild Solution of Impulsive Fractional Stochastic Integrodifferential Equations

**Theorem 6.1.** Assume that hypotheses \( (H_{6.1}) - (H_{6.6}) \) are satisfied, then the problem (6.1) has at least one mild solution on \( J \) provided that
\[
5\tilde{M}^2 \hat{q} \sum_{k=1}^{\hat{q}} M_k + 5\tilde{M}^2 b^2 \tilde{M}_f + 5\tilde{M}^2 b^2 Tr (Q) \Upsilon < 1.
\]
Proof. Define the map $\Phi : PC(J, L_2(\Omega, H)) \rightarrow PC(J, L_2(\Omega, H))$ by

$$(\Phi x)(t) = S(t)x_0 + \sum_{k=1}^{q} S(t-t_k)I_k(x(t_k)) + \int_0^t S(t-s)B(s)u(s)ds$$

$$+ \int_0^t S(t-s)f(s, x(s), x(a_1(s)), x(a_2(s)), \ldots, x(a_m(s)))ds$$

$$+ \int_0^t S(t-s)\left( \int_0^s \sigma(\tau, x(\tau), x(b_1(\tau)), x(b_2(\tau)), \ldots, x(b_n(\tau)))d\tau \right) ds. \tag{6.3}$$

To prove the existence of mild solution of the system (6.1), it is enough to show that the operator $\Phi$ has a fixed point in $PC(J, L_2(\Omega, H)).$ Let $B_r = \{ x \in PC(J, L_2(\Omega, H)) : \mathbb{E}\|x(t)\|_{PC}^2 \leq r, \ t \in J \}.$

**Step 1:** To show that $\Phi B_r \subseteq B_r.$ For each $x \in B_r$ from (6.3), one can have

$$\mathbb{E}\|\Phi x(t)\|^2 \leq 5\mathbb{E}\|S(t)x_0\|^2 + 5\mathbb{E}\left\| \sum_{k=1}^{q} S(t-t_k)I_k(x(t_k)) \right\|^2$$

$$+ 5\mathbb{E} \left\| \int_0^t S(t-s)B(s)u(s)ds \right\|^2$$

$$+ 5\mathbb{E} \left\| \int_0^t S(t-s)f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))ds \right\|^2$$

$$+ 5\mathbb{E} \left\| \int_0^t S(t-s)\left( \int_0^s \sigma(\tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)))d\tau \right) ds \right\|^2$$

$$\leq 5\tilde{M}^2\mathbb{E}\|x_0\|^2 + 5\tilde{M}^2q\sum_{k=1}^{q} \mathbb{E}\|I_k(x(t_k))\|^2$$

$$+ 5\tilde{M}^2\|B\|^2 \left[ \left( \int_0^t \mathbb{E}\|u(s)\|^pds \right)^{\frac{1}{p}} \left( \int_0^t ds \right)^{1-\frac{1}{p}} \right]^2$$

$$+ 5\tilde{M}^2b\int_0^t \mathbb{E}\|f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\|^2ds$$

$$+ 5\tilde{M}^2b\int_0^t \mathbb{E}\left\| \int_0^s \sigma(\tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)))d\tau \right\|^2ds$$

$$\leq 5\tilde{M}^2\mathbb{E}\|x_0\|^2 + 5\tilde{M}^2q\sum_{k=1}^{q} \tilde{M}_k\mathbb{E}\|x(t)\|^2 + 5\tilde{M}^2\|B\|^2\|u\|_{L_p(J,U)}^2 b^{\frac{2p-2}{p}}$$

$$+ 5\tilde{M}^2b\int_0^t \tilde{M}_f \left( \max_{i=0,1,2,\ldots,m} \mathbb{E}\|x_i\|^2 + 1 \right) ds$$

$$+ 5\tilde{M}^2b\int_0^t Tr(Q)\int_0^s \mathbb{E}\left\| \sigma(\tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau))) \right\|^2Q d\tau ds$$

$$\leq 5\tilde{M}^2\mathbb{E}\|x_0\|^2 + 5\tilde{M}^2\|B\|^2\|u\|_{L_p(J,U)}^2 b^{\frac{2p-2}{p}} + 5\tilde{M}^2b^2 \tilde{M}_f$$
\[
+ \left( 5\hat{M}^2 \tilde{q} \sum_{k=1}^{\tilde{q}} \hat{M}_k + 5\hat{M}^2 b^2 \hat{M}_f + 5\hat{M}^2 b Tr(Q) \bar{Y} \right) r
\leq r.
\]

One can deduce that \( \Phi B_r \subseteq B_r \) by choosing
\[
\left[ 5\hat{M}^2 \mathbb{E}\|x_0\|^2 + 5\hat{M}^2 \|B\|^2 \|u\|^2_{L^p(J,U)} b \frac{2\hat{p}-2}{\hat{r}} + 5\hat{M}^2 b^2 \hat{M}_f \right] \times \left[ 1 - \left( 5\hat{M}^2 \tilde{q} \sum_{k=1}^{\tilde{q}} \hat{M}_k + 5\hat{M}^2 b^2 \hat{M}_f + 5\hat{M}^2 b Tr(Q) \bar{Y} \right) \right]^{-1} \leq r.
\]

**Step 2:** To show that \( \Phi \) is continuous. Let \( \{x_{\tilde{n}}\} \) be a sequence such that \( x_{\tilde{n}} \rightarrow x \) in \( PC(J, L_2(\Omega, H)) \) as \( \tilde{n} \rightarrow \infty \), then for each \( t \in (t_k, t_{k+1}) \), one can obtain
\[
\mathbb{E}\| (\Phi x_{\tilde{n}})(t) - (\Phi x)(t) \|^2 \leq 3\mathbb{E} \left\| \sum_{k=1}^{\tilde{q}} \mathcal{S}(t - t_k)[I_k(x_{\tilde{n}}(t_k)) - I_k(x(t_k))] \right\|^2
+ 3\mathbb{E} \left\| \int_0^t \mathcal{S}(t - s) \left[ f\left( s, x_{\tilde{n}}(s), x_{\tilde{n}}(a_1(s)), \ldots, x_{\tilde{n}}(a_m(s)) \right)
- f\left( s, x(s), x(a_1(s)), \ldots, x(a_m(s)) \right) \right] ds \right\|^2
+ 3\mathbb{E} \left\| \int_0^t \mathcal{S}(t - s) \left( \int_0^s \left[ \sigma\left( \tau, x_{\tilde{n}}(\tau), x_{\tilde{n}}(b_1(\tau)), \ldots, x_{\tilde{n}}(b_n(\tau)) \right)
- \sigma\left( \tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)) \right) \right] dw(\tau) \right) ds \right\|^2
\leq 3\hat{M}^2 \tilde{q} \sum_{k=1}^{\tilde{q}} M_k \mathbb{E}\|x_{\tilde{n}} - x\|^2 + 3\hat{M}^2 b \int_0^t M_f \sup_{0 \leq s \leq b} \mathbb{E}\|x_{\tilde{n}}(s) - x(s)\|^2 ds
+ 3\hat{M}^2 b Tr(Q) M_\sigma \int_0^t \int_0^s \sup_{0 \leq \tau \leq b} \mathbb{E}\|x_{\tilde{n}}(\tau) - x(\tau)\|^2 d\tau ds.
\]
It is easy to see that \( \mathbb{E}\| (\Phi x_{\tilde{n}})(t) - (\Phi x)(t) \|^2 \rightarrow 0 \) as \( \tilde{n} \rightarrow \infty \). That is, \( \Phi \) is continuous.

**Step 3:** To prove that \( \Phi \) is equicontinuous on \( B_r \). Let \( 0 \leq \tau_1 < \tau_2 \leq b \), then for each \( x \in B_r \), one get
\[
\mathbb{E}\| (\Phi x)(\tau_2) - (\Phi x)(\tau_1) \|^2
\leq 8\mathbb{E}\| \mathcal{S}(\tau_2) - \mathcal{S}(\tau_1) - x_0 \|^2 + 8\mathbb{E} \left\| \sum_{k=1}^{\tilde{q}} [\mathcal{S}(\tau_2 - t_k) - \mathcal{S}(\tau_1 - t_k)] I_k(x(t_k)) \right\|^2
+ 8\mathbb{E} \left\| \int_0^{\tau_1} [\mathcal{S}(\tau_2 - s) - \mathcal{S}(\tau_1 - s)] B(s) u(s) ds \right\|^2 + 8\mathbb{E} \left\| \int_{\tau_1}^{\tau_2} \mathcal{S}(\tau_2 - s) B(s) u(s) ds \right\|^2
+ 8\mathbb{E} \left\| \int_0^{\tau_1} [\mathcal{S}(\tau_2 - s) - \mathcal{S}(\tau_1 - s)] f\left( s, x(s), x(a_1(s)), \ldots, x(a_m(s)) \right) ds \right\|^2
\]

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\[ + 8 \mathbb{E} \left| \int_{\tau_1}^{\tau_2} S(\tau - s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s))) \, ds \right|^2 \]
\[ + 8 \mathbb{E} \left| \int_0^{\tau_1} [S(\tau - s) - S(\tau_1 - s)] \left( \int_0^s \sigma(\tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau))) \, dw(\tau) \right) \, ds \right|^2 \]
\[ + 8 \mathbb{E} \left| \int_{\tau_1}^{\tau_2} S(\tau - s) \left( \int_0^s \sigma(\tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau))) \, dw(\tau) \right) \, ds \right|^2 \]
\[ \leq 8 \left\| S(\tau_2) - S(\tau_1) \right\|^2 \mathbb{E} \left\| x_0 \right\|^2 + 8 \delta \sum_{k=1}^{\delta} \left\| S(\tau_2 - t_k) - S(\tau_1 - t_k) \right\|^2 \hat{M}_k \mathbb{E} \left\| x(t) \right\|^2 \]
\[ + 8 \| B \|^2 \sup_{s \in [0, \tau_1 - \epsilon]} \left\| S(\tau_2 - s) - S(\tau_1 - s) \right\|^2 \tau_1^{\frac{2p-2}{p}} \| u \|^2_{L_p(J, U)} + 8 \hat{M}^2 \| B \|^2 \left\| (\tau_2 - \tau_1) \right\|^2 \tau_1^{\frac{2p-2}{p}} \| u \|^2_{L_p(J, U)} \]
\[ + 8 \tau_1 \hat{M} \int_{\tau_1}^{\tau_1} \left( \max_{i=0, 1, 2, \ldots, m} \mathbb{E} \left\| x_i \right\|^2 + 1 \right) \, ds \]
\[ + 8 \tau_1 \hat{M}^2 \int_{\tau_1}^{\tau_1} \left( \max_{i=0, 1, 2, \ldots, m} \mathbb{E} \left\| x_i \right\|^2 + 1 \right) \, ds \]
\[ + 8 \tau_1 \hat{M}^2 \int_{\tau_1}^{\tau_1} \left( \max_{i=0, 1, 2, \ldots, m} \mathbb{E} \left\| x_i \right\|^2 + 1 \right) \, ds \]
\[ \leq 8 \left\| S(\sigma) - S(\sigma_1) \right\|^2 \mathbb{E} \left\| x_0 \right\|^2 + 8 \delta \sum_{k=1}^{\delta} \hat{M}_k \left\| S(\tau_2 - t_k) - S(\tau_1 - t_k) \right\|^2 \]
\[ + 8 \| B \|^2 \tau_1^{\frac{2p-2}{p}} \sup_{s \in [0, \tau_1 - \epsilon]} \left\| S(\tau_2 - s) - S(\tau_1 - s) \right\|^2 \| u \|^2_{L_p(J, U)} \]
\[ + 8 \hat{M}^2 \| B \|^2 \left\| (\tau_2 - \tau_1) \right\|^2 \tau_1^{\frac{2p-2}{p}} \| u \|^2_{L_p(J, U)} + 8 \tau_1 \hat{M}^2 \| B \|^2 \int_{\tau_1}^{\tau_1} h_r(s) \, ds \]
\[ + 8 \tau_1 \hat{M}^2 \| B \|^2 \int_{\tau_1}^{\tau_1} h_r(s) \, ds \]

as \( \tau_2 \to \tau_1 \) and \( \epsilon \to 0 \), one can get \( \mathbb{E} \left\| (\Phi x)(\tau_2) - (\Phi x)(\tau_1) \right\|^2 \to 0 \), which implies that \( \Phi \) is equicontinuous.

**Step 4:** Now, show that \( (\Phi x)(t) \) is compact for \( 0 \leq t \leq b \). It is easy to see that \( (\Phi x)(0) \) is relatively compact in \( B_r \). Let \( 0 < t \leq b \) be fixed, for \( 0 < \epsilon < b \), \( x \in B_r \), define

\[ (\Phi^\epsilon x)(t) = S(t)x_0 + \sum_{k=1}^{\delta} S(t - t_k)I_k(x(t_k)) + \int_0^{t-\epsilon} S(t - s)B(s)u(s) \, ds \]
\[ + \int_0^{t-\epsilon} S(t - s)f(s, x(s), x(a_1(s)), \ldots, x(a_m(s))) \, ds \]

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\[ \begin{align*}
&+ \int_0^{t-\epsilon} S(t-s) \left( \int_0^s \sigma \left( \tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)) \right) \, dw(\tau) \right) \, ds \\
&= S(t)x_0 + \sum_{k=1}^{\hat{q}} S(t-t_k)I_k(x(t_k)) + S(\epsilon) \int_0^{t-\epsilon} S(t-s)B(s)u(s) \, ds \\
&+ S(\epsilon) \int_0^{t-\epsilon} S(t-s) f \left( s, x(s), x(a_1(s)), \ldots, x(a_m(s)) \right) \, ds \\
&+ S(\epsilon) \int_0^{t-\epsilon} S(t-s) \left( \int_0^s \sigma \left( \tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)) \right) \, dw(\tau) \right) \, ds.
\end{align*} \]

Then from the compactness of \( S(\epsilon) \ (\epsilon > 0) \), one get \( \{ (\Phi^\epsilon x)(t) : x \in B_r \} \) is relatively compact in \( H \), for every \( \epsilon \in (0, t) \). Furthermore, for every \( x \in B_r \),

\[ \begin{align*}
E \| (\Phi x)(t) - (\Phi^\epsilon x)(t) \|^2 &
\leq 6E \left\| \int_0^{t-\epsilon} [S(\epsilon)S(t-s) - S(t-s)]B(s)u(s) \, ds \right\|^2 \\
&+ 6E \left\| \int_{t-\epsilon}^t S(t-s)B(s)u(s) \, ds \right\|^2 \\
&+ 6E \left\| \int_0^{t-\epsilon} [S(\epsilon)S(t-s) - S(t-s)]f \left( s, x(s), x(a_1(s)), \ldots, x(a_m(s)) \right) \, ds \right\|^2 \\
&+ 6E \left\| \int_{t-\epsilon}^t S(t-s)f \left( s, x(s), x(a_1(s)), \ldots, x(a_m(s)) \right) \, ds \right\|^2 \\
&+ 6E \left\| \int_0^{t-\epsilon} [S(\epsilon)S(t-s) - S(t-s)] \left( \int_0^s \sigma \left( \tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)) \right) \, dw(\tau) \right) \, ds \right\|^2 \\
&+ 6E \left\| \int_{t-\epsilon}^t S(t-s) \left( \int_0^s \sigma \left( \tau, x(\tau), x(b_1(\tau)), \ldots, x(b_n(\tau)) \right) \, dw(\tau) \right) \, ds \right\|^2 \\
&\leq 6(t-\epsilon)\|B\|^2 \int_0^{t-\epsilon} \| S(\epsilon)S(t-s) - S(t-s) \|^2 E\|u(s)\|^2 \, ds \\
&+ 6\hat{M}^2\|B\|^2 \epsilon \frac{2m^2}{p} \|u\|^2_{L^p(I, \mathcal{V})} + 6\epsilon^2 \hat{M}^2 (1 + r) \\
&+ 6(t-\epsilon)(1 + r) \int_0^{t-\epsilon} \| S(\epsilon)S(t-s) - S(t-s) \|^2 \, ds \\
&+ 6(t-\epsilon)Tr(Q) \int_0^{t-\epsilon} \| S(\epsilon)S(t-s) - S(t-s) \|^2 h_r(s) \, ds \\
&+ 6\hat{M}^2 \epsilon Tr(Q) \int_{t-\epsilon}^t h_r(s) \, ds
\end{align*} \]

since, from Lemma 6.1 \((iii)\), \( S(\epsilon)S(t-s) - S(t-s) \to 0 \) as \( \epsilon \to 0 \) for \( s \in [0, t-\delta] \). Therefore, there are relatively compact sets to the set \( \{ (\Phi^\epsilon x)(t) : x \in B_r \} \). Hence, \( \Phi(t) \) is also relatively compact in \( B_r \). By Leray-Schauder fixed point theorem, one can conclude that \( \Phi \) has a fixed point \( x(\cdot) \) on \( B_r \). Therefore, the system (6.1) has a mild solution. \( \square \)
6.2.4 Existence of Optimal Controls of Impulsive Fractional Stochastic Integrodifferential Equations

Consider the following Lagrange problem \((\mathcal{P})\):

Find a control \((x^0, u^0) \in PC(J, L_2(\Omega, H)) \times \mathcal{A}_{ad}\) such that

\[
\mathcal{J}(x^0, u^0) \leq \mathcal{J}(x^u, u), \forall (x, u) \in PC(J, L_2(\Omega, H)) \times \mathcal{A}_{ad},
\]

where \(\mathcal{J}(x^u, u) := \mathbb{E}\left\{ \int_0^L \mathcal{L}(t, x^u(t), u(t))dt \right\}\) and \(x^u\) denotes the mild solution of the system (6.1) corresponding to the control \(u \in \mathcal{A}_{ad}\). In order to prove the existence of solution for the problem \((\mathcal{P})\), the following hypothesis is assumed.

\((H_{0.7})\) (i) The functional \(\mathcal{L} : J \times H \times U \rightarrow \mathbb{R} \cup \{\infty\}\) is \(\mathcal{F}_{\tau^*}\) measurable.

(ii) \(\mathcal{L}(t, \cdot, \cdot)\) is sequentially lower semicontinuous on \(H \times U\) for almost all \(t \in J\).

(iii) \(\mathcal{L}(t, x, \cdot)\) is convex on \(U\) for each \(x \in H\) and almost all \(t \in J\).

(iv) There exist constants \(d \geq 0, \ e > 0, \ \mu\) is nonnegative and \(\mu \in L^1(J, \mathbb{R})\) such that

\[\mathcal{L}(t, x, u) \geq \mu(t) + d\mathbb{E}\|x\|^2_H + e\|u\|^p_H.\]

**Theorem 6.2.** If hypothesis \((H_{0.7})\) and hypotheses of Theorem 6.1 hold, suppose that \(B\) is a strongly continuous operator, then the Lagrange problem \((\mathcal{P})\) admits at least one optimal pair, that is, there exists an admissible control pair \((x^0, u^0) \in PC(J, L_2(\Omega, H)) \times \mathcal{A}_{ad}\) such that

\[\mathcal{J}(x^0, u^0) = \mathbb{E}\left( \int_0^b \mathcal{L}(t, x^0(t), u^0(t))dt \right) \leq \mathcal{J}(x^u, u),\]

\[\forall (x^u, u) \in PC(J, L_2(\Omega, H)) \times \mathcal{A}_{ad}.\]

**Proof.** If \(\inf\\{\mathcal{J}(x^u, u) \mid (x^u, u) \in PC(J, L_2(\Omega, H)) \times \mathcal{A}_{ad}\} = +\infty\), there is nothing to prove. Without loss of generality, one can assume that infimum of \(\{\mathcal{J}(x^u, u) : (x^u, u) \in PC(J, L_2(\Omega, H)) \times \mathcal{A}_{ad}\} = \bar{\rho} < +\infty\). By using \((H_{0.7})\), one have \(\bar{\rho} > -\infty\). By definition of infimum there exists a minimizing sequence feasible pair \(\{(x^{\tilde{m}}, u^{\tilde{m}})\} \subset \mathcal{P}_{ad}\), where \(\mathcal{P}_{ad} = \{(x, u) : x\) is a mild solution of the system (6.1) corresponding to \(u \in \mathcal{A}_{ad}\}\) such that \(\mathcal{J}(x^{\tilde{m}}, u^{\tilde{m}}) \to \bar{\rho}\) as \(\tilde{m} \to +\infty\). Since \(\{u^{\tilde{m}}\} \subseteq \mathcal{A}_{ad}\), \(\tilde{m} = 1, 2, \ldots\) and is a bounded subset of the separable reflexive Banach space \(L_p(J, U)\), there exists a subsequence, relabeled as \(\{u^{\tilde{m}}\}\) and \(u^0 \in L_p(J, U)\) such that \(u^{\tilde{m}} \rightharpoonup u^0\) weakly in \(L_p(J, U)\). Since \(\mathcal{A}_{ad}\) is closed and convex, then by Marzur lemma \(u^0 \in \mathcal{A}_{ad}\). Let \(\{x^{\tilde{m}}\}\) denotes the sequence of solutions of the system (6.1)

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corresponding to \( \{u^\hat{n}\} \), \( x^0 \) is the mild solution of the system (6.1) corresponding to \( u^0 \). Then, \( x^\hat{n} \) and \( x^0 \) satisfy the following integral equations, respectively

\[
x^\hat{n}(t) = S(t)x_0 + \sum_{k=1}^{\hat{q}} S(t-t_k)I_k(x^\hat{n}(t_k)) + \int_0^t S(t-s)B(s)u^\hat{n}(s)ds \\
+ \int_0^t S(t-s)f\left(s, x^\hat{n}(s), x^\hat{n}(a_1(s)), \ldots, x^\hat{n}(a_m(s))\right)ds \\
+ \int_0^t S(t-s)\left(\int_0^s \sigma(\tau, x^\hat{n}(\tau), x^\hat{n}(b_1(\tau)), \ldots, x^\hat{n}(b_n(\tau)))d\tau\right)ds,
\]

and \( x^0(t) = S(t)x_0 + \sum_{k=1}^{\hat{q}} S(t-t_k)I_k(x^0(t_k)) + \int_0^t S(t-s)B(s)u^0(s)ds \\
+ \int_0^t S(t-s)f\left(s, x^0(s), x^0(a_1(s)), \ldots, x^0(a_m(s))\right)ds \\
+ \int_0^t S(t-s)\left(\int_0^s \sigma(\tau, x^0(\tau), x^0(b_1(\tau)), \ldots, x^0(b_n(\tau)))d\tau\right)ds.
\]

From the boundedness of \( \{u^\hat{n}\} \), \( \{u^0\} \) and Theorem 6.1, it follows that there exists a positive number \( \tilde{\delta} \) such that \( \mathbb{E}\|x^\hat{n}\|^2 \leq \tilde{\delta} \), \( \mathbb{E}\|x^0\|^2 \leq \tilde{\delta} \). For \( t \in J \), one can obtain

\[
\mathbb{E}\|x^\hat{n}(t) - x^0(t)\|^2 \leq 4\mathbb{E}\left|\sum_{k=1}^{\hat{q}} S(t-t_k)[I_k(x^\hat{n}(t_k)) - I_k(x^0(t_k))]\right|^2 \\
+ 4\mathbb{E}\left|\int_0^t S(t-s)[B(s)u^\hat{n}(s) - B(s)u^0(s)]ds\right|^2 \\
+ 4\mathbb{E}\left|\int_0^t S(t-s)f\left(s, x^\hat{n}(s), x^\hat{n}(a_1(s)), \ldots, x^\hat{n}(a_m(s))\right) \\
- f\left(s, x^0(s), x^0(a_1(s)), \ldots, x^0(a_m(s))\right)ds\right|^2 \\
+ 4\mathbb{E}\left|\int_0^t S(t-s)\left(\int_0^s \sigma(\tau, x^\hat{n}(\tau), x^\hat{n}(b_1(\tau)), \ldots, x^\hat{n}(b_n(\tau))) \\
- \sigma(\tau, x^0(\tau), x^0(b_1(\tau)), \ldots, x^0(b_n(\tau)))d\tau\right)ds\right|^2 \\
\leq 4\hat{M}^2\hat{q} \sum_{k=1}^{\hat{q}} \mathbb{E}\|I_k(x^\hat{n}(t_k)) - I_k(x^0(t_k))\|^2 \\
+ 4\hat{M}^2 \left[\left(\int_0^t \mathbb{E}\left\|B(s)u^\hat{n}(s) - B(s)u^0(s)\right\|^p ds\right)^{\frac{1}{p}} \left(\int_0^t ds\right)^{1-\frac{1}{p}}\right]^2 \\
+ 4\hat{M}^2 b \int_0^t \mathbb{E}\left|f\left(s, x^\hat{n}(s), x^\hat{n}(a_1(s)), \ldots, x^\hat{n}(a_m(s))\right) \\
- f\left(s, x^0(s), x^0(a_1(s)), \ldots, x^0(a_m(s))\right)\right|^2 ds
\]
\[ + 4 \hat{M}^2 b \text{Tr}(Q) \int_0^t \int_0^s \mathbb{E} \left\| \sigma \left( \tau, x_\ell^\alpha (\tau), x_\ell^0 (b_1(\tau)), \ldots, x_\ell^0 (b_n(\tau)) \right) \right\|^2 d\tau ds \]

\[ \leq 4 \hat{M}^2 q \sum_{k=1}^q M_k \mathbb{E} \| x_\ell^m - x_\ell^0 \|^2 + 4 \hat{M}^2 b^{2m-2} \mathbb{E} \| Bu_\ell^m - Bu_\ell^0 \|^2_{L_p(J,U)} \]

\[ + 4 \hat{M}^2 b \int_0^t M_i \mathbb{E} \| x_\ell^m - x_\ell^0 \|^2 ds + 4 \hat{M}^2 b^2 \text{Tr}(Q) \int_0^t M_p \mathbb{E} \| x_\ell^m - x_\ell^0 \|^2 ds \]

which implies that there exists a constant \( N^* > 0 \) such that

\[ \sup_{t \in J} \mathbb{E} \left\| x_\ell^m(t) - x_\ell^0(t) \right\|^2 \leq N^* \| Bu_\ell^m - Bu_\ell^0 \|^2_{L_p(J,U)} \text{ for } t \in J. \]

Since \( B \) is strongly continuous, one have \( \| Bu_\ell^m - Bu_\ell^0 \|^2_{L_p(J,U)} \overset{s}{\to} 0 \) as \( m \to \infty \). Then, \( \mathbb{E} \| x_\ell^m - x_\ell^0 \|^2 \overset{s}{\to} 0 \) as \( m \to \infty \), this yields that

\[ x_\ell^m \overset{s}{\to} x_\ell^0 \text{ in } PC(J, L_2(\Omega, H)) \text{ as } m \to \infty. \]

Note that (H6.7) implies the assumptions of Balder [18]. Hence, by Balder’s theorem, one can conclude that \( (x, u) \to \mathbb{E} \left( \int_0^b L(t, x(t), u(t)) dt \right) \) is sequentially lower semicontinuous in the weak topology of \( L_p(J, U) \subset L_1(J, U) \) and strong topology of \( L_1(J, H) \). Hence, \( J \) is weakly lower semicontinuous on \( L_p(J, U) \), and since by (H6.7)(iv), \( J > -\infty, J \) attains its infimum at \( u^0 \in A_{ad} \), that is,

\[ \tilde{\rho} = \lim_{m \to \infty} \mathbb{E} \left( \int_0^b L(t, x_\ell^m(t), u_\ell^m(t)) dt \right) \geq \mathbb{E} \left( \int_0^b L(t, x_\ell^0(t), u_\ell^0(t)) dt \right) = J(x_\ell^0, u^0) \geq \tilde{\rho}. \]

This completes the proof. \( \square \)

### 6.2.5 Example

Consider the following fractional stochastic integrodifferential equation to illustrate the obtained theory

\[ \frac{\partial^\alpha}{\partial t^\alpha} x(t, y) = \frac{\partial^2}{\partial y^2} x(t, y) + J^1_{-\alpha} \left[ \int_0^1 e(s) u(s, t) ds + \int_0^1 \mu_1(s, y) x(t \sin s, s) ds \right] \]

\[ + J^1_{-\alpha} \left[ \int_0^1 \sigma(s, x(s \sin s, y)) dw(s), t \in [0, 1], t \neq t_k, y \in [0, 1] \right] \]

(6.4)

\[ \Delta x(t_k, y) = I_k(x(t_k, y)), k = 1, 2, \ldots \hat{q}, \] (6.5)

\[ x(t, 0) = x(t, 1) = 0, t \in J, \] (6.6)

\[ x(0, y) = x_0(y), y \in [0, 1] \] (6.7)
where $0 < \alpha < 1$, let $H = U = L_2[0, 1]$, $w(t)$ is a standard cylindrical Wiener process in $H$ defined on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let the operator $A : D(A) \subseteq H \to H$ be defined by $A\zeta = \zeta''$ with the domain $D(A) = \{\zeta \in H : \zeta, \zeta'\text{ are absolutely continuous} \zeta'' \in H, \zeta(0) = \zeta(1) = 0\}$ then $A\zeta = \sum_{j=1}^{\infty} -j^2 \langle \zeta, e_j \rangle e_j$. Further, $A$ has discrete spectrum with eigenvalues of the form $-j^2$, $j \in \mathbb{N}$, and corresponding normalized eigenfunctions are given by $e_j(s) = \sqrt{\frac{2}{\pi}} \sin(js)$, $j = 1, 2, \cdots$. It is well known that, $A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on $H$, and is given by $T(t)\zeta = \sum_{j=1}^{\infty} e^{-j^2t} \langle \zeta, e_j \rangle e_j$ for all $\zeta \in H$, and every $t > 0$. It follows from these expressions that $\{T(t)\}_{t \geq 0}$ is uniformly bounded compact semigroup. Moreover, $\|T(t)\zeta\|^2 \leq \sum_{j=1}^{\infty} e^{-2j^2t} |\langle \zeta, e_j \rangle|^2 \leq \sum_{j=1}^{\infty} |\langle \zeta, e_j \rangle|^2 = \|\zeta\|^2$ (i.e) $\|T(t)\| \leq 1, \ t > 0$. 

(6.8)

It follows from the subordinate principle [43] that $A$ also generates a compact $\alpha$-order fractional analytic resolvent $S(t)$ of analyticity type $(w_0, \theta_0)$ for some $w_0$, $\theta_0$ and $S(t) = \int_0^{\infty} \Phi_\alpha(s)T(st^\alpha)ds$, $t > 0$, where

$$
\Phi_\alpha(s) = \sum_{j=0}^{\infty} \frac{(-s)^j}{j!\Gamma(-\alpha j + 1 - \alpha)}, \ 0 < \alpha < 1
$$

is the Wright function. It can be proved from (6.8) that there exists a constant $\hat{M} > 0$ such that $\|S(t)\| \leq \hat{M}$, $t \in [0, 1]$ (see [27]). Thus,

$$
\|S(t)\| \leq \int_0^{\infty} \|\Phi_\alpha(s)T(st^\alpha)\|ds \leq \int_0^{\infty} \Phi_\alpha(s)ds \leq 1.
$$

Take the functions $u : \Psi_x([0, 1]) \to \mathbb{R}$ as control, such that $u \in L_2(\Psi_x[0, 1])$, it means that $t \to u(t)$ is measurable. Set $A := \{u \in U : \|u\|_U \leq \kappa\}$, where $\kappa \in L_2(J, \mathbb{R}^+)$. One can restrict the admissible controls $A_{ad}$ to be all $u \in L_2(\Psi_x([0, 1]))$ such that $\|u(\cdot, t)\|_{L_2[0, 1]} \leq \kappa(t)$ a.e., $t \in J$.

In-order to represent the system (6.4)-(6.7) in the abstract form of system (6.1), one can consider

$$
x(t)(y) = x(t, y), \ t \in [0, 1], \ y \in [0, 1],
$$

$$
B(t)u(t)(y) = \int_0^1 e(y, s)u(s, t)ds,
$$

$$
f(t, \xi)(y) = \int_0^1 \mu_1(s, y)x(s)ds,
$$

$$
\sigma(t, \xi)(y) = \sigma(t, x(t \sin t, y)),
$$

$$
\phi(t)(y) = \phi_0(y), \ y \in [0, 1].
$$
Moreover, one should assume that

(i) The function $e : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous.

(ii) The function $\mu_1$ is measurable and $\int_0^1 \int_0^1 \mu_1(s, y)dsdy < \infty$.

(iii) The function $\frac{\partial}{\partial y}\mu_1(s, y)$ is measurable, $\mu_1(s, 0) = \mu_1(s, 1) = 0$ and let

$$\left(\int_0^1 \int_0^1 \left(\frac{\partial}{\partial y}\mu_1(s, y)\right)^2dsdy\right)^{\frac{1}{2}} < \infty.$$  

(iv) $w(t)$ denotes the one dimensional standard Brownian motion.

(v) For the function $\sigma : J \times \mathbb{R} \to \mathbb{R}$ the following conditions are hold

(a) for each $t \in J$, $\sigma(t, \cdot)$ is continuous,

(b) for each $x \in H$, $\sigma(\cdot, x)$ is measurable,

(c) there are positive functions $g_1$, $g_2 \in L^1(J)$ such that for each $(t, x) \in J \times H$,

$$\|\sigma(t, x)\| \leq g_1(t)\|x\| + g_2(t).$$

Now, consider the following cost function: $J(u) = \mathbb{E}\left\{\int_0^1 \mathcal{L}(t, x^u(t), u(t))dt\right\}$, with

$$\mathcal{L}(t, x^u(t), u(t))(y) := \int_0^1 \int_0^1 |x^u(t, y)|^2dydt + \int_0^1 \int_0^1 |u(t, y)|^2dydt. \text{ It is easy to see that all the hypotheses of Theorem 6.1 are satisfied, therefore the problem (6.4)-(6.7) has at least one optimal pair.}$

**6.3 The Solvability and Optimal Controls for FSDEs Driven by Poisson Jumps via Resolvent Operators**

**6.3.1 Problem Formulation**

In this subsection, consider the following form of FSDEs driven by Poisson jumps

$$d\left[J_t^{1-\alpha}(x(t) - x(0))\right] = [Ax(t) + B(t)u(t) + J_t^{1-\alpha} F(t, x(t))]dt$$

$$+ \int_\mathbb{Z} L(t, x(t-), z)\tilde{N}(dt, dz), \ t \in J = [0, b]$$

$$x(0) = x_0.$$
where \(0 < \alpha < 1\), \(x(\cdot)\) takes values in the separable Hilbert space \(H\), \(A: D(A) \subset H \to H\) is densely defined closed linear operator on \(H\). \(u\) is a given control function, it takes values from another separable reflexive Hilbert space \(U\), \(B\) is a linear operator from \(U\) into \(H\). Let \(q = q(t), t \in D_q\) is the stationary \(\mathcal{F}_t\)-adapted Poisson point process with a characteristic measure \(\lambda\). Let \(\tilde{N}(dt,dz)\) be the Poisson counting measure, associated with \(q\). Thus, \(N(t, Z) = \sum_{s \in D_\alpha, s \leq t} I_Z(q(s))\) with measurable set \(Z \in B(K - \{0\})\), which denotes the Borel \(\sigma\)-field of \(K - \{0\}\). Let \(\tilde{N}(dt,dz) = N(dt,dz) - dt\mu(dz)\) be the compensated Poisson measure that is independent of the Brownian motion. Let \(\beta([0, b] \times Z; H)\) be the space of all predictable mapping \(\chi: [0, b] \times Z \to H\) for which \(\int_0^b \int_Z E\|\chi(t, z)\|_H^p dt\mu(dz) < \infty\), then one can define the \(H\)-valued stochastic integral \(\int_0^b \int_Z \chi(t, z)\tilde{N}(dt,dz)\), which is centred \(p\)-integrable martingale. The functions \(F: J \times H \to H\) and \(L: J \times H \times (K - \{0\}) \to H\) are the appropriate functions.

### 6.3.2 Preliminaries and Hypotheses

Let \(C(J, L_p(\Omega, H))\) be the Banach space of all continuous maps from \(J\) into \(L_p(\Omega, H)\) satisfying

\[
\sup_{t \in J} \mathbb{E}\|x(t)\|^p < \infty.
\]

Consider the following FSDEs driven by Poisson jumps

\[
d\left[ J_t^{1-\alpha}(x(t) - x(0)) \right] = [Ax(t) + B(t)u(t) + J_t^{1-\alpha}F(t, x(t))]dt + \int_Z L(t, x(t^-), z)\tilde{N}(dt,dz)
\]

the above equation is equivalent to the following integral equation

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}Ax(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}B(s)u(s)ds + \int_0^t F(s, x(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z L(s, x(s^-), z)\tilde{N}(ds,dz) \right)
\]

this can be written in the following form

\[
x(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}Ax(s)ds, \quad t \geq 0, \quad (6.11)
\]

where

\[
h(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}B(s)u(s)ds + \int_0^t F(s, x(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z L(s, x(s^-), z)\tilde{N}(ds,dz) \right).
\]

Let us assume that the integral equation (6.11) has an associated resolvent operator \(\{S(t)\}_{t \geq 0}\) on \(H\).
Definition 6.5. [107] A one parameter family of bounded linear operators \{S(t)\}_{t \geq 0} on H is called a resolvent operator for (6.11), if the following conditions hold

(i) \(S(\cdot)x \in C([0, \infty), H)\) and \(S(0)x = x\) for all \(x \in H\),

(ii) \(S(t)D(A) \subset D(A)\) and \(A S(t)x = S(t)Ax\) for all \(x \in D(A)\) and every \(t \geq 0\),

(iii) for every \(x \in D(A)\) and \(t \geq 0\),
\[
S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS(s)x ds.
\]

Definition 6.6. [107] A resolvent operator \(\{S(t)\}_{t \geq 0}\) for (6.11) is called differentiable, if \(S(\cdot)x \in W^{1,1}_{loc}(\mathbb{R}^+, H)\) for all \(x \in [D(A)]\) and there exists \(\varphi_A \in L^1_{loc}(\mathbb{R}^+)\) such that \(\|S'(t)x\| \leq \varphi_A(t)\|x\|_{[D(A)]}\) for all \(x \in [D(A)]\), where the notation \([D(A)]\) stands the domain of the operator \(A\) provided with the graph norm \(\|x\|_{[D(A)]} = \|x\| + \|Ax\|\).

Definition 6.7. [107] A resolvent operator \(\{S(t)\}_{t \geq 0}\) for (6.11) is called analytic if the operator function \(S(\cdot) : (0, \infty) \to L(H)\) admits an analytic extension to a sector \(\sum = \{ \lambda \in \mathbb{C} : |\arg(\lambda)| < 0, \theta_0 \}\) for some \(0 < \theta_0 \leq \frac{\pi}{2}\).

Definition 6.8. A function \(x \in C(J, H)\) is called a mild solution of the integral equation (6.11) on \(J\), if \(\int_0^t (t-s)^{\alpha-1} x(s) ds \in [D(A)]\) for all \(t \in J\), \(h(t) \in C(J, H)\) and
\[
x(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} x(s) ds + h(t), \ \forall t \in J.
\]

Lemma 6.2. [19] Under the above conditions the following properties are valid

(i) if \(x(\cdot)\) is a mild solution of (6.11) on \(J\), then the function \(t \to \int_0^t S(t-s) h(s) ds\) is differentiable on \(J\) and
\[
x(t) = \frac{d}{dt} \int_0^t S(t-s) h(s) ds, \ \forall t \in J.
\]

(ii) if \(\{S(t)\}_{t \geq 0}\) is analytic and \(h \in C^\gamma(J, H)\) for some \(\gamma \in (0, 1)\), then the function defined by
\[
x(t) = S(t)(h(t) - h(0)) + \int_0^t S'(t-s)[h(s) - h(t)] ds + S(t)h(0), \ t \in J
\]
is a mild solution of (6.11) on \(J\).
(iii) if \( \{S(t)\}_{t \geq 0} \) is differentiable and \( h \in C(J, [D(A)]) \), then the function \( x : J \to H \) defined by
\[
x(t) = \int_0^t S'(t-s)h(s)ds + h(t), \ t \in J
\]
is a mild solution of (6.11) on \( J \).

Lemma 6.3. [7] For any \( p \geq 2 \) there exists \( c_p > 0 \) such that
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \int_Z H(\tau, z)\tilde{N}(d\tau, dz) \right|^p \leq c_p \left\{ \mathbb{E} \left[ (\int_0^s \int_Z ||H(\tau, z)||^2\mu(dz)d\tau)^{\frac{p}{2}} \right] \right\}.
\]

In-order to prove the existence result of the considered system, one need the following hypotheses

\( (H_{6.8}) \) The function \( F : J \times H \to H \) is continuous, and there exist the constants \( M_F > 0, \overline{M}_F > 0 \), for \( t \in J \) and \( x, y \in H \) such that
\[
\mathbb{E}||F(t, x)||^p \leq M_F(1 + \mathbb{E}||x||^p),
\]
\[
\mathbb{E}||F(t, x) - F(t, y)||^p \leq \overline{M}_F \mathbb{E}||x - y||^p.
\]

\( (H_{6.9}) \) The function \( L : J \times H \times (K - \{0\}) \to H \) satisfies the Lipschitz condition, and there exist constants \( M_L > 0, \overline{M}_L > 0 \) for \( t \in J \) and \( x, y \in H \) such that
\[
\int_Z \mathbb{E}||L(t, x(t-), z)||^p \mu(dz) \leq M_L \mathbb{E}||x||^p,
\]
\[
\int_Z \mathbb{E}||L(t, x(t-), z) - L(t, y(t-), z)||^p \mu(dz) \leq \overline{M}_L \mathbb{E}||x - y||^p.
\]

\( (H_{6.10}) \) Let \( u \in U \) be the control function and the operator \( B \in L_{\infty}(J, L(U, H)) \), \( \|B\| \) stands for the norm of the operator \( B \).

\( (H_{6.11}) \) The multivalued map \( A : J \to 2^U \setminus \{\emptyset\} \) has closed, convex and bounded values. \( A(\cdot) \) is graph measurable and \( A(\cdot) \subseteq \tilde{\Sigma} \), where \( \tilde{\Sigma} \) is a bounded set of \( U \).

### 6.3.3 Existence of Mild Solution of FSDEs Driven by Poisson Jumps

This subsection is devoted to find the existence result for the system (6.9)-(6.10), this problem is equivalent to the following integral equation
\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}Ax(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}B(s)u(s)ds
\]
\[
+ \int_0^t F(s, x(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z L(s, x(s-), z)\tilde{N}(ds, dz) \right).
\]

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By Lemma 6.2 and the above representation, the mild solution of (6.9)-(6.10) can be defined as follows.

**Definition 6.9.** An $H$-valued stochastic process $\{x(t)\}$ is said to be a mild solution of the problem (6.9)-(6.10) if

(i) $x(t)$ is measurable and $\mathcal{F}_t$ adapted for $t \geq 0$, 

(ii) $\int_0^t (t-s)\alpha^{-1}x(s)ds \in D(A)$ for all $t \in J$ and

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} A x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} B(s)u(s)ds \\
+ \int_0^t F(s,x(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} \left( \int_Z L(s,x(s-),z)\bar{N}(ds,dz) \right).
\]

(iii) Suppose that there exists a resolvent set $\{S(t)\}_{t\geq0}$, which is differentiable, and the functions $F$, $L$ are continuous, then by Lemma 6.2 (iii), if $x$ is a mild solution of (6.9)-(6.10), then

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} B(s)u(s)ds + \int_0^t F(s,x(s))ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} \left( \int_Z L(s,x(s-),z)\bar{N}(ds,dz) \right) \\
+ \int_0^t S'(t-s) \left[ x_0 + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)\alpha^{-1} B(\tau)u(\tau)d\tau + \int_0^s F(\tau,x(\tau))d\tau \\
+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)\alpha^{-1} \left( \int_Z L(\tau,x(\tau-),z)\bar{N}(d\tau,dz) \right) \right] ds.
\]

**Theorem 6.3.** Assume that the hypotheses $(H_{6.8}) - (H_{6.11})$ hold, then the problem (6.9)-(6.10) has a unique mild solution on $[0,b]$ provided that

\[
4^{p-1} \left\{ b^p M_F (1 + \|\varphi_A\|_{L^1(j)}^p b^p) + \frac{c_p}{\Gamma^p(\alpha)} \left( \frac{b^{op-p+1} M_L}{\alpha p - p + 1} + \frac{b^{op+2} M_L^2}{\alpha p - p + 1} \right) \right\} < 1. \tag{6.12}
\]

**Proof.** Consider the mapping $\Phi : C(J,L_F^p(\Omega,H)) \to C(J,L_F^p(\Omega,H))$ defined by

\[
(\Phi x)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} B(s)u(s)ds + \int_0^t F(s,x(s))ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)\alpha^{-1} \left( \int_Z L(s,x(s-),z)\bar{N}(ds,dz) \right)
\]
\begin{align*}
&\quad + \int_0^t S'(t-s) \left[ x_0 + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} B(\tau) u(\tau) \, d\tau + \int_0^s F(\tau, x(\tau)) \, d\tau \right] \, d\tau + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \left( \int_L L(\tau, x(\tau) - z) \, d\tau dz \right) \, d\tau \right] ds.

\text{Let } x \in C(J, L^p_p(\Omega, H)) \text{ from the hypotheses on } F(\cdot) \text{ and } L(\cdot), \text{ one can see that}

\begin{align*}
&\quad \mathbb{E} \left\| \int_0^t S'(t-s) \left[ x_0 + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} B(\tau) u(\tau) \, d\tau + \int_0^s F(\tau, x(\tau)) \, d\tau \right] \, d\tau + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \left( \int_L L(\tau, x(\tau) - z) \, d\tau dz \right) \, d\tau \right\|^p \\
&\quad \leq 4^{p-1} \mathbb{E} \left\| \int_0^t S'(t-s)x_0 ds \right\|^p \\
&\quad + 4^{p-1} \mathbb{E} \left\| \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} B(\tau) u(\tau) \, d\tau ds \right\|^p \\
&\quad + 4^{p-1} \mathbb{E} \left\| \int_0^t S'(t-s) \int_0^s F(\tau, x(\tau)) \, d\tau ds \right\|^p \\
&\quad + 4^{p-1} \mathbb{E} \left\| \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \left( \int_L L(\tau, x(\tau) - z) \, d\tau dz \right) \, d\tau \right\|^p \\
&\quad \leq 4^{p-1} \sum_{i=1}^4 I_{6,i}.
\end{align*}

Now,

\begin{align*}
I_{6.1} &= \mathbb{E} \left\| \int_0^t S'(t-s)x_0 ds \right\|^p \\
&\leq \| \varphi_A \|_{L^1(J)}^p b^{p-1} \int_0^t \mathbb{E} \| x_0 \|^p \, ds
\end{align*}

\begin{align*}
I_{6.2} &= \mathbb{E} \left\| \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} B(\tau) u(\tau) \, d\tau ds \right\|^p \\
&\leq \frac{\| \varphi_A \|_{L^1(J)}^p b^{p-1} \| B \|^p}{\Gamma(p)} \int_0^t \mathbb{E} \left\| \int_0^s (s-\tau)^{\alpha-1} u(\tau) \, d\tau \right\|^p \, ds
\end{align*}

\begin{align*}
I_{6.3} &= \mathbb{E} \left\| \int_0^t S'(t-s) \int_0^s F(\tau, x(\tau)) \, d\tau ds \right\|^p \\
&\leq \| \varphi_A \|_{L^1(J)}^p b^{p-1} \int_0^t \mathbb{E} \left\| \int_0^s F(\tau, x(\tau)) \, d\tau \right\|^p \, ds.
\end{align*}
\[ I_{6.4} = \mathbb{E} \left[ \left\| \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \left( \int_Z L(\tau, x(\tau-), z) \tilde{N}(d\tau, dz) \right) d\tau \right\|^p \right] \]

\[ \leq \frac{\| \varphi A \|^p}{\Gamma^p(\alpha)} \left( b^{p+1} M_L + \frac{b^{p+2} M_{\tilde{F}}^2}{\alpha p + 1} \right) \mathbb{E}\|x\|^p \]

It follows from the above estimations \( I_{6.1} - I_{6.4} \) and from the fact that \( u(\cdot), F(\cdot, \cdot) \) and \( L(\cdot, \cdot, \cdot) \) are continuous, \( \Phi \) is well defined. Moreover, for \( x, y \in C(J, L^p(\Omega, H)) \) and \( t \in [0, b] \), one get

\[ \mathbb{E}\| (\Phi x)(t) - (\Phi y)(t) \|^p \]

\[ \leq 4^{p-1} \mathbb{E} \left[ \left\| \int_0^t [F(s, x(s)) - F(s, y(s))] ds \right\|^p \right] \]

\[ + 4^{p-1} \mathbb{E} \left[ \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_Z [L(s, x(s-), z) - L(s, y(s-), z)] \tilde{N}(d\tau, dz) \right\|^p \right] \]

\[ + 4^{p-1} \mathbb{E} \left[ \left\| \int_0^t S'(t-s) \int_0^s [F(\tau, x(\tau)) - F(\tau, y(\tau))] d\tau ds \right\|^p \right] \]

\[ + 4^{p-1} \mathbb{E} \left[ \left\| \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \int_Z [L(\tau, x(\tau-), z) - L(\tau, y(\tau-), z)] \tilde{N}(d\tau, dz) \right\|^p \right] \]

\[ \leq 4^{p-1} \sum_{j=1}^4 J_{6,j}. \]

Now,

\[ J_{6.1} = \mathbb{E} \left[ \left\| \int_0^t [F(s, x(s)) - F(s, y(s))] ds \right\|^p \leq b^{p-1} \int_0^t \mathbb{E}\|F(s, x(s)) - F(s, y(s))\|^p ds \]

\[ \leq b^p M_F \mathbb{E}\|x - y\|^p, \]

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\[ J_{6.2} = \mathbb{E} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z [L(s, x(s-), z) - L(s, y(s-), z)] \tilde{N}(ds, dz) \right) \right\|^p \]
\[ \leq \frac{c_p}{\Gamma^p(\alpha)} \left\{ \int_0^t (t-s)^{p(\alpha-1)} \int_Z \mathbb{E}\|L(s, x(s-), z) - L(s, y(s-), z)\|^p \mu(dz) ds \right. \]
\[ + \left( \int_0^t (t-s)^{2(\alpha-1)} \int_Z \mathbb{E}\|L(s, x(s-), z) - L(s, y(s-), z)\|^2 \mu(dz) ds \right)^{\frac{p}{2}} \} \]
\[ \leq \frac{c_p}{\Gamma^p(\alpha)} \left\{ \int_0^t (t-s)^{p(\alpha-1)} \tilde{M}_L \mathbb{E}\|x - y\|^p ds + \left( \int_0^t (t-s)^{2(\alpha-1)} \tilde{M}_L \mathbb{E}\|x - y\|^2 ds \right)^{\frac{p}{2}} \} \]
\[ \leq \frac{c_p}{\Gamma^p(\alpha)} \left( \frac{b \alpha^p - \frac{\alpha + 2}{\alpha + 1} M_L}{\alpha p - p + 1} + \frac{b \alpha^p - \frac{\alpha + 2}{\alpha + 1} M_L}{\alpha p - p + 1} \right) \mathbb{E}\|x - y\|^p, \]

\[ J_{6.3} = \mathbb{E} \left\| \int_0^t S'(t-s) \int_0^s \left[ F(\tau, x(\tau)) - F(\tau, y(\tau)) \right] d\tau ds \right\|^p \]
\[ \leq \| \varphi_A \|^p_{L^1(\alpha)} b^{p-1} \int_0^t \mathbb{E} \left\| \int_0^s \left[ F(\tau, x(\tau)) - F(\tau, y(\tau)) \right] d\tau \right\|^p ds \]
\[ \leq \| \varphi_A \|^p_{L^1(\alpha)} b^{p-2} \int_0^t \int_0^s \mathbb{E}\|F(\tau, x(\tau)) - F(\tau, y(\tau))\|^p d\tau ds \]
\[ \leq \| \varphi_A \|^p_{L^1(\alpha)} b^{p-2} \tilde{M}_F \mathbb{E}\|x - y\|^p, \]

\[ J_{6.4} = \mathbb{E} \left\| \int_0^t S'(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \left( \int_Z [L(\tau, x(\tau-), z) - L(\tau, y(\tau-), z)] \tilde{N}(d\tau, dz) \right) d\tau \right\|^p \]
\[ \leq \frac{\| \varphi_A \|^p_{L^1(\alpha)} b^{p-1}}{\Gamma^p(\alpha)} \int_0^t \left\{ \int_0^s (s-\tau)^{p(\alpha-1)} \int_Z \mathbb{E}\|L(\tau, x(\tau-), z) - L(\tau, y(\tau-), z)\|^p \mu(dz) d\tau \right. \]
\[ + \left( \int_0^s (s-\tau)^{2(\alpha-1)} \int_Z \mathbb{E}\|L(\tau, x(\tau-), z) - L(\tau, y(\tau-), z)\|^2 \mu(dz) d\tau \right)^{\frac{p}{2}} \} ds \]
\[ \leq \frac{\| \varphi_A \|^p_{L^1(\alpha)} b^{p-1} c_p}{\Gamma^p(\alpha)} \int_0^t \left\{ \int_0^s (s-\tau)^{p(\alpha-1)} \tilde{M}_L \mathbb{E}\|x - y\|^p d\tau \right. \]
\[ + \left( \int_0^s (s-\tau)^{2(\alpha-1)} \tilde{M}_L \mathbb{E}\|x - y\|^2 d\tau \right)^{\frac{p}{2}} \} ds \]
\[ \leq \frac{\| \varphi_A \|^p_{L^1(\alpha)} c_p}{\Gamma^p(\alpha)} \left( \frac{b \alpha^p + \frac{\alpha + 2}{\alpha + 1} M_L}{\alpha p - p + 1} + \frac{b \alpha^p + \frac{\alpha + 2}{\alpha + 1} M_L}{\alpha p - p + 1} \right) \mathbb{E}\|x - y\|^p, \]

by combining \( J_{6.1} - J_{6.4} \), one have

\[ \mathbb{E}\|\Phi(x)(t) - (\Phi y)(t)\|^p \leq 4^{p-1} \left\{ b \tilde{M}_F (1 + \| \varphi_A \|^p_{L^1(\alpha)} b^p) + \frac{c_p}{\Gamma^p(\alpha)} \left( \frac{b \alpha^p + \frac{\alpha + 2}{\alpha + 1} M_L}{\alpha p - p + 1} + \frac{b \alpha^p + \frac{\alpha + 2}{\alpha + 1} M_L}{\alpha p - p + 1} \right) \right\} \mathbb{E}\|x - y\|^p. \]

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By assumption (6.12), it follows that $\Phi$ is contraction and there exists a unique fixed point $x(\cdot)$ of $\Phi$. Finally from Lemma 6.2 (iii), it can be concluded that $x(\cdot)$ is a mild solution of (6.9)-(6.10).

6.3.4 Existence of Optimal Controls of FSDEs Driven by Poisson Jumps

Consider the following Lagrange problem $(P)$:

Find a control $u^0 \in A_{ad}$ such that

$$J(u^0) \leq J(u), \quad \forall u \in A_{ad}$$

where

$$J(u) = \mathbb{E} \left\{ \int_0^b \mathcal{L}(t, x^u(t), u(t)) dt \right\}$$

and $x^u$ denotes the mild solution of the system (6.9)-(6.10) corresponding to the control $u \in A_{ad}$.

Assume the following hypothesis for existence of solutions of the problem $(P)$.

$$(H_{6.12}) \quad (i) \text{ The functional } \mathcal{L} : J \times H \times U \to \mathbb{R} \cup \{\infty\} \text{ is } \mathcal{F}_t\text{- measurable,}$$

$$(ii) \mathcal{L}(t, \cdot, \cdot) \text{ is sequentially lower semicontinuous on } H \times U \text{ for almost all } t \in J,$$

$$(iii) \mathcal{L}(t, x, \cdot) \text{ is convex on } U \text{ for each } x \in H \text{ and almost all } t \in J,$$

$$(iv) \text{there exist constants } d \geq 0, \ e > 0, \ \mu \text{ is nonnegative and } \mu \in L^1(J, \mathbb{R}) \text{ such that }$$

$$\mathcal{L}(t, x, u) \geq \mu(t) + d\mathbb{E}\|x\|_H^p + e\|u\|_U^p.$$  

Theorem 6.4. Let the hypotheses $(H_{6.8}) - (H_{6.12})$ and the inequality (6.12) hold. Suppose that $B$ is a strongly continuous operator, then the Lagrange problem $(P)$ admits at least one optimal pair, that is, there exists an admissible control $u^0 \in A_{ad}$ such that

$$J(u^0) = \mathbb{E} \left( \int_0^b \mathcal{L}(t, x^0(t), u^0(t)) dt \right) \leq J(u), \quad \forall u \in A_{ad}.$$

Proof. If $\inf \{ J(u) | u \in A_{ad} \} = +\infty$, there is nothing to prove. Without loss of generality, assume that $\inf \{ J(u) | u \in A_{ad} \} = \epsilon < +\infty$. Using $(H_{6.12})$, one have $\epsilon > -\infty$. By the definition of infimum, there exists a minimizing sequence feasible pair $\{(x^m, u^m)\} \subset P_{ad}$, where $P_{ad} = \{(x, u) : x \text{ is a mild solution of the system (6.9)-(6.10) corresponding to } u \in A_{ad}\}$ such that

$$J(x^m, u^m) \to \epsilon \text{ as } m \to +\infty.$$
Since \( \{u^m\} \subseteq A_{ad}, \ m = 1, 2, \cdots, \{u^m\} \) is a bounded subset of the separable reflexive Banach space \( L_p(J, U) \), there exists a subsequence, relabeled as \( \{u^m\} \) and \( u^0 \in L_p(J, U) \) such that \( u^m \xrightarrow{w} u^0 \) in \( L_p(J, U) \).

Since \( A_{ad} \) is closed and convex, then by Marzur lemma \( u^0 \in A_{ad} \). Let \( \{x^m\} \subset C(J, L^\infty_p(\Omega, H)) \) denote the corresponding sequence of solutions of the integral equation

\[
x^m(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B(s)u^m(s)ds + \int_0^t F(s, x^m(s))ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z L(s, x^m(s-z), z)\tilde{N}(ds, dz) \right) \\
+ \int_0^t \mathcal{S}'(t-s) \left[ x_0 + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} B(\tau)u^m(\tau)d\tau + \int_0^s F(\tau, x^m(\tau))d\tau \\
+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \left( \int_Z L(\tau, x^m(\tau-z), z)\tilde{N}(d\tau, dz) \right) \right] ds.
\]

It follows from the boundedness of \( \{u^m\} \) and Theorem 6.3, one can easily check that, there exists a \( \bar{\rho} > 0 \) such that

\[
\mathbb{E}\|x^m\|^p \leq \bar{\rho}, \ m = 0, 1, \cdots.
\]

For \( t \in J \),

\[
\mathbb{E}\|x^m(t) - x^0(t)\|^p \\
\leq 6^{p-1} \left\{ \mathbb{E}\left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [B(s)u^m(s) - B(s)u^0(s)]ds \right\|^p \\
+ \mathbb{E}\left\| \int_0^t [F(s, x^m(s)) - F(s, x^0(s))]ds \right\|^p \\
+ \mathbb{E}\left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z L(s, x^m(s-z), z) - L(s, x^0(s-z), z) \right)\tilde{N}(ds, dz) \right\|^p \\
+ \mathbb{E}\left\| \int_0^t \mathcal{S}'(t-s) \left[ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} [B(\tau)u^m(\tau) - B(\tau)u^0(\tau)]d\tau \right] ds \right\|^p \\
+ \mathbb{E}\left\| \int_0^t \mathcal{S}'(t-s) \left[ \int_0^s [F(\tau, x^m(\tau)) - F(\tau, x^0(\tau))]d\tau \right] ds \right\|^p \\
+ \mathbb{E}\left\| \int_0^t \mathcal{S}'(t-s) \left[ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \right. \\
\times \left. \left( \int_Z L(\tau, x^m(\tau-z), z) - L(\tau, x^0(\tau-z), z) \right)\tilde{N}(d\tau, dz) \right] ds \right\|^p \right\} \\
\leq 6^{p-1} \sum_{i=5}^{10} I_{6,i}.
\]
Now,

\[
I_{6.5} = \mathbb{E} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [B(s)u^m(s) - B(s)u^0(s)] ds \right\|^p \\
\leq \frac{1}{\Gamma^p(\alpha)} \left( \frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} b^{\frac{\alpha p - 1}{p}} \left( \int_0^t \mathbb{E} \left\| B(s)u^m(s) - B(s)u^0(s) \right\|^p ds \right)^{\frac{1}{p}} \\
\leq \frac{1}{\Gamma^p(\alpha)} \left( \frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} b^{\alpha p - 1} \left\| Bu^m - Bu^0 \right\|_{L^p(J,U)}^p,
\]

\[
I_{6.6} = \mathbb{E} \left\| \int_0^t [F(s,x^m(s)) - F(s,x^0(s))] ds \right\|^p \\
\leq b^{p-1} \int_0^t \mathbb{E} \left\| F(s,x^m(s)) - F(s,x^0(s)) \right\|^p ds \\
\leq b^{p-1} M_F \int_0^t \mathbb{E} \|x^m(s) - x^0(s)\|^p ds,
\]

\[
I_{6.7} = \mathbb{E} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_Z [L(s,x^m(s-),z) - L(s,x^0(s-),z)] d\bar{N}(ds,dz) \right) \right\|^p \\
\leq \frac{c_p}{\Gamma^p(\alpha)} \left\{ \int_0^t (t-s)^{p(\alpha-1)} \mathbb{E} \left\| L(s,x^m(s-),z) - L(s,x^0(s-),z) \right\|^p d\mu(dz) ds \\
+ \left( \int_0^t (t-s)^{2(\alpha-1)} \mathbb{E} \left\| L(s,x^m(s-),z) - L(s,x^0(s-),z) \right\|^2 d\mu(dz) ds \right)^{\frac{p}{2}} \right\} \\
\leq \frac{c_p}{\Gamma^p(\alpha)} \left\{ \int_0^t (t-s)^{p(\alpha-1)} \tilde{M}_F \mathbb{E} \|x^m(s) - x^0(s)\|^p ds \\
+ \tilde{M}_F b^{\frac{p}{2} - 1} \int_0^t (t-s)^{p(\alpha-1)} \mathbb{E} \|x^m(s) - x^0(s)\|^p ds \right\},
\]

\[
I_{6.8} = \mathbb{E} \left\| \int_0^t S'(t-s) \left[ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} [B(\tau)u^m(\tau) - B(\tau)u^0(\tau)] d\tau \right] ds \right\|^p \\
\leq b^{p-1} \|\varphi_A\|_{L^1(J)}^p \int_0^t \mathbb{E} \left\| \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} [B(\tau)u^m(\tau) - B(\tau)u^0(\tau)] d\tau \right\|^p ds \\
\leq \|\varphi_A\|_{L^1(J)}^p \frac{p-1}{\alpha p - 1} b^{\alpha p + p - 1} \left\| Bu^m - Bu^0 \right\|_{L^p(J,U)}^p,
\]

\[
I_{6.9} = \mathbb{E} \left\| \int_0^t S'(t-s) \left[ \int_0^s [F(\tau,x^m(\tau)) - F(\tau,x^0(\tau))] d\tau \right] ds \right\|^p \\
\leq \|\varphi_A\|_{L^1(J)}^p b^{p-1-1} \int_0^t \mathbb{E} \left\| \int_0^s [F(\tau,x^m(\tau)) - F(\tau,x^0(\tau))] d\tau \right\|^p ds \\
\leq \|\varphi_A\|_{L^1(J)}^p b^{2p-2} \int_0^t \mathbb{E} \left\| F(\tau,x^m(\tau)) - F(\tau,x^0(\tau)) \right\|^p d\tau ds \\
\leq \|\varphi_A\|_{L^1(J)}^p b^{2p-2} M_F \int_0^t \mathbb{E} \|x^m(\tau) - x^0(\tau)\|^p d\tau ds,
\]

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Thus by combining $I_{6.10}$, one have

$$
\mathbb{E} \left\| x^m(t) - x^0(t) \right\|^p \leq 6^{p-1} \left\{ \frac{b_0^{p-1}}{\Gamma(p)} \left( \frac{p - 1}{\alpha p - 1} \right)^{p-1} \left\| Bu^m - Bu^0 \right\|_{L_p(J,V)}^p + b^{p-1} \bar{M}_F \int_0^t \mathbb{E} \left\| x^m(s) - x^0(s) \right\|^p ds \\
+ \frac{c_p}{\Gamma(p)} \left( \int_0^t (t - s)^{p(\alpha - 1)} \bar{M}_L \mathbb{E} \left\| x^m(s) - x^0(s) \right\|^p ds \right) + \bar{M}_L^2 b^{p-1} \int_0^t (t - s) \mathbb{E} \left\| x^m(s) - x^0(s) \right\|^p ds \right) + \left\{ \frac{\| \varphi_A \|^p_{L_1(J)}}{\Gamma(p)} \left( \frac{p - 1}{\alpha p - 1} \right)^{p-1} b^{p+p-1} \left\| Bu^m - Bu^0 \right\|_{L_p(J,V)}^p \\
+ \left\{ \frac{\| \varphi_A \|^p_{L_1(J)}}{\Gamma(p)} b^{2p-2} \bar{M}_F \int_0^t \left( \int_0^s \mathbb{E} \left\| x^m(\tau) - x^0(\tau) \right\|^p d\tau \right) ds \right) \\
+ \bar{M}_L \int_0^t \left( \int_0^s (s - \tau)^{p(\alpha - 1)} \mathbb{E} \left\| x^m(\tau) - x^0(\tau) \right\|^p d\tau \right) ds \right) \right\}
$$

which implies

$$
\sup_{t \in J} \mathbb{E} \left\| x^m(t) - x^0(t) \right\|^p \leq 6^{p-1} \left\{ \frac{b_0^{p-1}}{\Gamma(p)} \left( \frac{p - 1}{\alpha p - 1} \right)^{p-1} \left\| Bu^m - Bu^0 \right\|_{L_p(J,V)}^p + b^p \bar{M}_F \sup_{t \in J} \mathbb{E} \left\| x^m(t) - x^0(t) \right\|^p
$$

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which implies that there exists a constant $N^{**} > 0$ such that

$$
\sup_{t \in J} \|x^m(t) - x^0(t)\|^p \leq N^{**} \|Bu^m - Bu^0\|^p_{L^p(J, U)} \quad \text{for } t \in J
$$

where

$$
N^{**} = \frac{6^{p-1}(1 + b^p\|\varphi_A\|^p_{L^1(J)})^{\frac{b^{p-1}}{\Gamma^p(\alpha)}} \left( \frac{p-1}{\alpha p-1} \right)^{p-1}}{1 - 6^{p-1}(1 + b^p\|\varphi_A\|^p_{L^1(J)}) \left\{ b^pM_F + c_p\tilde{M}_L (\frac{b^{p-1}}{\alpha p-1}) + c_p\tilde{M}_L^\frac{b^p}{\Gamma^p(\alpha)} \left( \frac{b^{p-1}}{\alpha p-1} \right) \right\}}
$$

and

$$
6^{p-1}(1 + b^p\|\varphi_A\|^p_{L^1(J)}) \left\{ b^pM_F + c_p\tilde{M}_L (\frac{b^{p-1}}{\alpha p-1}) + c_p\tilde{M}_L^\frac{b^p}{\Gamma^p(\alpha)} \left( \frac{b^{p-1}}{\alpha p-1} \right) \right\} < 1.
$$
Since, $B$ is strongly continuous, one have $\|Bu^m - Bu^0\|_{L^p(J,U)}^p \xrightarrow{s \to 0} 0$ as $m \to \infty$, thus, $E\|x^m - x^0\|^p \xrightarrow{s \to 0} 0$ as $m \to \infty$, this yields that $x^m \xrightarrow{s \to 0} x^0$ in $C(J, L^p_\mu(\Omega, H))$ as $m \to \infty$. Note that $(H_{6.12})$ implies the assumptions of Balder [18] are satisfied. Hence by Balder’s theorem, one can conclude that $(x, u) \xrightarrow{s \to 0} E\left(\int_0^b \mathcal{L}(t, x(t), u(t))dt\right)$ is sequentially lower semicontinuous in the strong topology of $L_1(J, H)$. Since $L_p(J, U) \subset L_1(J, U)$, $\mathcal{J}$ is weakly lower semicontinuous on $L_p(J, U)$, and since by $(H_{6.12})(iv)$, $\mathcal{J} > -\infty$, $\mathcal{J}$ attains its infimum at $u^0 \in \mathcal{A}_{ad}$, that is,

$$\epsilon = \lim_{m \to \infty} E\left(\int_0^b \mathcal{L}(t, x^m(t), u^m(t))dt\right) \geq E\left(\int_0^b \mathcal{L}(t, x^0(t), u^0(t))dt\right) = \mathcal{J}(x^0, u^0) \geq \epsilon.$$

\[\square\]

### 6.3.5 Example

Consider the following FSDE driven by Poisson jumps to illustrate the obtained theory

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dt} [J^{1-\alpha} x(t, v) - x_0] = \left[ \frac{\partial^2}{\partial v^2} x(t, v) + \int_0^1 e(v, s)u(s, t)ds + \frac{1}{\Gamma(1-\alpha)} \int_0^s (t-s)^{-\alpha} e^{-t} \left( e^{-s} \frac{1}{(t+10)} x(t, v) \right) ds \right] dt \\
x(t, 0) = x_0,
\end{array} \right.
\tag{6.13}
\end{align*}
\]

where $0 < \alpha < 1$, let $H = U = L_2[0, 1]$, $w(t)$ is a standard cylindrical Wiener process in $H$, $q(t)$ is the Poisson point process (independent of $w(t)$) with a $\sigma$-finite intensity measure $\mu(dz)$ defined on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and the function $e : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Denoting $\tilde{N}(dt, dz)$ as the Poisson counting measure associated with $q$ and the compensating martingale measure by $\tilde{N}(ds, dz) = N(ds, dz) - \mu(dz)ds$.

To rewrite the system (6.13) into the abstract form of the system (6.9)-(6.10). Let $A : D(A) \subset H \to H$ be defined by $A\zeta = \zeta''$ with the domain $D(A) = \{\zeta \in H : \zeta, \zeta' \text{ are absolutely continuous} \} \subset H$, $\zeta(0) = \zeta(1) = 0$ then,

$$A\zeta = \sum_{n=1}^{\infty} -n^2 \langle \zeta, e_n \rangle e_n.$$

Further, $A$ has discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenfunction is given by $e_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns), n = 1, 2, \cdots$. It is well known that, $A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on $H$, and is given by

$$T(t)\zeta = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \zeta, e_n \rangle e_n.$$
for all $\zeta \in H$, and every $t > 0$. It follows from these expressions that $\{T(t)\}_{t \geq 0}$ is uniformly bounded compact semigroup, thus $R(\hat{\lambda}, A) = (\hat{\lambda} - A)^{-1}$ is compact operator for all $\hat{\lambda} \in \rho(A)$.

From [107], one have that the integral equation

$$x(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)ds, \ s \geq 0$$

has an associated analytic resolvent operator $\{S(t)\}_{t \geq 0}$ on $H$, and is given by

$$S(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\hat{\theta}}} e^{\hat{\lambda} t} (\hat{\lambda}^\alpha - A)^{-1} d\hat{\lambda}, \ t > 0 \\ I, \ t = 0 \end{cases}$$

where $\Gamma_{r,\hat{\theta}}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\hat{\theta} \in (\frac{\pi}{2}, \pi)$. Since $S(t)$ is differentiable [9], and there exists a constant $\hat{M} > 0$ such that $\|S(t)x\| \leq \hat{M}\|x\|$ for $x \in D(A), \ t > 0$.

To represent the system (6.13) in the abstract form of the system (6.9)-(6.10), consider the functions

$$x(t)(v) = x(t, v), \ t \in [0, 1], \ v \in [0, 1],$$

$$F(x)(v) = e^{-t} + \frac{1}{(t + 10)} x(t, v),$$

$$L(x)(v) = (\cos t) x(t, v),$$

$$B(t)u(t)(v) = \int_0^1 e(v, s) u(s, t) ds$$

with the following cost functional

$$J(u) = E \left\{ \int_0^b \mathcal{L}(t, x^u(t), u(t)) dt \right\},$$

where

$$\mathcal{L}(t, x^u(t), u(t))(v) = \int_0^1 E\|x^u(t, v)\|^2 dv + \int_0^1 \|u(v, t)\|^2 dv$$

from above choices of nonlinear functions, it is not difficult to verify that all the hypotheses of Theorem 6.4 are satisfied, therefore the problem (6.13) has at least one optimal pair.

### 6.4 Conclusions and Future Directions

This chapter successfully addressed the question of existence of stochastic optimal control pairs for different class of fractional stochastic dynamical systems. The solvability and optimal controls of
impulsive fractional stochastic integrodifferential equations and FSDEs driven by Poisson jumps have been investigated in Hilbert space via resolvent operators and sufficient conditions have deduced for the existences of mild solutions respectively by using Leray-Schauder fixed point theorem and Banach contraction mapping principle. The existence of optimal controls of the corresponding Lagrange optimal control problems have been investigated. Finally, the derived theoretical results have been verified by respective examples.

There is increasing interests on investigations of fractional optimal controls for some other classes of FDEs with stochastic settings, such as fractional stochastic variational inequality, coupled FSDEs, FSDEs with multiple orders etc. In future, there is a crucial need to the natural extension of fractional optimal control problems for FSDEs and FSDIs with higher orders.