Chapter 3

Existence Result for FSDEs by Using Mainardi’s Function

3.1 Introduction

A class of differential equations, in which the delayed argument occurs in the derivative of the state variable as well as in the independent variable, is called neutral differential equations. The theory of neutral differential equations received great importance in both the theory and applications point of view. The neutral delay differential equations are used in the modelling of electrical networks, which contain lossless transmission lines. For instance, such networks can be found in high speed computers, where lossless transmission lines are used to interconnect switching circuits.

Some real life phenomena arising from the fields such as electronics, fluid dynamics, biology and chemical kinetics need description of phenomena through neutral integrodifferential equations. For example, the equation of rigid heat conduction with finite wave speeds, which was studied by Miller [97], can be well modelled in the form of an integrodifferential equation of neutral type with delay. Further, successful modelling of real life phenomena demands assurance of the existence of solution. In this regard, there has been considerable investigation on the existence results of various kinds of neutral differential equations of integer order (see [57]).

Hernandez and Henriquez [60] established existence results for a class of quasi-linear neutral functional differential equations with unbounded delay in Banach spaces by using semigroup theory
and Sadovski fixed point theorem. Balachandran and Sakthivel [11] proved the existence result for some class of neutral functional integrodifferential equations in Banach spaces with the aid of Schaefer fixed point theorem. Dauer and Balachandran [33] studied the existence of mild solutions for a nonlinear neutral integrodifferential equation in Banach space by using Schaefer fixed point theorem.

Fractional neutral differential equations received a little attention in the literature due to its various applications. Besides, there has been an increasing interest in the study of neutral stochastic differential equations that can be described with fractional order. Dabas and Chauhan [31] studied the existence and uniqueness result for an impulsive neutral fractional integrodifferential equation with infinite delay, by taking into account the \((1 - \alpha)\) order Riemann-Liouville fractional integral operator \(J^{1-\alpha}_t\). In general, the Riemann-Liouville’s fractional derivative is singular at the lower limit. Caputo overcomes this difficulty by subtraction the singular term and provide his definition for fractional derivative. Thus, the Caputo fractional derivative can be written in term of Riemann-Liouville’s fractional derivative. By using this concept, Li and Peng [78] considered a class of fractional neutral stochastic functional differential systems and studied its controllability result.

The notable property, which understood from the literature that the fundamental solution of the Cauchy problem for the standard linear diffusion equation gives a Gaussian probability density function. Schneider [116] achieved a general representation of all stable distributions in terms of special functions. The fundamental solutions of the fractional diffusion equations have been shown to provide probability density functions evolving in time or varying in space, which are related to the special class of stable distributions [93]. In this connection, Mainardi et al. [93] studied probability distributions generated by fractional diffusion equations, and derived the corresponding fundamental solution in terms of a special function of the Wright type (so called Mainardi function), also provided its basic characteristics. By taking the above Mainardi’s work as base, in the case of deterministic FDEs, Li et al. [81] established the existence results for nonlocal semilinear FDEs by using convex-power condensing operator, Mainardi’s function and fixed point theorem.

Inspired by the above discussions, this chapter is devoted to propose the new kind of mild solution for the considered fractional neutral stochastic integrodifferential equations and establish the existence result by using Mainardi’s function, fractional calculus and fixed point theorem technique.
3.2 Existence Result for Fractional Neutral Stochastic Integrodifferential Equations with Infinite Delay by using Mainardi’s Function

3.2.1 Problem Formulation

Consider the fractional neutral stochastic integrodifferential equations with infinite delay described by

\[
\frac{d}{dt} \left[ J_1^{1-\alpha} \left( x(t) - g(t, x_t) - \phi(0) + g(0, \phi) \right) \right] = A[x(t) - g(t, x_t)] + J_1^{1-\alpha} f(t, x_t) \\
+ \int_0^t \sigma(s, x_s)dw(s), \ t \in J = [0, b] \tag{3.1}
\]

\[
x(t) = \phi \in \mathcal{B}, \ t \in (-\infty, 0], \tag{3.2}
\]

where \( 0 < \alpha \leq 1 \), \( J_1^{1-\alpha} \) is the \((1 - \alpha)\) order Riemann-Liouville fractional integral operator, \( A : D(A) \subset H \to H \) is the infinitesimal generator of strongly continuous semigroup of bounded linear operators \( \{T(t), t \geq 0\} \) on a separable Hilbert space \( H \). The history \( x_t : (-\infty, 0] \to H, \ x_t(\theta) = x(t+\theta) \) for \( t \geq 0 \) belongs to some abstract space \( \mathcal{B} \). The functions \( f, g : J \times \mathcal{B} \to H \) are measurable in \( H \)-norm and \( \sigma : J \times \mathcal{B} \to L_2^0 \) is a measurable mapping, where \( L_2^0 = L_2(Q^{1/2}K, H) \) be the space of all Hilbert-Schmidt operators for a separable Hilbert space from \( Q^{1/2}K \) into \( H \) with norm \( \|\phi\|^p_{L_2^0} = Tr(\phi Q \phi^*) \).

3.2.2 Preliminaries and Hypotheses

The axiomatic definition of the phase space \( \mathcal{B} \) was introduced by Hale and Kato [56]. The axioms of the space \( \mathcal{B} \) are established for \( \mathcal{F}_0 \) measurable functions from \(( -\infty, 0] \) into \( H \) endowed with a seminorm \( \| \cdot \|_{\mathcal{B}} \). Further, the phase space \( \mathcal{B} \) satisfies the following axioms.

(a) If \( x : (-\infty, a) \to H, \ a > 0 \) is continuous on \([0, a]\) and \( x_0 \) in \( \mathcal{B} \), then for every \( t \in [0, a) \) the following conditions hold

(i) \( x_t \) is in \( \mathcal{B} \)

(ii) \( \| x(t) \| \leq L \| x_t \|_{\mathcal{B}} \)
(iii) \[\|x_t\|_B \leq K(t) \sup\{\|x(s)\| : 0 \leq s \leq t\} + N(t)\|x_0\|_B,\] where \(L > 0\) is a constant, \(K, N : [0, \infty) \to [0, \infty)\), \(K\) is continuous, \(N\) is locally bounded and \(L, K, N\) are independent of \(x(\cdot)\).

(aii) For the function \(x(\cdot)\) in (ai), \(x_t\) is a \(B\)-valued function on \([0, a)\).

(aiii) The space \(B\) is complete.

The collection of all strongly measurable, \(p\)-integrable, \(H\)-valued random variables denoted by \(L_p(\Omega, \mathcal{F}, \mathbb{P}, H) \equiv L_p(\Omega, H)\) is Banach space equipped with the norm
\[
\|x(\cdot)\|_{L_p} = (\mathbb{E}\|x(\cdot, w)\|^p)^{\frac{1}{p}},
\]
where the expectation \(\mathbb{E}\) is defined by \(\mathbb{E}(h) = \int_{\Omega} h(w) d\mathbb{P}\). Let \(C(\mathbb{R} \times \mathbb{R}, L_p(\Omega, H))\) be the Banach space of all continuous maps from \((-\infty, b\]) into \(L_p(\Omega, H)\) satisfying the condition
\[
\sup_{t \in (-\infty, b]} \mathbb{E}\|x(t)\|^p < \infty.
\]
Let \(Z\) be the closed subspace of all continuous process \(x\) that belongs to the space \(C(\mathbb{R} \times \mathbb{R}, L_p(\Omega, H))\) consisting of \(\mathcal{F}_t\)-adapted measurable process, such that the \(\mathcal{F}_0\) adapted process \(\phi \in L_p(\Omega, B)\). Let \(\| \cdot \|_Z\) be a seminorm defined by
\[
\|x\|_Z = \left(\sup_{t \in J} \|x_t\|_B^p\right)^{\frac{1}{p}}
\]
where \(\|x_t\|_B \leq \overline{N}\mathbb{E}\|\phi\|_B + \overline{K} \sup_{t \in J}\{\mathbb{E}\|x(s)\| : 0 \leq s \leq b\}\), with \(\overline{N} = \sup_{t \in J}\{N(t)\}\) and \(\overline{K} = \sup_{t \in J}\{K(t)\}\). It is easy to verify that \(Z\) furnished with the above defined norm topology is a Banach space.

**Definition 3.1.** [106] The Riemann-Liouville fractional integral of order \(\alpha > 0\) for the function \(x : J \to H\) is defined by
\[
J^\alpha_t x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds
\]
and whose Laplace transformation is given by
\[
L\{J^\alpha_t x(t)\} = \frac{1}{\lambda^{\alpha}} \hat{x}(\lambda), \quad \text{where} \quad \hat{x}(\lambda) = \int_0^{\infty} e^{-\lambda t} x(t) dt, \quad Re(\lambda) > w.
\]

**Definition 3.2.** [78] The Riemann-Liouville fractional derivative of order \(0 < \alpha < 1\) for the function \(x : J \to H\) can be defined as
\[
D^\alpha_t x(t) = \frac{d}{dt} J^{1-\alpha}_t x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s)ds.
\]
Definition 3.3. [78] The Caputo fractional derivative of order \(0 < \alpha < 1\) for the function \(x\) can be defined in terms of Riemann-Liouville fractional derivative as follows
\[
^{C}D^{\alpha}_{t}x(t) = \frac{d^{\alpha}}{dt^{\alpha}}(x(t) - x(0)).
\]
The Laplace transform of the Caputo fractional derivative of order \(0 < \alpha < 1\) is given by
\[
L\{^{C}D^{\alpha}_{t}x(t)\} = \lambda^{\alpha}x(\lambda) - \lambda^{\alpha-1}x(0).
\]

Definition 3.4. [116] The Mainardi’s function is defined by
\[
M_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^{n}}{n!(1-\alpha n - \alpha)}, \quad 0 < \alpha < 1, \ z \in \mathbb{C},
\]
it is clear that
\[
\int_{0}^{\infty} M_{\alpha}(r)dr = 1, \quad 0 < \alpha < 1.
\]
On the other hand, \(M_{\alpha}(z)\) satisfies the following equalities
\[
\int_{0}^{\infty} \frac{\alpha}{r^{\alpha+1}} M_{\alpha}\left(\frac{1}{r^{\alpha}}\right) e^{-\lambda r} dr = e^{-\lambda^{\alpha}}
\]
and
\[
\int_{0}^{\infty} r^{\delta} M_{\alpha}(r)dr = \frac{\Gamma(\delta + 1)}{\Gamma(\alpha \delta + 1)}, \quad \delta > -1, \ 0 < \alpha < 1.
\]
(3.3)

Lemma 3.1. [30] For any \(r \geq 1\) and for arbitrary \(L^{0}_{2}(K,H)\)-valued predictable process \(\Phi(\cdot)\)
\[
\sup_{s \in [0,t]} \mathbb{E}\left\| \int_{0}^{s} \Phi(u)dw(u) \right\|^{2r} \leq c_{r} \left( \int_{0}^{t} \mathbb{E}\|\Phi(s)\|_{L^{2}_{2}}^{2r} \right)^{\frac{1}{r}}, \quad t \geq 0,
\]
where \(c_{r} = (r(2r - 1))^{r}\).

Since the \(C_{0}\)-semigroup \(T(t)\) is operator generated by the infinitesimal \(A\), there exist constants \(\hat{N} \geq 1, \ w > 0\) such that \(\|T(t)\| \leq \hat{N}e^{wt}, \ t \geq 0\). It follows from the subordinate principle [9] that an exponentially bounded solution operator \(S_{\alpha}(t), \ 0 < \alpha < 1\) has been generated by \(A\), and which satisfying \(S_{\alpha}(0) = I\),
\[
\|S_{\alpha}(t)\| \leq \hat{N}e^{w^{1/\alpha}t}, \ t \geq 0
\]
and
\[
S_{\alpha}(t) = \int_{0}^{\infty} t^{-\alpha} M_{\alpha}(st^{-\alpha})T(s)ds = \int_{0}^{\infty} M_{\alpha}(r)T(t^{\alpha}r)dr, \ t > 0
\]
where \(M_{\alpha}(r)\) denotes the Mainardi function. Since \(\|S_{\alpha}(t)\| \leq \hat{N}e^{w^{1/\alpha}t}\), thus \(\{\lambda^{\alpha}: \lambda > w^{1/\alpha}\} \subset \rho(A)\), and
\[
\lambda^{\alpha-1}R(\lambda^{\alpha}, A)x = \int_{0}^{\infty} e^{-\lambda t}S_{\alpha}(t)dt, \ \lambda > w^{1/\alpha}, \ x \in H.
\]
The following hypotheses are assumed in order to obtain the main results.
(H3.1) The $C_0$- semigroup $T(t)$ generated by $A$ is compact for $t > 0$ and there exists $M > 0$ such that

$$\sup_{t \geq 0} \|T(t)\| \leq M, \text{ for all } t \geq 0. \quad (3.4)$$

(H3.2) The function $f : J \times \mathcal{B} \to H$ is continuous and there exist constants $M_1, M_2 > 0$ such that

$$\mathbb{E}\|f(t, x) - f(t, y)\|^p \leq M_1\|x - y\|_B^p,$$

$$\mathbb{E}\|f(t, x)\|^p \leq M_2(\|x\|_B^p + 1).$$

(H3.3) The function $g : J \times \mathcal{B} \to H$ is continuous and there exist constants $M_3, M_4 > 0$ such that

$$\mathbb{E}\|g(t, x) - g(t, y)\|^p \leq M_3\|x - y\|_B^p,$$

$$\mathbb{E}\|g(t, x)\|^p \leq M_4(\|x\|_B^p + 1).$$

(H3.4) The function $\sigma : J \times \mathcal{B} \to L^0_2$ satisfies

(a) for each $t \in J$, $\sigma(t, \cdot) : \mathcal{B} \to L^0_2$ is continuous and for each $x \in \mathcal{B}$, $\sigma(\cdot, x) : J \to L^0_2$ is strongly measurable,

(b) there is a positive integrable function, $m \in L^1([0, b])$ and a continuous non decreasing function $\Lambda_\sigma : [0, \infty) \to (0, \infty)$ such that for every $(t, x) \in J \times \mathcal{B}$, we have

$$\left( \int_0^t \left( \mathbb{E}\|\sigma(s, x)\|_{L^0_2}^p \right)^{\frac{q}{p}} ds \right)^{\frac{p}{q}} \leq m(t)\Lambda_\sigma(\|x\|_B^p), \liminf_{r \to 0} \frac{\Lambda_\sigma(r)}{r} = \Lambda < \infty.$$

(H3.5) Choose $\rho = 4^{p-1}\left[2^{p-1}\mathcal{K}^{p}(M_1 + b^pM_2M_2) + \frac{b^pM_2c_2\Lambda_\sigma}{(\alpha p - p + 1)\Gamma(\alpha)} \sup_{t \in J} m(t) \right],$

$$(1 - \rho)q = 4^{p-1}\left[-M^pM_3(\|\phi\|_B^p + 1) + (q' + 1)(M_1 + b^pM_2M_2) + \frac{b^pM_2c_2\Lambda_\sigma(q' + 1)}{(\alpha p - p + 1)\Gamma(\alpha)} \sup_{t \in J} m(t) \right].$$

3.2.3 Formation of Mild Solution for Fractional Neutral Stochastic Integrodifferential Equations with Infinite Delay

**Lemma 3.2.** An $H$-valued stochastic process $x(t) : t \in (-\infty, b]$ is said to be a mild solution of the system (3.1)-(3.2), if it satisfies the following integral equation

$$x(t) = S_\alpha(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t S_\alpha(t-s)f(s, x_s)ds$$

$$+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \left( \int_0^s \sigma(\tau, x_\tau)dw(\tau) \right) ds,$$
where

\[ S_\alpha(t)x = \int_0^\infty M_\alpha(r)T(t^\alpha r)xdr, \quad t \geq 0, \quad x \in H \quad (3.5) \]

and

\[ T_\alpha(t)x = \int_0^\infty \alpha r M_\alpha(r)T(t^\alpha r)xdr. \quad (3.6) \]

**Proof.** Consider

\[
\frac{d}{dt} \left[ J_t^{1-\alpha} \left( x(t) - g(t, x_t) - \phi(0) + g(0, \phi) \right) \right] = A[x(t) - g(t, x_t)] + J_t^{1-\alpha} f(t, x_t) + \int_0^t \sigma(s, x_s)dw(s).
\]

Now applying the Riemann-Liouville fractional integral operator on both sides of the above equation, one can obtain

\[
x(t) - g(t, x_t) - \phi(0) + g(0, \phi) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ag(s, x_s)ds + \int_0^t f(s, x_s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s \sigma(\tau, x_\tau)dw(\tau) \right) ds.
\]

by taking the Laplace transform on both sides of the above equation, one can get

\[ \hat{x}(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha, A)[\phi(0) - g(0, \phi)] + \hat{g}(\lambda) + \lambda^{\alpha-1}R(\lambda^\alpha, A)\hat{f}(\lambda) + R(\lambda^\alpha, A)\hat{\sigma}(\lambda), \quad (3.7)\]

where

\[
\hat{x}(\lambda) = \int_0^\infty e^{-\lambda t} x(t)dt, \quad \hat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t, x_t)dt,
\]

\[
\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t, x_t)dt, \quad \text{and} \quad \hat{\sigma}(\lambda) = \int_0^\infty e^{-\lambda t} \left( \int_0^t \sigma(s, x_s)dw(s) \right) dt.
\]

By taking inverse Laplace transform on both sides of (3.7), one can obtain

\[
L^{-1}[\hat{x}(\lambda)] = L^{-1}[\lambda^{\alpha-1}R(\lambda^\alpha, A)[\phi(0) - g(0, \phi)]] + L^{-1}[\hat{g}(\lambda)] + L^{-1}[\lambda^{\alpha-1}R(\lambda^\alpha, A)\hat{f}(\lambda)] + L^{-1}[R(\lambda^\alpha, A)\hat{\sigma}(\lambda)]. \quad (3.8)
\]

Now, since

\[
\lambda^{\alpha-1}R(\lambda^\alpha, A)\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} S_\alpha(t)\hat{f}(\lambda)dt = \hat{S}_\alpha(\lambda)\hat{f}(\lambda), \quad \text{where} \quad \hat{S}_\alpha(\lambda) = \int_0^\infty e^{-\lambda t} S_\alpha(t)dt.
\]

Taking inverse Laplace transform on both sides of the above equation yields

\[
L^{-1}[\lambda^{\alpha-1}R(\lambda^\alpha, A)\hat{f}(\lambda)] = L^{-1}[\hat{S}_\alpha(\lambda)\hat{f}(\lambda)]
\]

47
and by using the convolution theorem of Laplace transformation, one can get

\[
L^{-1}[\lambda^{\alpha-1}R(\lambda^\alpha, A)\hat{f}(\lambda)] = \int_0^t S_\alpha(t - s)f(s, x_s)ds. \quad (3.9)
\]

Now, \(L^{-1}[R(\lambda^\alpha, A)\hat{\sigma}(\lambda)]\) can be estimated as follows

\[
R(\lambda^\alpha, A)\hat{\sigma}(\lambda)
\]

\[
= \int_0^\infty e^{-\lambda t}T(t)\hat{\sigma}(\lambda)dt
\]

\[
= \int_0^\infty \int_0^\infty e^{-\lambda s}e^{-\lambda t}T(t) \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)dsdt
\]

\[
= \int_0^\infty \int_0^\infty \alpha t^{\alpha-1}e^{-\lambda t}T(t^\alpha) e^{-\lambda s} \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)dsdt
\]

\[
= \int_0^\infty \int_0^\infty \int_0^\infty \alpha t^{\alpha-1} \frac{e^{-\lambda t}T(t^\alpha)}{\alpha} e^{-\lambda s} \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)drdsdt
\]

\[
= \int_0^\infty \int_0^\infty \int_0^\infty \alpha t^{\alpha-1}M\alpha \left(\frac{1}{\tau}\right) e^{-\lambda t}T(t^\alpha) e^{-\lambda s} \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)drdsdt
\]

\[
= \int_0^\infty \alpha t^{\alpha-1}M\alpha \left(\frac{1}{\tau}\right) e^{-\lambda t}T(t^\alpha) e^{-\lambda s} \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)drdsdt
\]

\[
= \int_0^\infty e^{-\lambda t} \left(\int_0^t \alpha r(t - s)^{\alpha-1}M\alpha \left(\frac{1}{\tau}\right) e^{-\lambda t}T(t^\alpha) e^{-\lambda s} \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)drds\right)dt
\]

which implies

\[
L^{-1}[R(\lambda^\alpha, A)\hat{\sigma}(\lambda)] = \int_0^t \int_0^\infty \alpha t(t - s)^{\alpha-1}M\alpha \left(\frac{1}{\tau}\right) e^{-\lambda t}T(t^\alpha) e^{-\lambda s} \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)drds
\]

\[
= \int_0^t (t - s)^{\alpha-1}T\alpha(t - s) \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)ds \quad (3.10)
\]

From the equations (3.8)-(3.10), one can have

\[
x(t) = S\alpha(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t S\alpha(t - s)f(s, x_s)ds
\]

\[
+ \int_0^t (t - s)^{\alpha-1}T\alpha(t - s) \left(\int_0^s \sigma(\tau, x_\tau)d\tau\right)ds
\]

this completes the proof of the lemma. \(\square\)

Now, the definition of mild solution for the problem (3.1)-(3.2) can be stated as follows.

**Definition 3.5.** A stochastic process \(x(t) : J \times \Omega \to H\) is said to be a mild solution of the problem (3.1)-(3.2), if

(i) \(x(t)\) is measurable and \(\mathcal{F}_t\)-adapted for each \(t \geq 0\),
(ii) \(x(t) \in H\) has cadlag paths on \(t \in [0, b]\) a.s., and satisfies the following integral equation
\[
x(t) = S_{\alpha}(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t S_{\alpha}(t-s)f(s, x_s)ds
+ \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s) \left(\int_0^s \sigma(\tau, x_\tau)dw(\tau)\right)ds.
\]

3.2.4 Existence of Mild Solution of Fractional Neutral Stochastic Integrodifferential Equations with Infinite Delay

**Theorem 3.1.** Assume that the hypotheses \((H_{3.1}) - (H_{3.5})\) are satisfied, then there exists a mild solution for the Cauchy problem (3.1)-(3.2).

**Proof.** Let \(B_b\) be the space of all functions \(x : (-\infty, b] \to H\) such that \(x_0 \in B\) and the restriction \(x : J \to H\) is continuous. Let \(\| \cdot \|_b\) be the seminorm in \(B_b\) defined by
\[
\|x\|_b = \|x_0\|_B + \sup_{s \in [0, b]} (\mathbb{E}\|x(s)\|^p)^{\frac{1}{p}}, \ x \in B_b.
\]
Let \(Z_b = C((-\infty, b], L_p(\Omega, B_b))\). Considered the map \(\Phi : Z_b \to Z_b\) be defined by
\[
(\Phi x)(t) = \begin{cases}
\phi(t), & t \in (-\infty, 0], \\
S_{\alpha}(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t S_{\alpha}(t-s)f(s, x_s)ds \\
+ \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s) \left(\int_0^s \sigma(\tau, x_\tau)dw(\tau)\right)ds, & t \in [0, b].
\end{cases}
\]
One need to show that the operator \(\Phi\) has a fixed point, which is a mild solution of the Cauchy problem (3.1)-(3.2).

For \(\phi \in Z, \ y(\cdot) : (-\infty, b) \to Z_b\) the function defined by
\[
y(t) = \begin{cases}
\phi(t), & t \in (-\infty, 0], \\
S_{\alpha}(t)\phi(0), & t \in J
\end{cases}
\]
set \(x(t) = z(t) + y(t), \ t \in (-\infty, b]\). It is easy to check that \(x\) satisfies (3.1)-(3.2), if and only if \(z\) satisfies \(z_0 = 0\) and
\[
z(t) = -S_{\alpha}(t)g(0, \phi) + g(t, z_t + y_t) + \int_0^t S_{\alpha}(t-s)f(s, z_s + y_s)ds
+ \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s) \left(\int_0^s \sigma(\tau, z_\tau + y_\tau)dw(\tau)\right)ds.
\]

Let \(B^0_b = \{z \in B_b; \ z_0 = 0 \in B\}\). For any \(z \in B^0_b\), one can have
\[
\|z\|_b = \|z_0\|_B + \sup_{s \in [0, b]} (\mathbb{E}\|z(s)\|^p)^{\frac{1}{p}} = \sup_{s \in [0, b]} (\mathbb{E}\|z(s)\|^p)^{\frac{1}{p}}.
\]
Thus, if \( Z_b^0 = C((\infty, b], L_p(\Omega, B^0_b)) \), then \((Z_b^0, \| \cdot \|_b)\) is a Banach space. For each \( q \geq 0 \), set \( B_q = \{ z \in Z_b^0 : \| z \|_b^p \leq q \} \), then it is clear that \( B_q \) is bounded closed and convex set in \( Z_b^0 \). From the equations (3.3)-(3.5), one have \( \| S_\alpha(t) \| \leq M \). As well, from the equations (3.3), (3.4) and (3.6), \( \| T_\alpha(t) \| \leq M \). For \( z \in B_q \),

\[
\| z_t + y_t \|_B^p \leq 2^{p-1} \{ \| z_t \|_B^p + \| y_t \|_B^p \}
\leq 2^{p-1} \mathbf{K}^p q + 4^{p-1} \{ \mathbf{N}^p \| \phi \|_B^p + \mathbf{K}^{p(p-1)} M^p E \| \phi(0) \|_B^p \}
\leq 2^{p-1} \mathbf{K}^p q + q', \text{ where } q' = 4^{p-1} \{ \mathbf{N}^p \| \phi \|_B^p + \mathbf{K}^{p(p-1)} M^p E \| \phi(0) \|_B^p \}.
\]

Let the operator \( \Pi : Z_b^0 \rightarrow Z_b^0 \) be defined by

\[
(\Pi z)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -S_\alpha(t)g(0, \phi) + g(t, z_t + y_t) + \int_0^t S_\alpha(t - s)f(s, z_s + y_s)ds \\ + \int_0^t (t - s)^{\alpha - 1} T_\alpha(t - s) (\int_0^s \sigma(\tau, z_\tau + y_\tau)d\tau) d\tau \\ + \int_0^t (t - s)^{\alpha - 1} T_\alpha(t - s) (\int_0^s \sigma(\tau, z_\tau + y_\tau)d\tau) d\tau, & t \in J. \end{cases}
\]

Obviously, the operator \( \Phi \) has a fixed point is equivalent to prove that \( \Pi \) has a fixed point. Now, one shall show that the operator \( \Pi \) has a fixed point on \( B_q \), which implies the Cauchy problem (3.1)-(3.2) has a mild solution.

Decomposing \( \Pi \) as \( \Pi = \Pi_1 + \Pi_2 \), where \( \Pi_1 \) and \( \Pi_2 \) are defined on \( B_q \) respectively by

\[
(\Pi_1 z)(t) = -S_\alpha(t)g(0, \phi) + g(t, z_t + y_t)
\]

and \( (\Pi_2 z)(t) = \int_0^t S_\alpha(t - s)f(s, z_s + y_s)ds + \int_0^t (t - s)^{\alpha - 1} T_\alpha(t - s)
\)

\[
\times \left( \int_0^s \sigma(\tau, z_\tau + y_\tau)d\tau \right) ds.
\]

In-order to apply the Nussbaum fixed point theorem for the operator \( \Pi \), one need to prove the following assertions

(i) \( \Pi_1 \) and \( \Pi_2 \) are well defined,

(ii) \( \Pi_1 \) satisfies Lipschitz condition,

(iii) \( \Pi_2 \) is relatively compact,

(iv) \( \Pi B_q \subseteq B_q \).
Now, for $0 \leq t \leq b$

\[
E\|\Pi_1 z(t)\|^p \leq 2^{p-1}\{E\|\Pi_1 z(t)\|^p + E\|\Pi_2 z(t)\|^p \}
\]

and

\[
E\|\Pi_2 z(t)\|^p \leq 2^{p-1}\left\{ \int_0^t \left( \left( \int_0^{s} \sigma(\tau, z\tau + y\tau)d\tau \right) ds \right)^p + \int_0^t \left( \left( \int_0^{s} \sigma(\tau, z\tau + y\tau)d\tau \right) ds \right)^p \right\}
\]

Thus, one have

\[
\mathbb{E}\|Z(t)\|^p \leq 2^{p-1}\{E\|\Pi_1 z(t)\|^p + E\|\Pi_2 z(t)\|^p \}
\]

\[
\leq 4^{p-1}\left\{ - M^p M_4(\|\phi\|^p_B + 1) + M_4(2^{p-1}\mathcal{K}^q q + q' + 1) \right\}
\]

\[
+ b^p M^p M_2(2^{p-1}\mathcal{K}^q q + q' + 1) + \frac{b^{op} M^p_\sigma \Lambda_\sigma(2^{p-1}\mathcal{K}^q q + q' + 1)\sup m(t)}{(\alpha p - p + 1)\Gamma^p(\alpha)}
\]

\[
\leq 4^{p-1}\left\{ \left[ 2^{p-1}\mathcal{K}^q (M_4 + b^p M^p M_2) + \frac{b^{op} M^p_\sigma \Lambda_\sigma(2^{p-1}\mathcal{K}^q q + q' + 1)\sup m(t)}{(\alpha p - p + 1)\Gamma^p(\alpha)} \right] q \right. 
\]

\[+ \left. \left[ - M^p M_4(\|\phi\|^p_B + 1) + (q' + 1)(M_4 + b^p M^p M_2) + \frac{b^{op} M^p_\sigma \Lambda_\sigma(q' + 1)\sup m(t)}{(\alpha p - p + 1)\Gamma^p(\alpha)} \right] \right \}
\]

\[
\leq \rho q + (1 - \rho)q = q.
\]
Hence, $\Pi B_q \subseteq B_q$. Next, one need to prove that the operator $\Pi_1$ satisfies the Lipschitz condition. Take $z^{(1)}, z^{(2)} \in B_q$, then for each $t \in J$, one have

$$\mathbb{E} \left\| (\Pi_1 z^{(1)})(t) - (\Pi_1 z^{(2)})(t) \right\|^p \leq \mathbb{E} \left\| g(t, z^{(1)}_t + y_t) - g(t, z^{(2)}_t + y_t) \right\|^p \leq M_3 \left\| z^{(1)}_t - z^{(2)}_t \right\|^p_B \leq M_3 \sup_{0 \leq t \leq b} \mathbb{E} \left\| z^{(1)}(t) - z^{(2)}(t) \right\|^p_B.$$

Thus, $\Pi_1$ satisfies the Lipschitz condition provided that $M_3 < 1$. Finally, one need to prove that $\Pi_2$ is relatively compact in $B_q$. To prove this first one need to show that $\Pi_2$ maps $B_q$ into a precompact subset of $\Pi$. One can show that for every fixed $t \in J$, the set $V(t) = \{(\Pi_2 z)(t) : z \in B_q\}$ is precompact in $H$. Let $0 < \epsilon < b$ be fixed and $\epsilon$ be a real number satisfying $0 < \epsilon < t$. Define an operator $\Pi_2^{\epsilon, \delta}$ on $B_q$ by

$$(\Pi_2^{\epsilon, \delta} z)(t) = \int_0^{t-\epsilon} \int_{\delta}^{\infty} M_\alpha(r) T((t-s)^\alpha r) f(s, z_s + y_s) dr ds + \epsilon T(\alpha \delta) \int_{\delta}^{\infty} \int_0^{r} M_\alpha(r) f(s, z_s + y_s) dr ds$$

$$= T(\alpha \delta) \int_0^{t-\epsilon} \int_{\delta}^{\infty} M_\alpha(r) T((t-s)^\alpha r - \epsilon \delta) f(s, z_s + y_s) dr ds$$

$$+ \epsilon T(\alpha \delta) \int_{\delta}^{\infty} \int_0^{r} T((t-s)^\alpha r - \epsilon \delta) \left( \int_0^{s} \sigma(\tau, z_\tau + y_\tau) dw(\tau) \right) dr ds.$$

Since $T(t)$ is compact for $t > 0$, the set $\{(\Pi_2^{\epsilon, \delta} z)(t), z \in B_q\}$ is precompact in $H$ for every $\epsilon \in (0, t)$, $\delta > 0$.

Moreover, for each $z \in B_q$, one have

$$\mathbb{E} \left\| (\Pi_2 z)(t) - (\Pi_2^{\epsilon, \delta} z)(t) \right\|^p \leq 4^{p-1} \left\{ \mathbb{E} \left\| \int_0^t \int_{\delta}^{\epsilon} M_\alpha(r) T((t-s)^\alpha r) f(s, z_s + y_s) dr ds \right\|^p \right. \right.$$

$$+ \mathbb{E} \left\| \int_0^t \int_{\delta}^{t-\epsilon} M_\alpha(r) T((t-s)^\alpha r) f(s, z_s + y_s) dr ds \right\|^p$$

$$+ \mathbb{E} \left\| \int_0^t \int_0^{\delta} \alpha r (t-s)^{\alpha-1} M_\alpha(r) T((t-s)^\alpha r) \left( \int_0^{s} \sigma(\tau, z_\tau + y_\tau) dw(\tau) \right) dr ds \right\|^p$$

$$\left. + \mathbb{E} \left\| \int_{t-\epsilon}^t \int_0^{\delta} \alpha r (t-s)^{\alpha-1} M_\alpha(r) T((t-s)^\alpha r) \left( \int_0^{s} \sigma(\tau, z_\tau + y_\tau) dw(\tau) \right) dr ds \right\|^p \right\} \leq \sum_{i=1}^{4} I_{3,i}. \tag{3.11}$$
Now,

\[ I_{3.1} \leq \text{\[3.12\]} \int_{t-\epsilon}^{t} \epsilon^{-1} M^P \int_{0}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) ds \\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) \| p\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) \| p\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) \| p\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} M_\alpha(2^{p-1} K^p q + q' + 1) ds, \]

\[3.13\]

\[ I_{3.2} \leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) ds \\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) \| p\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) \| p\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} \int_{0}^{t} \int_{0}^{t} M_\alpha(r) f(s, z_s + y_s) \| p\| ds \\
\leq \epsilon^{-1} M^P \int_{t-\epsilon}^{t} M_\alpha(2^{p-1} K^p q + q' + 1) ds, \]

\[3.14\]

\[ I_{3.3} \leq \alpha^p \epsilon^{-1} M^P \int_{0}^{t} (t-s)^{\alpha-1}p \left( \int_{0}^{\delta} r M_\alpha(r) dr \right) p \int_{0}^{t} \sigma(\tau, z_\tau + y_\tau) dw(\tau) \\| ds \\
\leq \alpha^p \epsilon^{-1} M^P c_p \int_{0}^{t} (t-s)^{\alpha-1}p \left( \int_{0}^{\delta} r M_\alpha(r) dr \right) p m(s) \Lambda_\alpha(\| z_s + y_s \|_E^p) ds \\
\leq \alpha^p \epsilon^{-1} M^P c_p \int_{0}^{t} (t-s)^{\alpha-1}p \left( \int_{0}^{\delta} r M_\alpha(r) dr \right) p m(s) \Lambda_\alpha(\| z_s + y_s \|_E^p) ds \\
\leq \alpha^p \epsilon^{-1} M^P c_p \int_{0}^{t} (t-s)^{\alpha-1}p m(s) \Lambda_\alpha(2^{p-1} K^p q + q') ds, \]

\[3.15\]

It follows from the equations \(3.11\)-\(3.15\) that for each \(z \in B_q\), \(\| (\Pi z)(t) - (\Pi_{\epsilon}^{\delta} z)(t) \| \rightarrow 0 \) as \(\epsilon \rightarrow 0^+, \delta \rightarrow 0^+\). Therefore, there are relatively compact sets arbitrarily closed to the set \(V(t) = \{(\Pi z)(t), z \in B_q\}\) and thus \(V(t)\) is also relatively compact in \(B_q\). Now, one needs to prove that \(\{(\Pi z)(t), z \in B_q\}\) be an equicontinuous family of functions.

Let \(0 < \epsilon < t < b\) and \(\delta > 0\) such that \(\| S_\alpha(s_1) - S_\alpha(s_2) \| < \epsilon_1\) and \(\| T_\alpha(s_1) - T_\alpha(s_2) \| < \epsilon_2\) with \(|s_1 - s_2| < \delta\). For \(z \in B_q\), \(0 < |h| < \delta\), \(t + h \in J\), one have

\[ \| \Pi z(t + h) - \Pi z(t) \| \leq 5^p - 1 \left\{ \begin{array}{l}
\mathbb{E} \left[ \left| \int_{0}^{t} \left[ S_\alpha(t + h - s) - S_\alpha(t - s) \right] f(s, z_s + y_s) ds \right| \right]^p \\
\mathbb{E} \left[ \left| \int_{0}^{t} S_\alpha(t + h - s) f(s, z_s + y_s) ds \right| \right]^p 
\end{array} \right. 
\]

53
(3.1)-(3.2). Let 
\[ (\Omega \times \Pi)^3 \]
3.2.5 Application

\[ \text{Consider the following fractional neutral stochastic integrodifferential equations with infinite delay to illustrate the obtained theory} \]
\[ + E \left\| \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T_\alpha (t+h-s) \left( \int_0^s \sigma(\tau, z_\tau + y_\tau) dw(\tau) \right) ds \right\|^p \]
\[ + E \left\| \int_t^{t+h} (t+h-s)^{\alpha-1} T_\alpha (t+h-s) \left( \int_0^s \sigma(\tau, z_\tau + y_\tau) dw(\tau) \right) ds \right\|^p \]
\[ + E \left\| \int_0^t (t-s)^{\alpha-1} [T_\alpha (t+h-s) - T_\alpha (t-s)] \left( \int_0^s \sigma(\tau, z_\tau + y_\tau) dw(\tau) \right) ds \right\|^p \}
\[ \leq 5^{p-1} \left\{ b^{p-1} \epsilon_1^p \int_0^t M_2 (2^{-p} K^p q + q' + 1) ds \right. \]
\[ + h^{p-1} M^{p} \int_t^{t+h} M_2 (2^{-p} K^p q + q' + 1) ds \]
\[ + \frac{b^{p-1} M^{p} c_p}{\Gamma(p)} \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] \Lambda_\sigma (2^{-p} K^p q + q') m(s) ds \]
\[ + \frac{h^{p-1} M^{p} c_p}{\Gamma(p)} \int_t^{t+h} (t+h-s)^{(\alpha-1)p} \Lambda_\sigma (2^{-p} K^p q + q') m(s) ds \]
\[ \left. + b^{p-1} c_p \int_0^t (t-s)^{(\alpha-1)p} \Lambda_\sigma (2^{-p} K^p q + q') m(s) ds \right\} . \]

Therefore, for sufficiently small \( \epsilon_1, \epsilon_2 \) the right hand side of the above inequality tends to zero as \( h \to 0 \). On the other-hand, the compactness of \( T(t), t > 0 \) implies the continuity in the uniform operator topology. Thus, the set \( \{ \Pi z, z \in B_q \} \) is equicontinuous. Hence, \( \Pi_2 \) maps \( B_q \) into a equicontinuous family of functions. Also, \( \Pi_2 (B_q) \) is bounded in \( Z \) and so by Arzela-Ascoli theorem, \( \Pi_2 (B_q) \) is precompact. Hence, it follows from the Nussbaum fixed point theorem there exists a fixed point \( z(\cdot) \) for \( \Pi \) on \( B_q \) such that \( \Pi z(t) = z(t) \). This completes the proof of the theorem. \( \square \)

3.2.5 Application

Consider the following fractional neutral stochastic integrodifferential equations with infinite delay to illustrate the obtained theory

\[
\begin{align*}
\frac{d}{dt} \left[ J_t^{1-\alpha} (v(t, x) - g(t, v(t-h, x)) - \phi(0, x) + g(0, v(-h, x))) \right] &= \frac{\delta^2}{\delta x^2} [v(t, x) - g(t, v(t-h, x))] \\
+ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s, v(s-h, x)) ds + \int_0^t \sigma(s, v(s-h, x)) dw(s) \\
v(t, 0) &= v(t, \pi) = 0, \ t \in J \\
v(t, x) &= \phi(t, x), \ t \in (-\infty, 0]
\end{align*}
\]

(3.16)

Let \( H = L_2[0, \pi] \), \( w(t) \) is a standard cylindrical Wiener process in \( H \) defined on a stochastic basis \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, P)\). One can rewrite the system (3.16) into the abstract form of the system (3.1)-(3.2). Let \( A : H \to H \) be defined by \( Az = z'' \) with the domain
\(D(A) = \{ z \in H : \text{z, } z' \text{ are absolutely continuous } z'' \in H, \ z(0) = z(\pi) = 0 \},\) then

\[ A\zeta = \sum_{n=1}^{\infty} -n^2 \langle \zeta, e_n \rangle e_n, \ \zeta \in D(A), \]

where \(e_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns), n = 1, 2, \ldots\) is the complete orthonormal set of eigenvalues of \(A\). It is well known that \(A\) is the infinitesimal generator of a compact analytic semigroup \(\{T(t)\}_{t \geq 0}\) in \(H\) and is given by

\[ T(t)\zeta = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \zeta, e_n \rangle e_n \]

for all \(\zeta \in H\) and every \(t > 0\). The subordinate principle of solution operator implies that \(A\) is the infinitesimal generator of a solution operator \(\{S_\alpha(t)\}_{t \geq 0}\) (see [9]). Since \(S_\alpha(t)\) is strongly continuous on \([0, \infty)\) by uniform bounded theorem, there exists a constant \(M > 0\) such that \(\|S_\alpha(t)\| \leq M\) for \(t \in [0, b]\).

Now, the special phase space \(\mathcal{B}\) is defined as follows. Let \(h(s) = e^{2s}, s < 0\) then \(l = \int_{-\infty}^{0} h(s) ds = \frac{1}{2}\) and define

\[ \|\xi\|_{\mathcal{B}} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s, 0]} \|\xi(\theta)\|_{L_2} ds. \]

Hence for \((t, \xi) \in [0, b] \times \mathcal{B},\) where \(\xi(\theta)(\zeta) = \phi(\theta, \zeta), (\theta, \zeta) \in (-\infty, 0] \times [0, \pi].\) Set

\[ f(t, \xi)(x) = \int_{-\infty}^{0} \mu_1(t, x, \theta) \Theta_1(\xi(\theta)(x)) d\theta, \]
\[ g(t, \xi)(x) = \int_{-\infty}^{0} \mu_2(t, x, \theta) \Theta_2(\xi(\theta)(x)) d\theta, \]
\[ \sigma(t, \xi)(x) = \int_{-\infty}^{0} \mu_3(t, x, \theta) \Theta_3(\xi(\theta)(x)) d\theta, \]

with these settings the equation (3.16) can be rewritten in the abstract form of the system (3.1)-(3.2). By imposing suitable conditions on \(\mu_i, i = 1, 2, 3\) and \(\Theta_i, i = 1, 2, 3,\) it is easy to verify that the hypotheses \((H_{3.1}) \sim (H_{3.5}).\) Thus, there exists a mild solution for the system (3.16).

### 3.3 Conclusions and Future Directions

In this chapter, a class of fractional neutral stochastic integrodifferential equations with infinite delay has been considered in Hilbert space by using the idea that Caputo fractional derivative can be written in terms of Riemann-Liouville fractional derivative. Some new kind of mild solution has
been proposed for the considered system, which include the one sided probability density function in terms of the special function of fractional calculus so called Mainardi’s function. Further, the existence of mild solution has been proved for the considered system by employing Nussbaum fixed point theorem, solution operator and fractional calculus. Finally, an illustrative example has been given to verify the acquired theoretical results.

The concept of mild solution obtained in this chapter could be adopted to FSDIs, and one can study the basic fundamental qualitative properties of solutions of differential inclusions namely controllability and stability analysis. In particular, nowadays there has been increasing interests in the evolving topics Mittag-Leffler stability, Hyers-Ulam stability etc.

On the other hand, it has been observed from the literature that, one cannot apply semigroup theory directly to solve linear or nonlinear FSDEs in terms of a variation of constants formula. Therefore, an unified functional analytic approach need to be formulated to derive a variation of constants formula for a wide class of FSDEs, which covers the theories of $C_0$-semigroups and cosine families as particular cases, which is the most fundamental issue in the study of FSDEs. In future some other special functions of fractional calculus namely Mittag-Leffler function, Wright function etc., can be used to find the unified approach to solve the FSDEs, since these special functions of fractional calculus may provide different insights in the concept of solution of FSDEs, the motivation comes from the work of Mainardi and Pagnini [92], which provided solutions of the time-fractional diffusion equation with the help of Wright functions.