CHAPTER 2
DISTANCE TWO LABELING

2.1 INTRODUCTION

Although many results on $L(2, 1)$ labeling (distance two labeling) have already been published, still some problems remain unsolved. A sufficient literature survey on $L(2, 1)$ labeling and $L(3, 2, 1)$ labeling is already given in the section 1.2. Our work is motivated by the conjecture of Griggs and Yeh [9] that asserts that $\lambda_{2,1}(G) \leq \Delta^2$ for every graph $G$ with maximum degree $\Delta \geq 2$.

In this chapter we study distance two labeling and its span value for planar graph with maximal edges $Pl_n$, circulant graph $Ci_n(1, 2)$, $G \times P_m$ graphs by varying $G$ with path, cycle, star, grid and $Pl_n$. Further upper bound span value is determined for $G \times P_m$ simple graphs. Also we have found few results on $L(3, 2, 1)$ labeling for $G \times P_m$ graphs.

2.2 DISTANCE TWO LABELING FOR SOME PLANAR GRAPHS

In this section two types of planar graphs are taken. Distance two labeling is given for planar graph with maximal edges $Pl_n$ and circulant graph $Ci_n(1, 2)$ with even $n$.

Distance two labeling for planar graph with maximal edges $Pl_9$ is given in the Figure 2.1 with span $\lambda = 10$. 
Algorithm 2.2.1

Input: Number of vertices $n$ of $P_l n = (V, E)$

Output: Distance two labeling of $P_l n$

begin

$V = \{ v_1, v_2, \ldots, v_n \}$

$E = \{ v_i v_{i+1}, v_j v_{j+1} : 1 \leq i \leq n - 2 \} \cup \{ v_j v_{j+1} : 1 \leq j \leq n - 3 \} \cup \{ v_{n-1} v_n \}$

if $(n - 2) \mod 2 = 0$ then $m = (n - 2) / 2$;
else $m = (n - 1) / 2$;

$f (v_m) = 0; f (v_{m-1}) = 3; f (v_{m-2}) = 1$;

$k = 0$;

for $i = m - 3$ to 1

$\{$

$f (v_i) = 5 + k$ ;

$k = k + 2$;

$\}$

$j = 1$

for $i = m + 1$ to $n - 2$
\{  
  f(v_i) = 2j;  
  j = j+1;  
\}  

f(v_n) = n+1; f(v_{n-1}) = n-1;  
end.

**Theorem 2.2.2.** The class of planar graphs with maximal edges $P_l_n$ has distance two labeling with span $\lambda = n+1$.

**Proof.** Consider the graph $P_l_n = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_iv_{i+1}, v_iv_n : 1 \leq i \leq n-2\} \cup \{v_jv_{j+1} : 1 \leq j \leq n-3\} \cup \{v_{n-1}v_n\}$. Consider the function $f: V(G) \to \{0, 1, 2, \ldots, n+1\}$. We prove this theorem in two cases.

**Case 1.** If $d(v_i, v_j) = 1$ then to prove $|f(v_i) - f(v_j)| \geq 2$

$d(v_i, v_j) = 1$ occur only when $i = n$ or $j = n - 1$ and $j = i + 1$ or $i - 1$.

**Sub case 1.1.** When $i = n$ or $n - 1$

if $i = n$ and $j = n - 1$ then we have from Algorithm 2.2.1,

$|f(v_i) - f(v_j)| = |f(v_n) - f(v_{n-1})| = |(n+1) - (n-1)| = 2$.  

if $i = n$ and $j \neq n - 1$, then we choose any constant $k$ which is less than $n - 1$,

$f(v_j) = k$ we have $|f(v_i) - f(v_j)| = |f(v_n) - f(v_j)| = |(n+1) - k| = |n - k + 1|$.  

Since $|k| < n - 1$ and the labels are non negative positive integer we have $k < n - 1 \Rightarrow n - k > 1 \Rightarrow n - k + 1 > 2$, this implies, $|f(v_n) - f(v_j)| > 2$.

Hence $|f(v_n) - f(v_j)| \geq 2$ if $d(v_n, v_j) = 1$ for all $j$.

Similarly we can prove $|f(v_{n-1}) - f(v_j)| \geq 2$ if $d(v_n, v_j) = 1$ for all $j$.  

Sub case 1.2. When \( j = i+1 \) or \( i -1 \)

If \( j = i+1 \) then \( d(v_i, v_{i+1}) = 1 \) for all \( i: 1 \leq i \leq n-3 \), we have
\[
|f(v_i) - f(v_{i+1})| = 2 \quad \text{for } i \neq m - 1 \quad \text{where } m = (n - 2) / 2 \quad \text{if } n \text{ is even and}
\]
\[
m = (n - 1) / 2 \quad \text{if } n \text{ is odd. Also } |f(v_{m-1}) - f(v_m)| = |3 - 0| > 2.
\]
Similarly when \( j = i -1 \), we have \( |f(v_{i-1}) - f(v_i)| = 2 \) for all \( i: 2 \leq i \leq n - 2 \).
Hence we have \( |f(v_i) - f(v_{i+1})| \geq 2 \) if \( d(v_i, v_{i+1}) = 1 \) for all \( i \).

Case 2. If \( d(v_i, v_j) = 2 \) then to prove \( |f(v_i) - f(v_j)| \geq 1 \)

\( d(v_i, v_j) = 2 \) occurs only when \( j = i+2 \) and \( j \neq i+1 \) or \( i -1 \).

In both the cases we have \( |f(v_i) - f(v_{i+2})| \geq 1 \) and \( |f(v_i) - f(v_j)| \geq 1 \), since the labeling of all the vertices are distinct. In this case we get \( |f(v_i) - f(v_j)| \geq 1 \).

The maximum number use for this labeling is \( n+1 \). Hence the planar Graph \( Pl_n \) has distance two labeling and \( \lambda = n+1 \) for \( n > 5 \). \( \square \)

Example 2.2.3. The distance two labeling of 4- regular planar graph \( Ci_{18}(1, 2) \) with \( \lambda = 8 \) is given in Figure 2.2.

![Figure 2.2 Distance two labeling of \( Ci_{18}(1, 2) \)]
When $n$ is even, the 4-regular circulant graph $C_{i_n}(1, 2)$ is planar. When $n$ is odd the 4-regular circulant graph $C_{i_n}(1, 2)$ is non planar.

**Algorithm 2.2.4.**

**Input:** 4-regular planar graph $C_{i_n}(1, 2)$ with even number of ($n > 6$) vertices

**Output:** Distance two labeling of 4-regular planar graph $C_{i_n}(1, 2)$ with $n > 6$ vertices

begin

**Step 1:** Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i, v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1, v_nv_2, v_{n-1}v_1\} \cup \{v_jv_{j+2} : 1 \leq j \leq n-2\}$;

**Step 2:** for $n \equiv 0 \pmod{4}$ and $\frac{n}{4}$ is even, the function $f: V \to \{0, 1, 2, \ldots, 7\}$ defined as follows

\[
\begin{align*}
    f(v_{8i+1}) &= 0 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \quad f(v_{8i+2}) = 2 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \\
    f(v_{8i+3}) &= 4 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \quad f(v_{8i+4}) = 6 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \\
    f(v_{8i+5}) &= 1 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \quad f(v_{8i+6}) = 3 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \\
    f(v_{8i+7}) &= 5 \text{ for } 0 \leq i \leq \frac{n}{8} - 1; \quad f(v_{8i+8}) = 7 \text{ for } 0 \leq i \leq \frac{n}{8} - 1;
\end{align*}
\]

**Step 3:** for $n \equiv 2 \pmod{4}$; $n \equiv 0 \pmod{4}$ and $\frac{n}{4}$ is odd, the function $f: V \to \{0, 1, 2, \ldots, 7\}$ defined as follows

\[
\begin{align*}
    f(v_{8i+1}) &= 0 \text{ for } 0 \leq i \leq \frac{n}{5} - 1; \quad f(v_{8i+2}) = 2 \text{ for } 0 \leq i \leq \frac{n}{5} - 1;
\end{align*}
\]
\[ f(v_{s_i-3}) = 4 \text{ for } 0 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor - 1; \]

\[ f(v_{s_i-4}) = 6 \text{ for } 0 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor - 1; \]

\[ f(v_{s_i-5}) = 8 \text{ for } 0 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor - 1; \]

\[
\text{if } \left( n - 5 \left\lfloor \frac{n}{5} \right\rfloor \right) = 1 \text{ then } f \left( v_{s_i-1} \right) = 10; \\
\text{else if } \left( n - 5 \left\lfloor \frac{n}{5} \right\rfloor \right) = 2 \text{ then } f \left( v_{s_i-1} \right) = 3; f \left( v_{s_i-2} \right) = 5; \\
\text{else if } \left( n - 5 \left\lfloor \frac{n}{5} \right\rfloor \right) > 2 \text{ then } \\
\quad f \left( v_{s_i-} \right) = 2i - 1; \text{ for } i = 1 \text{ to } n - 5 \left\lfloor \frac{n}{5} \right\rfloor; \\
\text{end.}

**Theorem 2.2.5.** The class \( Ci_n(1, 2) \) of 4-regular planar graph, admits distance two labeling for \( n \geq 6 \) and the span is given as follows

i. for \( n = 6 \), \( \lambda(Ci_6(1,2)) = 10 \)

ii. for \( n \equiv 0 \pmod{4} \) and \( \frac{n}{4} \) is even, \( \lambda(Ci_n(1,2)) = 7 \)

iii. for \( n \equiv 2 \pmod{4} \); \( n \equiv 0 \pmod{4} \) and \( \frac{n}{4} \) is odd
\[ \hat{\lambda}(C_{i_n}(1,2)) = \begin{cases} 8 & \text{for } \left( n - 5 \left\lfloor \frac{n}{5} \right\rfloor \right) = 0, 2, 3, 4 \\ 10 & \text{for } \left( n - 5 \left\lfloor \frac{n}{5} \right\rfloor \right) = 1 \end{cases} \]

**Proof:** One can easily verify distance two labeling and span value for \( n < 6 \).

Consider the graph \( G(V, E) = C_{i_n}(1, 2) \) with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_nv_{n+2}, v_{n-1}v_1\} \cup \{v_jv_{j+2} : 1 \leq j \leq n - 2\} \). The function \( f : V(G) \rightarrow \{0, 1, 2, \ldots, \frac{n}{2} - 1\} \) defined as in Algorithm 2.2.4. We prove this theorem in two cases for \( n \geq 8 \).

**Case 1.** If \( d(v_i, v_j) = 1 \) then to prove \(|f(v_i) - f(v_j)| \geq 2\).

The situation \( d(v_i, v_j) = 1 \) occur only when \( j = i - 1 \) or \( i+1 \) and \( j = i - 2 \) or \( i + 2 \)

**Sub case 1.1.** When \( j = i + 1 \) or \( i - 1 \)

If \( j = i + 1 \) then \( d(v_i, v_j) = 1 \) for all \( i : 1 \leq i \leq n - 1 \).

According to the Algorithm 2.2.4 we have \(|f(v_i) - f(v_j)| = |f(v_i) - f(v_{i+1})| \geq 2\) for all \( i : 1 \leq i \leq n - 1 \) and for \( i = n \) and \( j = 1 \), we have \(|f(v_n) - f(v_1)| > 2\). Thus \(|f(v_i) - f(v_j)| \geq 2\) for all \( i = j + 1 \).

Similarly when \( j = i - 1 \), we have \(|f(v_{i-1}) - f(v_i)| > 2\) for all \( i : 2 \leq i \leq n \).

Hence we have, \(|f(v_i) - f(v_j)| \geq 2\) for \( j = i - 1 \) or \( i + 1 \).

**Sub case 1.2.** When \( j = i + 2 \) or \( i - 2 \)

If \( j = i + 2 \) then \( d(v_i, v_j) = 1 \) for all \( i : 1 \leq i \leq n - 2 \). According to the algorithm we have \(|f(v_i) - f(v_j)| = |f(v_i) - f(v_{i+2})| \geq 4 > 2\) for all \( i : 1 \leq i \leq n - 2 \) and for the vertices \( \{v_nv_2, v_{n-1}v_1\} \) we have \(|f(v_n) - f(v_2)| > 2\) and \(|f(v_{n-1}) - f(v_1)| > 2\).
Thus $|f(v_i) - f(v_j)| \geq 2$ for all $j = i+2$. Similarly when $j = i - 2$, we have $|f(v_{i-2}) - f(v_i)| > 2$ for all $i : 3 \leq i \leq n$. Hence $|f(v_i) - f(v_j)| \geq 2$ for $j = i-2$ or $i+2$.

From both the sub cases we have $|f(v_i) - f(v_j)| \geq 2$.

**Case 2.** If $d(v_i, v_j) = 2$ then to prove $|f(v_i) - f(v_j)| \geq 1$.

This situation $d(v_i, v_j) = 2$ occurs only when $j = i+3$ or $i-3$ and $j \neq i+2$ or $i-2$, $j \neq i+1$ or $i-1$. i.e., $d(v_i, v_{i+3}) = 2$ for $1 \leq i \leq n - 3$ or $d(v_{i-3}, v_i) = 2$ for $4 \leq i \leq n$ and $d(v_i, v_j) = 2$ for $j \neq i+2$ or $i-2$, $j \neq i+1$ or $i-1$.

In both the cases we have $|f(v_i) - f(v_j)| \geq 1$, since according to the Algorithm 2.2.4, the vertices with distance two have distinct labeling. Thus we have $|f(v_i) - f(v_j)| \geq 1$ if $d(v_i, v_j) = 2$. One can easily verify the maximum labeling number or span used for the graph $C_i \in (1, 2)$ which is given in hypothesis, using the Algorithm 2.2.4. Hence the 4-regular planar graph $C_i \in (1, 2)$ has distance two labeling.

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**2.3 CONSTRUCTION OF $G \times P_m$ GRAPHS**

If the given number of stations (vertices) is too large and we have to assign a channel (label) for these stations without interference then we can arrange the stations in $m$ layers and each station of any layer is connected to corresponding station in the succeeding and preceding layers by a link or edge. This idea creates a construction of $G \times P_m$ graph. Consider $m$ copies of simple graph ‘$G$’ with $n$ vertices. Arrange $G$ in $m$ layers and connect the corresponding vertices of each layer by $mn$ edges like pillars of a Multi-storey building.
Formally we define a $G \times P_m$ graph in the following way that is suitable for our labeling purpose.

\[
\bigcup_{i=1}^{m} G^j = \left(V(G^j), E(G^j)\right) \quad \text{where} \quad V(G^j) = \{v_1^j, v_2^j, \ldots, v_n^j\} \quad \text{for} \quad j = 1, 2, \ldots, m \quad \text{and} \\
E(G^j) = \{v_i^j v_k^j : v_i^j v_k^j \in E(G)\}. \quad \text{Now we construct a new graph} \quad G^M = (V, E) \text{ called } G \times P_m \text{ where } V(G^M) = V(G^1) \cup V(G^2) \cup \ldots \cup V(G^m) = \bigcup_{j=1}^{m} V(G^j) \quad \text{and} \\
E(G^M) = \bigcup_{j=1}^{m} E(G^j) \cup E' \quad \text{where} \quad E' = \{v_i^j v_k^{j+1} : 1 \leq i \leq n; 1 \leq k \leq m - 1\}.
\]

The construction is given in the Figure 2.3.

![Figure 2.3 $G \times P_m$ Graph $G^M$](image)

Number of vertices in this $G \times P_m$ graph $G^M$ is $m|V|$, where $m$ is the number of copies of $G$ and $|V|$ is the number of vertices of a simple graph $G$. Number of edges in this $G \times P_m$ Graph $G^M$ is $(m - 1) |V| + m|E|$. In general the
maximum degree of a graph $G$ is denoted as $\Delta$. In our $G \times P_m$ graph maximum degree is denoted as

$$\Delta_M = \begin{cases} 
\Delta + 1 & \text{if } M = 2 \\
\Delta + 2 & \text{if } M \geq 3
\end{cases} \leq |V| + 1.$$

If $M = 1$, then $G^M$ becomes a simple graph so we consider the graph $G^M$ with $M$ layers where $M \geq 2$.

For example, the graph $P^M_n$ will give the structure of grid. David Kuo and Jing-Ho Yan (2004) show that distance two labeling for the product of paths is given as

$$\lambda(P^M_n) = \lambda(P_n \times P_m) = \begin{cases} 5 & \text{if } n = 2 \text{ and } m \geq 3 \\
6 & \text{if } m, n \geq 3.
\end{cases}$$

### 2.4 DISTANCE TWO LABELING FOR SOME CLASS OF CONNECTED GRAPHS $G^M$

In this section distance two labeling for $G^M$ by varying $G$ as cycle, star and planar graph with maximal edges are discussed.

In the following algorithm, we give the construction and distance two labeling for $C^M_n$. Distance two labeling of $C^M_n$ is discussed in two cases. In the first case, distance two labeling for $C^M_n$ with $M < 3$ is given. In the second case, distance two labeling for $C^M_n$ with $M \geq 3$ is given under three sub cases $M \equiv 1 \pmod{3}$, $M \equiv 2 \pmod{3}$ and $M \equiv 0 \pmod{3}$.

**Algorithm 2.4.1**

**Input:** Number of vertices $mn$ of $C^M_n$  
**Output:** Distance two labeling of vertices of the graph $C^M_n$

begin
\[ V(C_n^M) = \left\{ \bigcup_{j=1}^{m} V(C_n^j) : V(C_n^j) = \{ v_1^j, v_2^j, \ldots, v_n^j \} \right\} \]

\[ E(C_n^M) = \bigcup_{j=1}^{m} E(C_n^j) \cup E' \text{ where } E(C_n^j) = \{ v_i^j v_{i+1}^j : 1 \leq i \leq n-1 ; 1 \leq j \leq m \} \cup \{ v_n^j v_1^j \} \]

\[ E' = \{ v_i^j v_{j+1}^j : 1 \leq i \leq n ; 1 \leq k \leq m-1 \} \]

if \((m < 3)\)

for \(j = 1\) to \(2\)

for \(i = 1\) to \(n-1\)

\{ \n
\hspace{1cm} f(v_i^j) = 2(i-1) + 3(j-1); \\
\hspace{1cm} f(v_n^j) = 2(n-1); \\
\hspace{1cm} f(v_n^2) = 1; \\
\}

else if \((m \geq 3)\) and \((m \equiv 1 \mod 3)\)

for \(j = 1\) to \(\frac{m-1}{3}\)

for \(i = 1\) to \(n-1\)

\{ \n
\hspace{1cm} f(v_i^{3j-2}) = f(v_i^m) = 2(i-1); \\
\hspace{1cm} f(v_i^{3j-1}) = 2(i-1) + 3; \\
\hspace{1cm} f(v_i^{3j}) = 2(i-1) + 6; \\
\hspace{1cm} f(v_n^{3j-2}) = f(v_n^m) = 2(n-1); \\
\hspace{1cm} f(v_n^{3j-1}) = 1; \\
\hspace{1cm} f(v_n^{3j}) = 4; \\
\}

else if \((m \geq 3)\) and \((m \equiv 2 \mod 3)\)

for \(j = 1\) to \(\frac{m-2}{3}\)
for $i = 1$ to $n - 1$
{
    $f(v_{i}^{3j-2}) = f(v_{i}^{m-1}) = 2(i - 1)$;
    $f(v_{i}^{3j-1}) = f(v_{i}^{m}) = 2(i - 1) + 3$;
    $f(v_{i}^{3j}) = 2(i - 1) + 6$;
    $f(v_{n}^{3j-2}) = f(v_{n}^{m-1}) = 2(n - 1)$;
    $f(v_{n}^{3j-1}) = f(v_{n}^{m}) = 1$;
    $f(v_{n}^{3j}) = 4$;
}
else if $(m \geq 3)$ and $(m = 0 \mod 3)"
for $j = 1$ to $\frac{m}{3}$
{
for $i = 1$ to $n - 1$
{
    $f(v_{i}^{3j-2}) = 2(i - 1)$;
    $f(v_{i}^{3j-1}) = 2(i - 1) + 3$;
    $f(v_{i}^{3j}) = 2(i - 1) + 6$;
    $f(v_{n}^{3j-2}) = 2(n - 1)$;
    $f(v_{n}^{3j-1}) = 1$;
    $f(v_{n}^{3j}) = 4$;
}
end.

**Theorem 2.4.2.** The $C_{n} \times P_{m}$ graph $C_{n}^{M}$ has distance two labeling with span

$$\hat{\lambda}(C_{n}^{M}) = \hat{\lambda}(C_{n} \times P_{m}) = \begin{cases} 
2n - 1 & \text{for } m < 3 \\
2n + 2 & \text{for } m \geq 3
\end{cases}.$$ 

**Proof:** Consider the graph $C_{n}^{M}$ with vertexset, edge set and the function $f: V(C_{n}^{M}) \rightarrow \{0, 1, 2, \ldots, 2n+2\}$ as defined in the above Algorithm 2.4.1. One
can easily verify that \( \lambda = 2n - 1 \) for \( m < 3 \) using the Algorithm 2.4.1. For \( m \geq 3 \), the theorem is proved in two possible cases.

**Case 1.** If \( d(v_i, v_j) = 1 \) then to prove \( |f(v_i) - f(v_j)| \geq 2 \).

\[ d(v_i, v_j) = 1 \] occur for the edges \( \{v_i^j v_{i+1}^j : 1 \leq i \leq n - 1; 1 \leq j \leq m\} \cup \{v_n^1 v_i^1\} \);

\( \{v_{i+1}^j v_i^j : 2 \leq i \leq n; 1 \leq j \leq m\} \); \( \{v_i^{j+1} v_i^j : 1 \leq i \leq n; 1 \leq j \leq m - 1\} \);

\( \{v_i^j v_i^{j-1} : 1 \leq i \leq n; 2 \leq j \leq m\} \)

By Algorithm 2.4.1, in all cases, we get \( |f(v_n^j) - f(v_{i+1}^j)| \geq 2 \).

\[ |f(v_i^j) - f(v_{i+1}^j)| = 3 \text{ for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m - 1. \]

\[ |f(v_i^j) - f(v_{i+1}^j)| = 3 \text{ for } 1 \leq i \leq n - 1 \text{ and } 2 \leq j \leq m. \]

\[ |f(v_n^j) - f(v_{i+1}^j)| = |f(v_n^j) - f(v_{n+1}^j)| \geq 3 \text{ for } 1 \leq j \leq m - 1 \text{ and } 2 \leq j \leq m. \]

\[ |f(v_i^j) - f(v_{i+1}^j)| \geq 2 \text{ for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m; \]

\[ |f(v_{i+1}^j) - f(v_i^j)| \geq 2 \text{ for } 2 \leq i \leq n \text{ and } 1 \leq j \leq m. \]

Thus for all \( i \) and \( j \), we have \( |f(v_i) - f(v_j)| \geq 2 \).

**Case 2.** If \( d(v_i, v_j) = 2 \) then to prove \( |f(v_i) - f(v_j)| \geq 1 \).

\[ d(v_i, v_j) = 2 \] occur for the edges \( \{v_i^{j-2} v_i^j : 1 \leq i \leq n - 1; 1 \leq j \leq m - 2\} \);

\( \{v_i^{j-2} v_i^j : 1 \leq i \leq n - 1; 3 \leq j \leq m\} \); \( \{v_i^{j-2} v_{i-2}^j, v_i^j v_i^j : 1 \leq i \leq n - 2; 1 \leq j \leq m\} \)

\( \{v_i^{j-1} v_i^j : 3 \leq i \leq n; 1 \leq j \leq m\} \); \( \{v_i^{j+1} v_i^j : 1 \leq i \leq n - 1; 1 \leq j \leq m - 1\} \)

\( \{v_i^{j+1} v_{i+1}^j : 1 \leq i \leq n - 1; 2 \leq j \leq m\} \); \( \{v_i^{j+1} v_{i+1}^j : 2 \leq i \leq n; 1 \leq j \leq m - 1\} \);

\( \{v_i^{j+1} v_{i+1}^j : 2 \leq i \leq n; 2 \leq j \leq m\} \).

all the vertices at distance two have distinct labeling according to the Algorithm 2.4.1. Thus in this case we have \( |f(v_i) - f(v_j)| \geq 1 \) for all \( i, j \).
By Algorithm 2.4.1, the vertices \( v_{n-1}^{3j} \) for \( j \equiv 0 \pmod{3} \) has the maximum label, that is \( f(v_{n-1}^{3j}) = 2n+2 \) for \( j \equiv 0 \pmod{3} \). Hence the graph \( C_n^M \) has distance two labeling and

\[
\lambda(C_n^M) = \lambda(C_n \times P_m) = \begin{cases} 
2n-1 & \text{for } m < 3 \\
2n+2 & \text{for } m \geq 3.
\end{cases}
\]

In the following algorithm, we give the construction and distance two labeling for the graph \( K_{1,n}^M \) for \( n \geq 5 \). The star \( K_{1,n} \) is placed in \( M \) stores and the \( n+1 \) vertices of star linked to the corresponding \( n+1 \) vertices of the stars in succeeding and preceding layers. Next distance two labeling of \( K_{1,n}^M \) is discussed in two cases. In the first case, distance two labeling for \( K_{1,n}^M \) with \( M < 4 \) is given. In the second case, distance two labeling for \( K_{1,n}^M \) with \( M \geq 4 \) is given under four sub cases \( M \equiv 1 \pmod{4}, M \equiv 2 \pmod{4}, M \equiv 3 \pmod{4} \) and \( M \equiv 0 \pmod{4} \).

**Algorithm 2.4.3**

**Input:** Number of vertices of \( K_{1,n}^M \)

**Output:** Distance two labeling of vertices of the graph \( K_{1,n}^M \)

begin

\[ V(K_{1,n}^M) = \bigcup_{j=1}^{m} V(K_{1,n}^{j}) \text{ where } V(K_{1,n}^{j}) = \{v_1^j, v_2^j, \ldots, v_n^j, v_{n+1}^j\} \]

\[ E(K_{1,n}^M) = \bigcup_{j=1}^{m} E(K_{1,n}^{j}) \cup E' \text{ where } E(K_{1,n}^{j}) = \{v_i^j v_{i+1}^j : 1 \leq i \leq n; 1 \leq j \leq m\} \cup \]

\[ E' = \{v_i^j v_{i+1}^j : 1 \leq i \leq n + 1; 1 \leq j \leq m - 1\} \]

if \( m < 4 \)

for \( j = 1 \) to \( m \)
for $i = 1$ to $n$
{
\[ f(v_i^j) = 2(i - 1) + 3(j - 1); \]
\[ f(v_{n-1}^1) = 2(n - 1); \]
\[ f(v_{n-1}^2) = 1; \]
\[ f(v_{n-1}^3) = 4; \]
}

if ($m \geq 4$) and ($m \equiv 1$ (mod$4$))

for $j = 1$ to $\frac{m-1}{4}$

for $i = 1$ to $n$
{
\[ f(v_i^{j-3}) = f(v_i^m) = 2(i - 1); \]
\[ f(v_i^{j-2}) = 2(i - 1) + 3; \]
\[ f(v_i^{j-1}) = 2(i - 1) + 6; \]
\[ f(v_i^j) = 2(i - 1) + 9; \]
\[ f(v_{n-1}^{j-3}) = f(v_{n-1}^m) = 2n; \]
\[ f(v_{n-1}^{j-2}) = 1; \]
\[ f(v_{n-1}^{j-1}) = 4; \]
\[ f(v_{n-1}^j) = 7; \]
}

else if ($m \geq 4$) and ($m \equiv 2$ (mod$4$))

for $j = 1$ to $\frac{m-2}{4}$

for $i = 1$ to $n$
{

\[ f(v_i^{4j-3}) = f(v_i^{m-1}) = 2(i - 1); \]
\[ f(v_i^{4j-2}) = f(v_i^{m}) = 2(i - 1) + 3; \]
\[ f(v_i^{4j-1}) = 2(i - 1) + 6; \]
\[ f(v_i^{4j}) = 2(i - 1) + 9; \]
\[ f(v_{n-1}^{4j-3}) = f(v_{n-1}^{m-1}) = 2n; \]
\[ f(v_{n-1}^{4j-2}) = f(v_{n-1}^{m}) = 1; \]
\[ f(v_{n-1}^{4j-1}) = 4; \]
\[ f(v_{n-1}^{4j}) = 7; \]

} else if \((m \geq 4) \text{ and } (m \equiv 3 \pmod{4})\)

\[ \text{for } j = 1 \text{ to } \frac{m-3}{4} \]
\[ \text{for } i = 1 \text{ to } n \]
\[ \{ \]
\[ f(v_i^{4j-3}) = f(v_i^{m-2}) = 2(i - 1); \]
\[ f(v_i^{4j-2}) = f(v_i^{m-1}) = 2(i - 1) + 3; \]
\[ f(v_i^{4j-1}) = f(v_i^{m}) = 2(i - 1) + 6; \]
\[ f(v_i^{4j}) = 2(i - 1) + 9; \]
\[ f(v_{n-1}^{4j-3}) = f(v_{n-1}^{m-2}) = 2n; \]
\[ f(v_{n-1}^{4j-2}) = f(v_{n-1}^{m-1}) = 1; \]
\[ f(v_{n-1}^{4j-1}) = f(v_{n-1}^{m}) = 4; \]
\[ f(v_{n-1}^{4j}) = 7; \]
\[ \} \]
else if \((m \geq 4)\) and \((m \equiv 0 \pmod{4})\)

\[
\text{for } j = 1 \text{ to } \frac{m}{4}
\]

\[
\text{for } i = 1 \text{ to } n
\]

\[
\{
\]

\[
f(v^{4j-3}_i) = 2(i - 1);
\]

\[
f(v^{4j-2}_i) = 2(i - 1) + 3;
\]

\[
f(v^{4j-1}_i) = 2(i - 1) + 6;
\]

\[
f(v^{4j}_i) = 2(i - 1) + 9;
\]

\[
f(v^{4j-3}_{n-1}) = 2n;
\]

\[
f(v^{4j-2}_{n-1}) = 1;
\]

\[
f(v^{4j-1}_{n-1}) = 4;
\]

\[
f(v^{4j}_{n-1}) = 7;
\]

\[
\}
\]

end.

**Theorem 2.4.4.** The \(K_{1,n} \times P_m\) graph \(K^M_{1,n}\) has distance two labeling with span

\[
\lambda(K^M_{1,n}) = \lambda(K_{1,n} \times P_m) = \begin{cases} 
2n + 1 & \text{for } m = 2 \\
2n + 4 & \text{for } m = 3 \\
2n + 7 & \text{for } m \geq 4
\end{cases}
\]

**Proof:** Consider the graph \(K^M_{1,n}\) with vertex set, edge set and the function \(f: V(K^M_{1,n}) \rightarrow \{0, 1, 2, \ldots, 2n + 7\}\) be defined as in Algorithm 2.4.3. One can easily verify that \(\lambda = 2n+1\) for \(m = 2\) and \(\lambda = 2n+4\) for \(m = 3\) using above Algorithm 2.4.3. For \(m \geq 4\), the theorem is proved in two possible cases.
Case 1. If \( d(v_i, v_j) = 1 \) then to prove \(|f(v_i) - f(v_j)| \geq 2\).

\( d(v_i, v_j) = 1 \) occur for the edges \( \{v_i^j v_j^j : 1 \leq i \leq n; 1 \leq j \leq m\} \) and
\( \{v_i^j v_j^{i+1} : 1 \leq i \leq n+1; 1 \leq j \leq m - 1\}; \{v_i^j v_j^{i-1} : 1 \leq i \leq n-1; 2 \leq j \leq m\} \).

By Algorithm 2.4.3, in all cases,

We get \(|f(v_{i+1}^j) - f(v_i^j)| \geq 2 \) for \( 1 \leq i \leq n \); and \( 1 \leq j \leq m \).

\[ |f(v_i^j) - f(v_{i+1}^j)| = 3 \] for \( 1 \leq i \leq n + 1 \) and \( 1 \leq j \leq m - 1 \).

\[ |f(v_i^j) - f(v_{i-1}^j)| = 3 \] for \( 1 \leq i \leq n + 1 \) and \( 2 \leq j \leq m \).

Thus for all \( i \) and \( j \), we have \(|f(v_i) - f(v_j)| \geq 2\).

Case 2. If \( d(v_i, v_j) = 2 \) then to prove \(|f(v_i) - f(v_j)| \geq 1\).

\( d(v_i, v_j) = 2 \) occur for the edges \( \{v_i^j v_{i+1}^j : 1 \leq i \leq n; 1 \leq j \leq m - 1\} \),
\( \{v_i^j v_{i-1}^j : 1 \leq i \leq n; 2 \leq j \leq m\}; \{v_i^j v_{i+2}^j : 1 \leq i \leq n + 1; 1 \leq j \leq m - 2\} \); 
\( \{v_i^j v_{i-2}^j : 1 \leq i \leq n + 1; 3 \leq j \leq m\} \) and \( \{v_i^j v_{i+k}^j : 1 \leq i, k \leq n \) and \( i \neq k; 1 \leq j \leq m\} \).

All the vertices at distance two have distinct labeling. Thus in this case we have \(|f(v_i) - f(v_j)| \geq 1 \) for all \( i \) and \( j \). By Algorithm 2.4.3, the vertices \( v_n^{4j} \), for \( j = 0 \) (mod 4) has the maximum label, that is \( f(v_n^{4j}) = 2n + 7 \) for \( j = 0 \) (mod 4). Hence the graph \( K_{1,n}^M \) has distance two labeling and

\[
\lambda(K_{1,n}^M) = \begin{cases} 
2n + 1 & \text{for } m = 2 \\
2n + 4 & \text{for } m = 3 \\
2n + 7 & \text{for } m \geq 4. 
\end{cases}
\]

In the following algorithm, we give the construction and distance two labeling for \( Pl_n^M \) for \( n > 5 \). The planar graph \( Pl_n \) is placed in \( M \) stores and the \( n \) vertices of \( Pl_n \) linked to the corresponding \( n \) vertices of \( Pl_n \) in succeeding and preceding layers. The distance two labeling of \( Pl_n^M \) is discussed in two cases. In the first case, distance two labeling for \( Pl_n^M \) with \( M < 4 \) is given. In
the second case, distance two labeling for $Pl_n^M$ with $M \geq 4$ is given under four sub cases $M \equiv 1 \pmod{4}$, $M \equiv 2 \pmod{4}$, $M \equiv 3 \pmod{4}$ and $M \equiv 0 \pmod{4}$.

Algorithm 2.4.5.

**Input:** Number of vertices $mn$ of $Pl_n^M$

**Output:** Distance two labeling of vertices of the graph $Pl_n^M$

begin

$$V(Pl_n^M) = \bigcup_{j=1}^{m} V(Pl_n^j) : V(Pl_n^j) = \{v_1^j, v_2^j, ..., v_n^j\}$$

$$E(Pl_n^M) = \bigcup_{j=1}^{m} E(Pl_n^j) \cup E'$$

where $E(Pl_n^j) = \{v_i^j v_{i+1}^j, v_i^j v_{i+2}^j : 1 \leq i \leq n - 2\} \cup \{v_i^j v_{i+1}^j : 1 \leq k \leq n - 3\} \cup \{v_{n-1}^j v_n^j\}$

$$E' = \{v_i^j v_{i+1}^j : 1 \leq i \leq n; 1 \leq k \leq m - 1\}$$

if $(m < 4)$

for $j = 1$ to $m$

for $i = 1$ to $n - 1$

$$f(v_i^j) = 2(i - 1) + 3(j - 1);$$

$$f(v_n^j) = 1;$$

$$f(v_i^2) = 4;$$

$$f(v_i^3) = 7;$$

} if $(m \geq 4)$ and $(m = 1 \pmod{4})$

for $j = 1$ to $\frac{m-1}{4}$
for $i = 1$ to $n - 1$

\[
\begin{align*}
  f(v_i^{4j-3}) &= f(v_i^{m}) = 2(i - 1); \\
  f(v_i^{4j-2}) &= 2(i - 1) + 3; \\
  f(v_i^{4j-1}) &= 2(i - 1) + 6; \\
  f(v_i^{4j}) &= 2(i - 1) + 9; \\
  f(v_n^{4j-3}) &= f(v_n^{m}) = 2(n - 1); \\
  f(v_n^{4j-2}) &= 1; \\
  f(v_n^{4j-1}) &= 4; \\
  f(v_n^{4j}) &= 7;
\end{align*}
\]

} else if \((m \geq 4)\) and \((m \equiv 2(\text{mod}4))\)

for \(j = 1\) to \(\frac{m - 2}{4}\)

for \(i = 1\) to \(n - 1\)

\[
\begin{align*}
  f(v_i^{4j-3}) &= f(v_i^{m-1}) = 2(i - 1); \\
  f(v_i^{4j-2}) &= f(v_i^{m}) = 2(i - 1) + 3; \\
  f(v_i^{4j-1}) &= 2(i - 1) + 6; \\
  f(v_i^{4j}) &= 2(i - 1) + 9; \\
  f(v_n^{4j-3}) &= f(v_n^{m-1}) = 2(n - 1); \\
  f(v_n^{4j-2}) &= f(v_n^{m}) = 1; \\
  f(v_n^{4j-1}) &= 4; \\
  f(v_n^{4j}) &= 7;
\end{align*}
\]
else if \((m \geq 4)\) and \((m \equiv 3 \text{(mod4)})\)

\[
\text{for } j = 1 \text{ to } \frac{m-3}{4}
\]

\[
\text{for } i = 1 \text{ to } n - 1
\]

\{

\begin{align*}
    f(v^{4j-3}_i) &= f(v^{m-2}_i) = 2(i-1); \\
    f(v^{4j-2}_i) &= f(v^{m-1}_i) = 2(i-1) + 3; \\
    f(v^{4j-1}_i) &= f(v^m_i) = 2(i-1) + 6; \\
    f(v^4_i) &= 2(i-1) + 9; \\
    f(v^{4j-3}_n) &= f(v^{m-2}_n) = 2(n-1); \\
    f(v^{4j-2}_n) &= f(v^{m-1}_n) = 1; \\
    f(v^{4j-1}_n) &= f(v^m_n) = 4; \\
    f(v^4_n) &= 7;
\end{align*}
\}

else if \((m \geq 4)\) and \((m \equiv 0 \text{(mod4)})\)

\[
\text{for } j = 1 \text{ to } \frac{m}{4}
\]

\[
\text{for } i = 1 \text{ to } n - 1
\]

\{

\begin{align*}
    f(v^{4j-3}_i) &= 2(i-1); \\
    f(v^{4j-2}_i) &= 2(i-1) + 3; \\
    f(v^{4j-1}_i) &= 2(i-1) + 6; \\
    f(v^4_i) &= 2(i-1) + 9; \\
    f(v^{4j-3}_n) &= 2(n-1); \\
    f(v^{4j-2}_n) &= 1;
\end{align*}
\}
\[ f(v_{n-1}^i) = 4; \]
\[ f(v_n^i) = 7; \]
}
end.

**Theorem 2.4.6.** The \( P_l \times P_m \) graph \( P^M_l \) has distance two labeling with span

\[ \lambda(PL^M_n) = \lambda(P_l \times P_m) = \begin{cases} 
2n - 1 & \text{for } m = 2 \\
2n + 2 & \text{for } m = 3 \\
2n + 5 & \text{for } m \geq 4.
\end{cases} \]

**Proof:** Consider the graph \( P^M_l \) with vertex set, edge set and the function \( f : V(PL^M_l) \to \{0, 1, 2, \ldots, 2n+5\} \) be defined as in Algorithm 2.4.5. One can easily verify that \( \lambda = 2n - 1 \) for \( m = 2 \) and \( \lambda = 2n+2 \) for \( m = 3 \) using above Algorithm 2.4.5. For \( m \geq 4 \), the theorem is proved in two possible cases.

**Case 1.** If \( d(v_i, v_j) = 1 \) then to prove \( |f(v_i) - f(v_j)| \geq 2 \).

\( d(v_i, v_j) = 1 \) occur for the edges \( \{v_i^j, v_{i+1}^j, v_{i-1}^j, v_n^j : 1 \leq i \leq n - 3; 1 \leq j \leq m\} \);

\( \{v_i^{j+1} : 1 \leq i \leq n - 3; 1 \leq j \leq m\} \);

\( \{v_i^j : 1 \leq k \leq m\} \);

\( \{v_i^j, v_{i-1}^j : 1 \leq i \leq n; 1 \leq j \leq m-1\} \) and

\( \{v_i^j, v_{i+1}^j : 1 \leq i \leq n; 2 \leq j \leq m\} \)

By Algorithm 2.4.5, We get \( |f(v_i^j) - f(v_{i-1}^j)| \geq 2 \) and \( |f(v_i^j) - f(v_n^j)| \geq 2 \) for \( 1 \leq i \leq n - 2 \) and \( 1 \leq j \leq m \).

\[ |f(v_i^j) - f(v_{i+1}^j)| = 2 \text{ for } 1 \leq i \leq n - 3 \text{ and } 1 \leq j \leq m; \]
\[ |f(v_i^j) - f(v_{i-1}^j)| = 2 \text{ for } 2 \leq i \leq n - 2 \text{ and } 1 \leq j \leq m. \]
\[ |f(v_{i-1}^j) - f(v_n^j)| = 2 \text{ for } 1 \leq j \leq m. \]
\[ |f(v_n^j) - f(v_{i+1}^j)| \geq 2 \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m - 1. \]
\[ |f(v_i) - f(v_{i+1})| \geq 2 \text{ for } 1 \leq i \leq n \text{ and } 2 \leq j \leq m. \]

Thus for all \( i \) and \( j \), we have \( |f(v_i) - f(v_j)| \geq 2 \).

**Case 2.** If \( d(v_i, v_j) = 2 \) then to prove \( |f(v_i) - f(v_j)| \geq 1 \).

\( d(v_i, v_j) = 2 \) occur for the edges
\[ \{v_i^j v_{i+2}^j : 1 \leq i \leq n - 4; 1 \leq j \leq m\}; \{v_i^j v_{i+2}^j : 3 \leq i \leq n - 4; 1 \leq j \leq m\}; \]
\[ \{v_i^j v_{i+1}^j : 1 \leq i \leq n - 3; 1 \leq j \leq m - 1\}; \{v_i^j v_{i+1}^j : 2 \leq i \leq n - 3; 1 \leq j \leq m - 1\} \]
\[ \{v_i^j v_{i+1}^j : 1 \leq i \leq n - 3; 2 \leq j \leq m\}; \{v_i^j v_{i+1}^j : 2 \leq i \leq n - 3; 2 \leq j \leq m\} \]
\[ \{v_n^j v_n^{j+1} ; v_n^j v_{n+1}^{j+1} : 1 \leq j \leq m - 1\}; \{v_n^j v_n^{j+1} ; v_n^j v_{n+1}^{j+1} : 2 \leq j \leq m\} \text{ and} \]
\[ \{v_i^j v_{i+2}^j : 1 \leq i \leq n; 1 \leq j \leq m - 2\}; \{v_i^j v_{i+2}^j : 1 \leq i \leq n; 3 \leq j \leq m\}. \]

All the vertices at difference two have distinct labeling. Thus in this case we have \( |f(v_i) - f(v_j)| \geq 1 \) for all \( i \) and \( j \). By Algorithm 2.4.5, the vertices \( v_{n-1}^{4j}, j \equiv 0 \text{ (mod 4)} \) has the maximum label, that is \( f(v_{n-1}^{4j}) = 2n+5 \), for \( j \equiv 0 \text{ (mod 4)} \). Hence the \( Pl_n \times P_m \) graph \( Pl_n^M \) has distance two labeling and

\[ \hat{\lambda}(Pl_n^M) = \hat{\lambda}(Pl_n \times P_m) = \begin{cases} 2n-1 & \text{for } m = 2 \\ 2n+2 & \text{for } m = 3 \\ 2n+5 & \text{for } m \geq 4. \end{cases} \]

In \( K_n \times P_m \) graph \( K_n^M \), the complete graph \( K_n \) is placed in \( M \) stores and the \( n \) vertices of \( K_n \) linked to the corresponding \( n \) vertices of the planar \( K_n \)'s in succeeding layers. The distance two labeling of \( K_n^M \) for \( n > 5 \) is discussed in two cases. In the first case, distance two labeling for the graph \( K_n^M \) with \( M < 4 \) is given. In the second case, distance two labeling for the graph \( K_n^M \) with \( M \geq 4 \) is given under four sub cases \( M \equiv 1 \text{ (mod 4)} \), \( M \equiv 2 \text{ (mod 4)} \), \( M \equiv 3 \text{ (mod 4)} \) and \( M \equiv 0 \text{ (mod 4)} \). The labeling of the vertices of \( K_n^M \) is same as given in Algorithm 2.4.5.
**Theorem 2.4.7.** The $K_n \times P_m$ graph $K_n^M$ has distance two labeling with

$$\text{span } \lambda(K_n^M) = \lambda(K_n \times P_m) = \begin{cases} 2n-1 & \text{for } m = 2 \\ 2n+2 & \text{for } m = 3 \\ 2n+5 & \text{for } m \geq 4. \end{cases}$$

**Theorem 2.4.8.** If $G$ is a simple graph with atleast one vertex having maximum degree $|V|-1$ and $|V| \leq 5$, then $G \times P_m$ that is $G^M$ will have distance two labeling.

**Proof:** Consider the simple graph $G$ with atleast one vertex have maximum degree $|V|-1$ and $|V| \leq 5$. The distance two labeling for $G \times P_m$ that is $G^M$ will be given as follows

**Case 1.** For $M \leq 4$ and $|V| \leq 5$

When $M = 2$, the labeling numbers will be $\{0, 2, 4, 6, 8\}; \{3, 5, 7, 9, 1\}$. Thus the maximum number used for two stores is $\lambda \leq 9$ for $|V| \leq 5$.

When $M = 3$, the labeling numbers will be $\{0, 2, 4, 6, 8\}; \{3, 5, 7, 9, 1\}; \{6, 8, 10, 12, 4\}$. Thus the maximum number used for two stores is $\lambda \leq 12$ for $|V| \leq 5$.

When $M = 4$, the labeling numbers will be $\{0, 2, 4, 6, 8\}; \{3, 5, 7, 9, 1\}; \{6, 8, 10, 12, 4\}; \{9, 11, 13, 15, 7\}$.

Thus the maximum number used for two stores is $\lambda \leq 15$ for $|V| \leq 5$.

**Case 2.** For $M > 4$

When $M = 1 \pmod{4}$, the labeling numbers will be $\{0, 2, 4, 6, 8\}$.

When $M = 2 \pmod{4}$, the labeling numbers will be $\{3, 5, 7, 9, 11\}$.

When $M = 3 \pmod{4}$, the labeling numbers will be $\{6, 8, 10, 12, 14\}$.

When $M = 0 \pmod{4}$, the labeling numbers will be $\{9, 11, 13, 15, 17\}$.
Then the maximum labeling number used or span is \( \lambda \leq 2\Delta_M + 5 \) for \( G^M \), where 
\( G \) is the graph with at least one vertex having maximum degree 
\( |V| - 1 \) and \( |V| \leq 5 \), where \( \Delta_M = |V| + 1 \).

\[ \square \]

2.5 DISTANCE TWO LABELING OF \( (P_n \times P_n)^M \)

In this section, we investigate distance two labeling for \( P_n \times P_n \times P_m \) that is \( (P_n \times P_n)^M \).

![Figure 2.4 Construction of \( (P_n \times P_n)^4 \)]](image)

The Construction of the graph \( (P_n \times P_n)^M \) is shown in Figure 2.4.
Example 2.5.1.

Distance two labeling of \((P_n \times P_n)^5\) is shown in below Matrix form

Stores \(M = 1\)

\[
\begin{pmatrix}
0 & 2 & 4 & 0 & 2 \\
6 & 8 & 10 & 6 & 8 \\
12 & 14 & 16 & 12 & 14 \\
0 & 2 & 4 & 0 & 2 \\
6 & 8 & 10 & 6 & 8
\end{pmatrix}
\begin{pmatrix}
3 & 5 & 7 & 3 & 5 \\
9 & 11 & 13 & 9 & 11 \\
15 & 17 & 19 & 15 & 17 \\
3 & 5 & 7 & 3 & 5 \\
9 & 11 & 13 & 9 & 11
\end{pmatrix}
\begin{pmatrix}
6 & 8 & 10 & 6 & 8 \\
12 & 14 & 16 & 12 & 14 \\
18 & 20 & 22 & 18 & 20 \\
6 & 8 & 10 & 6 & 8 \\
12 & 14 & 16 & 12 & 14
\end{pmatrix}
\]

Stores \(M = 4\)

\[
\begin{pmatrix}
9 & 11 & 13 & 9 & 11 \\
15 & 17 & 19 & 15 & 17 \\
21 & 23 & 25 & 21 & 23 \\
9 & 11 & 13 & 9 & 11 \\
15 & 17 & 19 & 15 & 17
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 4 & 0 & 2 \\
6 & 8 & 10 & 6 & 8 \\
12 & 14 & 16 & 12 & 14 \\
0 & 2 & 4 & 0 & 2 \\
6 & 8 & 10 & 6 & 8
\end{pmatrix}
\]

The entries of the matrices are the labels given to the vertices of the graph. There are 5 matrices which represent the labeling of the five different layers of the \((P_n \times P_n)^M\) grid. The \(ij^{th}\) entry of the \(k^{th}\) matrices represent the distance two labeling of \(ij^{th}\) vertex lying in the \(k^{th}\) layer of the \((P_n \times P_n)^M\) square grid. After the 4th store the same values of the first four levels will be repeated. In general for all values of \(n \geq 3\) and \(M \geq 4\) the maximum value used to label (span) the \((P_n \times P_n)^M\) graph is \(\lambda = 25\).

Distance two labeling of first store in the \((P_n \times P_n)^M\) grid is given in below Algorithm 2.5.2. In this algorithm, the distance two labeling for grid \(P_n \times P_n\) is given in three cases \(n \equiv 1 \pmod{3}\), \(n \equiv 2 \pmod{3}\) and \(n \equiv 0 \pmod{3}\).
Algorithm 2.5.2

**Input:** Number of vertices of $P_n \times P_n$

**Output:** Distance two labeling $f(v_{ij})$ of $n^2$ vertices of the graph $P_n \times P_n$

begin

$V(P_{n,n}) = \bigcup_{i,j=1}^{n} v_{ij}$;

$E(P_{n,n}) = \{v_{ij}, v_{ij-1} : 1 \leq i \leq n; 1 \leq j \leq n-1\} \cup \{v_{ij}, v_{i,j+1} : 1 \leq i \leq n-1; 1 \leq j \leq n\};$

if $(n \geq 3)$ and $(n \equiv 1 \pmod{3})$

for $i = 1$ to $\frac{n-1}{3}$

for $j = 1$ to $\frac{n-1}{3}$

{}

$f(v_{3i-2,3j-2}) = f(v_{n,3j-2}) = f(v_{3i-2,n}) = f(v_{n,n}) = 0;$

$f(v_{3i-2,3j-1}) = f(v_{n,3j-1}) = 2;$

$f(v_{3i-2,3j}) = f(v_{n,3j}) = 4;$

$f(v_{3i-1,3j-2}) = f(v_{3i-1,3j-1}) = 6;$

$f(v_{3i-1,3j-1}) = 8;$

$f(v_{3i-1,3j}) = 10;$

$f(v_{3i-2,3j}) = f(v_{3i,n}) = 12;$

$f(v_{3i,3j-1}) = 14;$

$f(v_{3i,3j}) = 16;$

}

elseif $(n \geq 3)$ and $(n \equiv 2 \pmod{3})$

for $i = 1$ to $\frac{n-2}{3}$


for \( j = 1 \) to \( \frac{n-2}{3} \)

\[
\{ \\
\begin{align*}
  f(v_{3i-2,3j-2}) &= f(v_{n-1,3j-2}) = f(v_{3i-2,n-1}) = f(v_{n-1,n-1}) = 0; \\
  f(v_{3i-2,3j-1}) &= f(v_{3i-2,n}) = f(v_{n-1,n}) = 2; \\
  f(v_{3i-2,3j}) &= f(v_{n-1,3j}) = 4; \\
  f(v_{3i-1,3j-2}) &= f(v_{3i-1,n-1}) = f(v_{n,3j-2}) = f(v_{n,n-1}) = 6; \\
  f(v_{3i-1,3j-1}) &= f(v_{3i-1,n}) = f(v_{n,3j-1}) = f(v_{n,n}) = 8; \\
  f(v_{3i-1,3j}) &= f(v_{n,3j}) = 10; \\
  f(v_{3i,3j-2}) &= f(v_{3i,n-1}) = 12; \\
  f(v_{3i,3j-1}) &= f(v_{3i,n}) = 14; \\
  f(v_{3i,3j}) &= 16; \\
\} \\
\text{elseif} \ (n \geq 3) \text{ and } (n \equiv 0 \text{ (mod} 3) )
\]

for \( i = 1 \) to \( \frac{n}{3} \)

for \( j = 1 \) to \( \frac{n}{3} \)

\[
\{ \\
\begin{align*}
  f(v_{3i-2,3j-2}) &= 0; \\
  f(v_{3i-2,3j-1}) &= 2; \\
  f(v_{3i-2,3j}) &= 4; \\
  f(v_{3i-1,3j-2}) &= 6; \\
  f(v_{3i-1,3j-1}) &= 8; \\
  f(v_{3i-1,3j}) &= 10; \\
\} \\
\]
\[ f(v_{3,3j-2}) = 12; \]
\[ f(v_{3,3j-1}) = 14; \]
\[ f(v_{3,3j}) = 16; \]

end.

In Algorithm 2.5.3, we give the construction of \((P_n \times P_n)^M\) grid and assign the distance two labeling. The grids \(P_n \times P_n\) are placed in \(M\) stores and the \(n\) vertices of grid linked to the corresponding \(n\) vertices of the grids in succeeding and preceding layers. In the first step we define vertex set and edge sets. In the successive steps 2 to 5 we assign distance two labeling of \((P_n \times P_n)^M\), \(M \geq 4\) for four cases \(M \equiv 1 \pmod{4}\), Error! Objects cannot be created from editing field codes., \(M \equiv 3 \pmod{4}\) and \(M \equiv 0 \pmod{4}\) respectively.

Algorithm 2.5.3

Input: Number of \(mn^2\) vertices of \((P_n \times P_n)^M\)

Output: Distance two labelling \(f(v_{ij}^m)\) of \(mn^2\) vertices of the graph \((P_n \times P_n)^M\)

begin

Step 1: Take vertex and edge set of grid as \(V(P_{n,n}^M) = \left\{ \bigcup_{k=1}^{m} V(P_{n,n}^k) : V(P_{n,n}^k) = \bigcup_{i,j,k} v_{ij}^k \right\} \)

\[
E(P_{n,n}^M) = \bigcup_{k=1}^{m} E(P_{n,n}^k) \bigcup E' \text{ where}
\]

\[
E(P_{n,n}^k) = \left\{ v_{ij}^k v_{ij-1}^k : 1 \leq i \leq n; 1 \leq j \leq n - 1 \right\} \cup \left\{ v_{ij}^k v_{ij+1}^k : 1 \leq i \leq n - 1; 1 \leq j \leq n \right\}
\]

\[
E' = \left\{ v_{ij}^k v_{ij}^{k-1} : 1 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m - 1 \right\}
\]
Step 2: Consider the distance two labeling $f(v_{ij})$ from the above Algorithm 2.5.2., When $m \geq 4$ and $m \equiv 1 \pmod{4}$, the labeling of $(P_n \times P_n)^M$ grid will be as follows

$$\text{for } k = 1 \text{ to } \frac{m-1}{4}$$

$$\{$$
$$f(v_{ij}^{4k-3}) = f(v_{ij}^m) = f(v_{ij});$$
$$f(v_{ij}^{4k-2}) = f(v_{ij}) + 3;$$
$$f(v_{ij}^{4k-1}) = f(v_{ij}) + 6;$$
$$f(v_{ij}^{4k}) = f(v_{ij}) + 9;$$

$$\}$$

Step 3: When $m \geq 4$ and $m \equiv 2 \pmod{4}$, the labeling of $(P_n \times P_n)^M$ grid will be as follows

$$\text{for } k = 1 \text{ to } \frac{m-2}{4}$$

$$\{$$
$$f(v_{ij}^{4k-3}) = f(v_{ij}^{m-1}) = f(v_{ij});$$
$$f(v_{ij}^{4k-2}) = f(v_{ij}^m) = f(v_{ij}) + 3;$$
$$f(v_{ij}^{4k-1}) = f(v_{ij}) + 6;$$
$$f(v_{ij}^{4k}) = f(v_{ij}) + 9;$$

$$\}$$
Step 4: When $m \geq 4$ and $m \equiv 3 \text{(mod 4)}$, the labeling of $(P_n \times P_n)^M$ grid will be as follows

for $k = 1$ to $\frac{m-3}{4}$

\[
\begin{align*}
\{ & f(v_{ij}^{4k-3}) = f(v_{ij}^m) = f(v_{ij}); \\
& f(v_{ij}^{4k-2}) = f(v_{ij}^{m-1}) = f(v_{ij}) + 3; \\
& f(v_{ij}^{4k-1}) = f(v_{ij}^m) = f(v_{ij}) + 6; \\
& f(v_{ij}^{4k}) = f(v_{ij}) + 9; \\
\}
\]

Step 5: When $m \geq 4$ and $m \equiv 0 \text{(mod 4)}$, the labeling of $(P_n \times P_n)^M$ grid will be as follows

for $k = 1$ to $\frac{m}{4}$

\[
\begin{align*}
\{ & f(v_{ij}^{4k-3}) = f(v_{ij}); \\
& f(v_{ij}^{4k-2}) = f(v_{ij}) + 3; \\
& f(v_{ij}^{4k-1}) = f(v_{ij}^m) = f(v_{ij}) + 6; \\
& f(v_{ij}^{4k}) = f(v_{ij}) + 9; \\
\}
\]

end.
Theorem 2.5.4. The $P_n\times P_n\times P_m$ graph $(P_n\times P_n)^M$ grid, $M \geq 4$ has distance two labeling and with span $\lambda((P_n\times P_n)^M) = 25$.

Proof: Consider the graph $(P_n\times P_n)^M$ with the vertex and edge set defined as in Algorithm 2.5.3. Let the function $f : V((P_n\times P_n)^M) \rightarrow \{0, 1, 2, \ldots, 25\}$ be defined as in Algorithm 2.5.3. This theorem is proved in possible two cases.

Case 1. If $d(v_i, v_j) = 1$ then to prove $|f(v_i) - f(v_j)| \geq 2$.

$d(v_i, v_j) = 1$ occur for the edges $\{v_i^k v_j^k : 1 \leq i \leq n; 1 \leq j \leq n - 1; 1 \leq k \leq m\}$;

$\{v_i^k v_{i+1}^k : 1 \leq i \leq n; 2 \leq j \leq n; 1 \leq k \leq m\}; \{v_i^k v_{i-1}^k : 1 \leq i \leq n - 1; 1 \leq j \leq n; 1 \leq k \leq m\};$

$\{v_i^k v_{i+1}^k : 2 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m\}; \{v_i^k v_{i-1}^k : 1 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m - 1\}$ and

$\{v_i^k v_{i+1}^k : 1 \leq i \leq n; 1 \leq j \leq n; 2 \leq k \leq m\}$.

By Algorithm 2.5.3, in all cases,

We get $|f(v_i^k) - f(v_{i+1}^k)| \geq 2$ for $1 \leq i \leq n; 1 \leq j \leq n - 1$ and $1 \leq k \leq m$.

$|f(v_i^k) - f(v_{i+1}^k)| \geq 2$ for $1 \leq i \leq n; 2 \leq j \leq n$ and $1 \leq k \leq m$.

$|f(v_i^k) - f(v_{i+1}^k)| > 2$ for $1 \leq i \leq n - 1; 1 \leq j \leq n$ and $1 \leq k \leq m$.

$|f(v_i^k) - f(v_{i-1}^k)| > 2$ for $1 \leq i \leq n; 2 \leq j \leq n$ and $1 \leq k \leq m$.

$|f(v_i^k) - f(v_{i+1}^{k+1})| = 3$ for $1 \leq i \leq n; 1 \leq j \leq n$ and $1 \leq k \leq m - 1$.

$|f(v_i^k) - f(v_{i-1}^{k+1})| = 3$ for $1 \leq i \leq n; 1 \leq j \leq n$ and $2 \leq k \leq m$.

Thus for all $i$ and $j$, we have $|f(v_i) - f(v_j)| \geq 2$.

Case 2. If $d(v_i, v_j) = 2$ then to prove $|f(v_i) - f(v_j)| \geq 1$.

$d(v_i, v_j) = 2$ occur for the edges $\{v_i^k v_{j-2}^k : 1 \leq i \leq n; 1 \leq j \leq n - 2; 1 \leq k \leq m\}$;

$\{v_i^k v_{j-2}^k : 1 \leq i \leq n; 3 \leq j \leq n; 1 \leq k \leq m\}$;

$\{v_i^k v_{j-2}^k : 1 \leq i \leq n - 2; 1 \leq j \leq n; 1 \leq k \leq m\}$;
\begin{align*}
\{v^k_y v^k_{i-2j} : 3 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m\}; \\
\{v^k_y v^k_{i-1j} : 1 \leq i \leq n; 1 \leq j \leq n-1; 1 \leq k \leq m-1\}; \\
\{v^k_y v^{k-1}_{i-1j} : 1 \leq i \leq n; 2 \leq j \leq n; 1 \leq k \leq m-1\}; \\
\{v^k_y v^{k-1}_{i-1j} : 1 \leq i \leq n-1; 1 \leq j \leq n; 1 \leq k \leq m-1\}; \\
\{v^k_y v^{k-1}_{i-1j} : 2 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m-1\}; \\
\{v^k_y v^{k-1}_{i-1j} : 1 \leq i \leq n; 2 \leq j \leq n; 2 \leq k \leq m\}; \\
\{v^k_y v^{k-1}_{i-1j} : 1 \leq i \leq n-1; 1 \leq j \leq n; 2 \leq k \leq m\}; \\
\{v^k_y v^{k-1}_{i-1j} : 2 \leq i \leq n; 1 \leq j \leq n; 2 \leq k \leq m\}; \\
\{v^k_y v^{k-2}_{i-1j} : 1 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m-2\}; \\
\{v^k_y v^{k-2}_{i-1j} : 1 \leq i \leq n; 1 \leq j \leq n; 3 \leq k \leq m\}.
\end{align*}

All the vertices at difference two have distinct labeling. Thus in this case we have \(|f(v_i) - f(v_j)| \geq 1\) for all \(i\) and \(j\). By Algorithm 2.5.3, the maximum labeling number used is 25. Hence the graph \((P_n \times P_n)^M\) has distance two labeling and \(\lambda((P_n \times P_n)^M) = 25\). \(\square\)

### 2.6 UPPER BOUND SPAN FOR \(G^M\) SIMPLÊE GRAPHS

The maximum degree for the \(G \times P_m\) graph \(G^M\) is given as

\[
\Delta_M = \begin{cases} 
\Delta + 1 & \text{if } M = 2 \\
\Delta + 2 & \text{if } M \geq 3
\end{cases} \leq |V| + 1, \tag{2.1}
\]
where $\Delta$ is the maximum degree of the simple graph $G$ and $|V|$ is the number of vertices in the simple graph $G$.

Griggs and Yeh (1992) has given upper bound span for any graph $G$ with maximum degree $\Delta$ to be $\lambda(G) \leq \Delta^2 + 2\Delta$ \hspace{1cm} (2.2)

By (2.1), $\Delta = \Delta_M - 2$. Substituting $\Delta$ in (2.2) we get,

$$\lambda(G^M) \leq \Delta^2 + 2\Delta = (\Delta_M - 2)^2 + 2(\Delta_M - 2) = \Delta_M^2 - 2\Delta_M.$$ 

Hence the upper bound span for $G^M$ in terms of maximum degree $\Delta_M$ is

$$\lambda(G^M) \leq \Delta_M^2 - 2\Delta_M.$$ 

2.7 $L(3, 2, 1)$ LABELING FOR $G^M$ SIMPLE GRAPHS

In this section we find $L(3, 2, 1)$ Labeling for square grids and $(P_n \times P_n)^M$ grids. Also upper bound span values for $G^M$ simple graphs are discussed.

The graph $P_n^M$ will give the structure of grid $P_m \times P_n$. A two-dimensional grid graph is a graph which is cartesian product of the path $P_m$ and $P_n$. A square grid is a grid formed by tiling the plane regularly with squares. In the following algorithm $L(3, 2, 1)$ labeling for square grid $P_n \times P_n$ is discussed in following four cases for $n \geq 4$ and $n = 1 (mod 4)$, $n = 2 (mod 4)$, $n = 3 (mod 4)$, $n = 0 (mod 4)$ respectively.

**Algorithm 2.7.1**

**Input**: Number of $n^2$ vertices of $P_n \times P_n$

**Output**: $L(3, 2, 1)$- labeling $f(v_{i,j})$ of $n^2$ vertices of the graph $P_n \times P_n$

begin

$$V(P_n \times P_n) = \bigcup_{i,j=1}^{n} V_{i,j};$$


$$E(P_{n,n}) = \{v_{i,j} \mid 1 \leq i \leq n; 1 \leq j \leq n-1\} \cup \{v_{i,j} \mid 1 \leq i \leq n-1; 1 \leq j \leq n\}$$

if \((n \geq 4)\) and \((n = 1 \text{mod } 4)\)

for \(i = 1\) to \(\frac{n-1}{4}\)

for \(j = 1\) to \(\frac{n-1}{4}\)

\[
\begin{align*}
f(v_{4i-3,4j-3}) &= f(v_{n-4,4j-3}) = f(v_{4i-3,n}) = f(v_{n-4,n}) = 1; \\
f(v_{4i-3,4j-2}) &= f(v_{n-4,4j-2}) = 4; \\
f(v_{4i-3,4j-1}) &= f(v_{n-4,4j-1}) = 7; \\
f(v_{4i-3,4j}) &= f(v_{n-4,4j}) = 10; \\
f(v_{4i-2,4j-3}) &= f(v_{4i-2,n}) = 6; \\
f(v_{4i-2,4j-2}) &= 9; f(v_{4i-2,4j-1}) = 12; f(v_{4i-2,4j}) = 15; \\
f(v_{4i-1,4j-3}) &= f(v_{4i-1,n}) = 11; \\
f(v_{4i-1,4j-2}) &= 14; f(v_{4i-1,4j-1}) = 17; f(v_{4i-1,4j}) = 20; \\
f(v_{4i,4j-3}) &= f(v_{4i,n}) = 16; \\
f(v_{4i,4j-2}) &= 19; f(v_{4i,4j-1}) = 2; f(v_{4i,4j}) = 13;
\end{align*}
\]

elseif \((n \geq 4)\) and \((n = 2 \text{mod } 4)\)

for \(i = 1\) to \(\frac{n-2}{4}\)

for \(j = 1\) to \(\frac{n-2}{4}\)

\[
\begin{align*}
f(v_{4i-3,4j-3}) &= f(v_{n-4,4j-3}) = f(v_{4i-3,n}) = f(v_{n-4,n}) = 1; \\
f(v_{4i-3,4j-2}) &= f(v_{n-4,4j-2}) = 4; \\
f(v_{4i-3,4j-1}) &= f(v_{n-4,4j-1}) = 7;
\end{align*}
\]
\[ f(v_{4i,3,j}) = f(v_{n-1,i,j}) = 10; \]
\[ f(v_{4i-2,4,j}) = f(v_{4i-2,n-1}) = f(v_{n,4,j-3}) = f(v_{n,n-1}) = 6; \]
\[ f(v_{4i-2,4,j-2}) = f(v_{4i-2,n}) = f(v_{n,4j-2}) = f(v_{n,n}) = 9; \]
\[ f(v_{4i-2,4,j-4}) = f(v_{n,4j-4}) = 12; \]
\[ f(v_{4i-2,4,j}) = f(v_{n,4j}) = 15; \]
\[ f(v_{4i-1,4,j-3}) = f(v_{4i-1,n-1}) = 11; \]
\[ f(v_{4i-1,4,j-2}) = f(v_{4i-1,n}) = 14; \]
\[ f(v_{4i-1,4,j-1}) = f(v_{4i-1,n-1}) = 20; \]
\[ f(v_{4i-1,4,j}) = f(v_{4i,n-1}) = 16; \]
\[ f(v_{4i,4,j-2}) = f(v_{4i,n}) = 19; \]
\[ f(v_{4i,4,j-1}) = f(v_{4i,n-1}) = 13; \]

elseif \((n \geq 4)\) and \((n = 3(\text{mod} 4))\)

for \(i = 1\) to \(\frac{n-3}{4}\)

for \(j = 1\) to \(\frac{n-3}{4}\)

\[ f(v_{4i-3,4,j-3}) = f(v_{n-2,4,j-3}) = f(v_{4i-3,n-2}) = f(v_{n-2,n-2}) = 1; \]
\[ f(v_{4i-3,4,j-2}) = f(v_{4i-3,n-1}) = f(v_{n-2,4,j-2}) = f(v_{n-2,n-1}) = 4; \]
\[ f(v_{4i-3,4,j-1}) = f(v_{n-2,4,j-1}) = f(v_{n-2,n}) = f(v_{4i-3,n}) = 7; \]
\[ f(v_{4i-3,4,j}) = f(v_{n-2,4,j}) = 10; \]
\[ f(v_{4i-2,4,j-3}) = f(v_{4i-2,n-2}) = f(v_{n-1,4,j-3}) = f(v_{n-1,n-2}) = 6; \]
\[ f(v_{4i-2,4,j-2}) = f(v_{4i-2,n-1}) = f(v_{n-1,4,j-2}) = f(v_{n-1,n-1}) = 9; \]
\[ f(v_{4i-2,4,j-1}) = f(v_{n-1,4,j-1}) = f(v_{4i-2,n}) = f(v_{n-1,n}) = 12; \]
\[ f(v_{4i-2,4,j}) = f(v_{n-1,4,j}) = 15; \]
\begin{align*}
 f(v_{4i-1,4j-3}) &= f(v_{4i-1,n-2}) = f(v_{n,4j-3}) = f(v_{n,n-2}) = 11; \\
 f(v_{4i-1,4j-2}) &= f(v_{4i-1,n-1}) = f(v_{n,4j-2}) = f(v_{n,n-1}) = 14; \\
 f(v_{4i-1,4j-1}) &= f(v_{4i-1,n}) = f(v_{n,4j-1}) = f(v_{n,n}) = 17; \\
 f(v_{4i-1,4j}) &= f(v_{n,4j}) = 20; \\
 f(v_{4i,4j-3}) &= f(v_{4i,n-2}) = 16; \\
 f(v_{4i,4j-2}) &= f(v_{4i,n-1}) = 19; \\
 f(v_{4i,4j-1}) &= f(v_{4i,n}) = 2; \\
 f(v_{4i,4j}) &= 13; \\
\end{align*}

elseif (n \geq 4) and (n = 0(\text{mod} 4))

for \( i = 1 \) to \( n/4 \)

for \( j = 1 \) to \( n/4 \)

\{ 

\begin{align*}
 f(v_{4i-3,4j-3}) &= 1; f(v_{4i-3,4j-2}) = 4; \\
 f(v_{4i-3,4j-1}) &= 7; f(v_{4i-3,4j}) = 10; \\
 f(v_{4i-2,4j-3}) &= 6; f(v_{4i-2,4j-2}) = 9; \\
 f(v_{4i-2,4j-1}) &= 12; f(v_{4i-2,4j}) = 15; \\
 f(v_{4i-1,4j-3}) &= 11; f(v_{4i-1,4j-2}) = 14; \\
 f(v_{4i-1,4j-1}) &= 17; f(v_{4i-1,4j}) = 20; \\
 f(v_{4i,4j-3}) &= 16; f(v_{4i,4j-2}) = 19; \\
 f(v_{4i,4j-1}) &= 2; f(v_{4i,4j}) = 13; \\
\end{align*}

\}

end.
**Theorem 2.7.2.** The square grid \((P_n \times P_n), n \geq 4\) has \(L(3, 2, 1)\)-labeling and with span \(\lambda(P_n \times P_n) = 20\).

**Proof.** Consider the square grid \((P_n \times P_n)\) with the vertex and edge set defined as in Algorithm 2.7.1. Let the function \(f : V(P_n \times P_n) \to \{1, 2, \ldots, 20\}\) be defined as in Algorithm 2.7.1. This theorem is proved in possible three cases.

**Case 1.** If \(d(v_i, v_j) = 1\) then to prove \(|f(v_i) - f(v_j)| \geq 3\).

\(d(v_i, v_j) = 1\) occur for the edges \(\{v_{i,j}, v_{i-1,j+1} : 1 \leq i \leq n; 2 \leq j \leq n\}\);
\(\{v_{i,j}, v_{i,j+1} : 1 \leq i \leq n - 1; 1 \leq j \leq n\}\);
\(\{v_{i,j}, v_{i+1,j} : 1 \leq i \leq n; 1 \leq j \leq n - 1\}\)

and \(\{v_{i,j}, v_{i-1,j+1} : 2 \leq i \leq n; 1 \leq j \leq n\}\).

By Algorithm 2.7.1, in all cases,

We get \(|f(v_{i,j}) - f(v_{i,j+1})| \geq 3\) for \(1 \leq i \leq n; 1 \leq j \leq n - 1\).

\(|f(v_{i,j}) - f(v_{i,j-1})| \geq 3\) for \(1 \leq i \leq n; 2 \leq j \leq n\).

\(|f(v_{i,j}) - f(v_{i+1,j})| > 3\) for \(1 \leq i \leq n - 1; 1 \leq j \leq n\).

\(|f(v_{i,j}) - f(v_{i-1,j})| > 3\) for \(1 \leq i \leq n; 2 \leq j \leq n\).

Thus for all \(i\) and \(j\), we have \(|f(v_i) - f(v_j)| \geq 3\).

**Case 2.** If \(d(v_i, v_j) = 2\) then to prove \(|f(v_i) - f(v_j)| \geq 2\).
$d(v_i, v_j) = 2$ occur for the edges
\[
\left\{ v_{i,j} v_{i,j+2} : 1 \leq i \leq n; 1 \leq j \leq n - 2 \right\}; \left\{ v_{i,j} v_{i,j-2} : 1 \leq i \leq n; 3 \leq j \leq n \right\};
\]
\[
\left\{ v_{i,j} v_{i-2,j} : 1 \leq i \leq n-2; 1 \leq j \leq n \right\}; \left\{ v_{i,j} v_{i+2,j} : 3 \leq i \leq n; 1 \leq j \leq n \right\};
\]
\[
\left\{ v_{i,j} v_{i+1,j-1} : 1 \leq i \leq n-1; 1 \leq j \leq n-1 \right\}; \left\{ v_{i,j} v_{i+1,j+1} : 2 \leq i \leq n; 1 \leq j \leq n-1 \right\};
\]
\[
\left\{ v_{i,j} v_{i-1,j+1} : 1 \leq i \leq n-1; 2 \leq j \leq n \right\}; \left\{ v_{i,j} v_{i+1,j-1} : 2 \leq i \leq n; 2 \leq j \leq n \right\}
\]

By Algorithm 2.7.1, in all cases,

We get $|f(v_{i,j}) - f(v_{i,j+2})| \geq 6$ for $1 \leq i \leq n; 1 \leq j \leq n - 2$.

\[
|f(v_{i,j}) - f(v_{i,j+2})| \geq 6 \quad \text{for} \quad 1 \leq i \leq n; 3 \leq j \leq n.
\]
\[
|f(v_{i,j}) - f(v_{i+2,j})| \geq 2 \quad \text{for} \quad 1 \leq i \leq n-2; 1 \leq j \leq n.
\]
\[
|f(v_{i,j}) - f(v_{i+2,j})| \geq 2 \quad \text{for} \quad 3 \leq i \leq n; 1 \leq j \leq n.
\]
\[
|f(v_{i,j}) - f(v_{i+1,j+1})| \geq 4 \quad \text{for} \quad 1 \leq i \leq n-1; 1 \leq j \leq n-1.
\]
\[
|f(v_{i,j}) - f(v_{i+1,j+1})| \geq 2 \quad \text{for} \quad 2 \leq i \leq n; 1 \leq j \leq n-1.
\]
\[
|f(v_{i,j}) - f(v_{i+1,j-1})| \geq 2 \quad \text{for} \quad 1 \leq i \leq n-1; 2 \leq j \leq n.
\]
\[
|f(v_{i,j}) - f(v_{i+1,j-1})| \geq 4 \quad \text{for} \quad 2 \leq i \leq n; 2 \leq j \leq n.
\]

Thus for all $i$ and $j$, we have $|f(v_i) - f(v_j)| \geq 2$.

**Case 3.** If $d(v_i, v_j) = 3$ then to prove $|f(v_i) - f(v_j)| \geq 1$.

$d(v_i, v_j) = 3$ occur for the edges Error! Objects cannot be created from editing field codes.; Error! Objects cannot be created from editing field codes.

Error! Objects cannot be created from editing field codes.;
\[
\left\{ v_{i,j} v_{i+2,j} : 1 \leq i \leq n-2; 2 \leq j \leq n \right\}; \left\{ v_{i,j} v_{i+2,j} : 3 \leq i \leq n; 2 \leq j \leq n \right\};
\]
\[
\left\{ v_{i,j} v_{i-1,j+2} : 1 \leq i \leq n-1; 1 \leq j \leq n-2 \right\}; \left\{ v_{i,j} v_{i-1,j+2} : 2 \leq i \leq n; 1 \leq j \leq n-2 \right\};
\]
\[
\left\{ v_{i,j} v_{i+1,j-2} : 1 \leq i \leq n-1; 3 \leq j \leq n \right\}; \left\{ v_{i,j} v_{i+1,j-2} : 2 \leq i \leq n; 3 \leq j \leq n \right\}
\]
all the vertices at difference three have distinct labeling. Thus in this case we have \( |f(v_i) - f(v_j)| \geq 1 \) for all \( i, j \). By Algorithm 2.7.1, the maximum labelling number is 20. Hence the square Grid \( P_n \times P_n \), \( n \geq 4 \) has \( L(3, 2, 1) \)-labeling and \( \lambda(P_n \times P_n) = 20 \).

Example 2.7.3. \( L(3, 2, 1) \) labeling of \( (P_5 \times P_5)^5 \) is shown in below Matrix form

<table>
<thead>
<tr>
<th>Stores ( M = 1 )</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 4 7 10 1</td>
<td>8 11 14 17 8</td>
</tr>
<tr>
<td></td>
<td>16 19 22 25 16</td>
<td>23 26 29 32 23</td>
</tr>
<tr>
<td></td>
<td>11 14 17 20 11</td>
<td>18 21 24 27 18</td>
</tr>
<tr>
<td></td>
<td>6 9 12 15 6</td>
<td>13 16 19 22 13</td>
</tr>
<tr>
<td></td>
<td>1 4 7 10 1</td>
<td>8 11 14 17 8</td>
</tr>
<tr>
<td></td>
<td>1 4 7 10 1</td>
<td>8 11 14 17 8</td>
</tr>
<tr>
<td></td>
<td>22 25 28 31 22</td>
<td>1 4 7 10 1</td>
</tr>
<tr>
<td></td>
<td>37 40 43 46 37</td>
<td>16 19 22 25 16</td>
</tr>
<tr>
<td></td>
<td>32 35 38 41 32</td>
<td>11 14 17 20 11</td>
</tr>
<tr>
<td></td>
<td>27 30 33 36 27</td>
<td>6 9 12 15 6</td>
</tr>
<tr>
<td></td>
<td>22 25 28 31 22</td>
<td>1 4 7 10 1</td>
</tr>
</tbody>
</table>

The entries of the matrices are the labels given to the vertices of the graph. There are five matrices which represent the labeling of the five different layers of the graph \( (P_n \times P_n)^M \) grid. The \( ij^{th} \) entry of the \( k^{th} \) matrices represent the \( L(3, 2, 1) \) labeling of \( ij^{th} \) vertex lying in the \( k^{th} \) layer of the \( (P_n \times P_n)^M \) graph. After the 4\(^{th} \) store the same values of the first four stores will be repeated. In general for all values of \( n \geq 4 \) and \( M \geq 4 \) the maximum value used to label (span) the \( (P_n \times P_n)^M \) is \( \lambda = 46 \).

In Algorithm 2.7.4, we give the construction of \( (P_n \times P_n)^M \) and assign the \( L(3, 2, 1) \) labeling. The grids \( P_n \times P_n \) are placed in \( M \) stores and the \( n \) vertices of grid linked to the corresponding \( n \) vertices of the grids in
succeeding and preceding layers. In the first step we define vertex set and edge sets. In the successive steps 2 to 6 we assign $L(3, 2, 1)$ labeling of $(P_n \times P_n)^M$, $n \geq 4$, $M \geq 4$ for four cases $M \equiv 1(\text{mod}\ 4)$, $M \equiv 2(\text{mod}\ 4)$, $M \equiv 3(\text{mod}\ 4)$ and $M \equiv 0(\text{mod}\ 4)$ respectively.

Algorithm 2.7.4

Input: Number of $mn^2$ vertices of $(P_n \times P_n)^M$

Output: $L(3, 2, 1)$ labeling $f(v_{i,j}^m)$ of $mn^2$ vertices of the graph $(P_n \times P_n)^M$

begin

Step1: Take vertex and edge set of grid as

$$V(P_n^M) = \bigcup_{k=1}^{m} V(P_n^k) : V(P_n^k) = \bigcup_{i,j=1}^{n} v_{i,j}^k$$

$$E(P_n^M) = \bigcup_{k=1}^{m} E(P_n^k) \cup E'$$

where $E' = \{v_{i,j}^k v_{i,j}^{k-1} : 1 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m-1\}$

$$E(P_n^k) = \{v_{i,j}^k v_{i,j-1}^k : 1 \leq i \leq n; 1 \leq j \leq n-1\} \cup \{v_{i,j}^k v_{i-1,j}^k : 1 \leq i \leq n-1; 1 \leq j \leq n\}$$

Step 2: Consider the $L(3, 2, 1)$ labeling $f(v_{i,j})$ from the above Algorithm 2.7.2, making small changes in labels given as follows

In the case $n \geq 4$ and $n \equiv 1 (\text{mod}\ 4)$ change the label of the vertices $f(v_{4i,4j-1})$, $f(v_{4i,4j})$ as 22 and 25 respectively.

In the case $n \geq 4$ and $n \equiv 2 (\text{mod}\ 4)$ change the label of the vertices $f(v_{4i,4j-1})$, $f(v_{4i,4j})$ as 22 and 25 respectively.

In the case $n \geq 4$ and $n \equiv 3 (\text{mod}\ 4)$ change the label of the vertices $f(v_{4i,4j-1}), f(v_{4i,n})$ as 22 and $f(v_{4i,4j})$ as 25 respectively.

In the case $n \geq 4$ and $n \equiv 0 (\text{mod}\ 4)$ change the label of the vertices $f(v_{4i,4j-1}), f(v_{4i,4j})$ as 22 and 25 respectively.
Step 3: When $m \geq 4$ and $m \equiv 1 \pmod{4}$, the labeling of $(P_n \times P_n)^M$ will be as follows

for $k = 1$ to $\frac{m-1}{4}$

\[
\begin{align*}
    f(v_{i,j}^{4k-3}) &= f(v_{i,j}^m) = f(v_{i,j}); \\
    f(v_{i,j}^{4k-2}) &= f(v_{i,j}) + 7; \\
    f(v_{i,j}^{4k-1}) &= f(v_{i,j}) + 14; \\
    f(v_{i,j}^{4k}) &= f(v_{i,j}) + 21;
\end{align*}
\]

Step 4: When $m \geq 4$ and $m \equiv 2 \pmod{4}$, the labeling of $(P_n \times P_n)^M$ will be as follows

for $k = 1$ to $\frac{m-2}{4}$

\[
\begin{align*}
    f(v_{i,j}^{4k-3}) &= f(v_{i,j}^{m-1}) = f(v_{i,j}); \\
    f(v_{i,j}^{4k-2}) &= f(v_{i,j}^m) = f(v_{i,j}) + 7; \\
    f(v_{i,j}^{4k-1}) &= f(v_{i,j}) + 14; \\
    f(v_{i,j}^{4k}) &= f(v_{i,j}) + 21;
\end{align*}
\]

Step 5: When $m \geq 4$ and $m \equiv 3 \pmod{4}$, the labeling of $(P_n \times P_n)^M$ will be as follows

for $k = 1$ to $\frac{m-3}{4}$
\[
\{( \\
\quad f(v_{i,j}^{4k-3}) = f(v_{i,j}^{m-2}) = f(v_{i,j}); \\
\quad f(v_{i,j}^{4k-2}) = f(v_{i,j}^{m-1}) = f(v_{i,j}) + 14; \\
\quad f(v_{i,j}^{4k-1}) = f(v_{i,j}^m) = f(v_{i,j}) + 21; \\
\quad f(v_{i,j}^{4k}) = f(v_{i,j}) + 21; \\
\}
\]

**Step 6:** When \( m \geq 4 \) and \( m \equiv 0 \text{ (mod 4)} \), the labeling of \((P_n \times P_n)^M\) will be as follows

for \( k = 1 \) to \( \frac{m}{4} \)

\[
\{( \\
\quad f(v_{i,j}^{4k-3}) = f(v_{i,j}); \\
\quad f(v_{i,j}^{4k-2}) = f(v_{i,j}) + 14; \\
\quad f(v_{i,j}^{4k-1}) = f(v_{i,j}) + 21; \\
\quad f(v_{i,j}^{4k}) = f(v_{i,j}) + 21; \\
\}
\]

end.

**Theorem 2.7.5.** The \( P_n \times P_n \times P_m \) graph \((P_n \times P_n)^M\) has \( L(3, 2, 1) \) labeling for \( n \geq 4, M \geq 4 \) with span \( \lambda((P_n \times P_n)^M) = 46 \).
Proof: Consider the graph \((P_n \times P_n)^M\) with the vertex and edge set defined as in Algorithm 2.7.4. Let the function \(f: V((P_n \times P_n)^M) \to \{1, 2, \ldots, 46\}\) be defined as in Algorithm 2.7.4. This theorem is proved in possible three cases.

Case 1. If \(d(v_i, v_j) = 1\) then to prove \(|f(v_i) - f(v_j)| \geq 3\).

\(d(v_i, v_j) = 1\) occur for the edges

\[\{v_{i,j}^k, v_{i,j+1}^k : 1 \leq i \leq n; 1 \leq j \leq n-1; 1 \leq k \leq m\}\; ;
\]
\[\{v_{i,j}^k, v_{i,j-1}^k : 1 \leq i \leq n; 2 \leq j \leq n; 1 \leq k \leq m\};
\]
\[\{v_{i,j}^k, v_{i-1,j}^k : 1 \leq i \leq n-1; 1 \leq j \leq n; 1 \leq k \leq m\};
\]
\[\{v_{i,j}^k, v_{i,j+1}^k : 2 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m\};
\]
\[\{v_{i,j}^k, v_{i,j-1}^k : 1 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m-1\}\] and
\[\{v_{i,j}^k, v_{i,j}^{k-1} : 1 \leq i \leq n; 1 \leq j \leq n; 2 \leq k \leq m\}\].

By Algorithm 2.7.4, in all cases,

We get \(|f(v_{i,j}^k) - f(v_{i,j+1}^k)| \geq 3\) for \(1 \leq i \leq n; 1 \leq j \leq n-1\) and \(1 \leq k \leq m\).

\(|f(v_{i,j}^k) - f(v_{i,j-1}^k)| \geq 3\) for \(1 \leq i \leq n; 2 \leq j \leq n\) and \(1 \leq k \leq m\).

\(|f(v_{i,j}^k) - f(v_{i-1,j}^k)| \geq 5\) for \(1 \leq i \leq n-1; 1 \leq j \leq n\) and \(1 \leq k \leq m\).

\(|f(v_{i,j}^k) - f(v_{i,j}^{k+1})| \geq 5\) for \(1 \leq i \leq n; 2 \leq j \leq n\) and \(1 \leq k \leq m\).

\(|f(v_{i,j}^k) - f(v_{i,j}^{k-1})| = 7\) for \(1 \leq i \leq n; 1 \leq j \leq n\) and \(1 \leq k \leq m-1\).

\(|f(v_{i,j}^k) - f(v_{i,j}^{k+1})| = 7\) for \(1 \leq i \leq n; 1 \leq j \leq n\) and \(2 \leq k \leq m\).

Thus for all \(i\) and \(j\), we have \(|f(v_i) - f(v_j)| \geq 3\).

Case 2. If \(d(v_i, v_j) = 2\) then to prove \(|f(v_i) - f(v_j)| \geq 2\).

\(d(v_i, v_j) = 2\) occur for the edges

Error! Objects cannot be created from editing field codes.
By Algorithm 2.7.4, in all cases,

We get \[ |f(v_{ij}^k) - f(v_{i,j+2}^k)| \geq 6 \quad \text{for } 1 \leq i \leq n; 1 \leq j \leq n - 2 \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i,j-2}^k)| \geq 6 \quad \text{for } 1 \leq i \leq n; 3 \leq j \leq n \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i-j}^k)| \geq 2 \quad \text{for } 1 \leq i \leq n-2; 1 \leq j \leq n \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i-j}^k)| \geq 2 \quad \text{for } 3 \leq i \leq n; 1 \leq j \leq n \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i,j+1}^k)| \geq 4 \quad \text{for } 1 \leq i \leq n - 1; 1 \leq j \leq n - 1 \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i,j+1}^k)| \geq 2 \quad \text{for } 2 \leq i \leq n; 1 \leq j \leq n - 1 \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i+1,j}^k)| \geq 2 \quad \text{for } 1 \leq i \leq n - 1; 2 \leq j \leq n \text{ and } 1 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i-1,j}^k)| \geq 4 \quad \text{for } 2 \leq i \leq n; 2 \leq j \leq n \text{ and } 1 \leq k \leq m. \]
\[ |f(v_{ij}^k) - f(v_{ij+1}^{k+1})| \geq 2 \text{ for } 1 \leq i \leq n; 1 \leq j \leq n - 1 \text{ and } 1 \leq k \leq m - 1. \]

\[ |f(v_{ij}^k) - f(v_{ij}^{k+1})| \geq 4 \text{ for } 1 \leq i \leq n; 2 \leq j \leq n \text{ and } 1 \leq k \leq m - 1. \]

\[ |f(v_{ij}^k) - f(v_{ij+1}^{k+1})| \geq 2 \text{ for } 1 \leq i \leq n; 1 \leq j \leq n - 1 \text{ and } 2 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{ij}^{k+1})| \geq 4 \text{ for } 1 \leq i \leq n; 2 \leq j \leq n \text{ and } 2 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i+1,j}^{k+1})| \geq 8 \text{ for } 1 \leq i \leq n - 1; 1 \leq j \leq n \text{ and } 1 \leq k \leq m - 1. \]

\[ |f(v_{ij}^k) - f(v_{i+1,j}^k)| \geq 2 \text{ for } 2 \leq i \leq n; 1 \leq j \leq n \text{ and } 1 \leq k \leq m - 1. \]

\[ |f(v_{ij}^k) - f(v_{i+1,j}^{k+1})| \geq 8 \text{ for } 1 \leq i \leq n - 1; 1 \leq j \leq n \text{ and } 2 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i+1,j}^k)| \geq 2 \text{ for } 2 \leq i \leq n; 1 \leq j \leq n \text{ and } 2 \leq k \leq m. \]

\[ |f(v_{ij}^k) - f(v_{i,j}^{k+2})| = 14 \text{ for } 1 \leq i \leq n; 1 \leq j \leq n \text{ and } 1 \leq k \leq m - 2. \]

\[ |f(v_{ij}^k) - f(v_{i,j}^{k+2})| = 14 \text{ for } 1 \leq i \leq n; 1 \leq j \leq n \text{ and } 3 \leq k \leq m. \]

Thus in this case we have \( |f(v_i) - f(v_j)| \geq 2 \) for all \( i \) and \( j \).

**Case 3.** If \( d(v_i, v_j) = 3 \) then to prove \( |f(v_i) - f(v_j)| \geq 1 \).

\( d(v_i, v_j) = 3 \) occur for the edges

\[ \{v_{i,j}^k : 1 \leq i \leq n; 1 \leq j \leq n - 3; 1 \leq k \leq m\}; \]

\[ \{v_{i,j}^k : 1 \leq i \leq n; 4 \leq j \leq n; 1 \leq k \leq m\}; \]

\[ \{v_{i,j}^k : 1 \leq i \leq n - 3; 1 \leq j \leq n; 1 \leq k \leq m\}; \]

\[ \{v_{i,j}^k : 4 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m\}; \]

\[ \{v_{i,j}^k : 1 \leq i \leq n - 2; 1 \leq j \leq n - 1; 1 \leq k \leq m\}; \]

\[ \{v_{i,j}^k : 3 \leq i \leq n; 1 \leq j \leq n - 1; 1 \leq k \leq m\}; \]

\[ \{v_{i,j}^k : 1 \leq i \leq n - 2; 2 \leq j \leq n; 1 \leq k \leq m\}; \]
\{v_{i,j}^{k}v_{i+1,j}^{k} : 3 \leq i \leq n; 2 \leq j \leq n; 1 \leq k \leq m\};
\{v_{i,j}^{k}v_{i+1,j-1}^{k} : 1 \leq i \leq n - 1; 1 \leq j \leq n - 2; 1 \leq k \leq m\};
\{v_{i,j}^{k}v_{i,j+1}^{k} : 2 \leq i \leq n; 1 \leq j \leq n - 2; 1 \leq k \leq m\};
\{v_{i,j}^{k}v_{i,j-1}^{k} : 1 \leq i \leq n - 1; 3 \leq j \leq n; 1 \leq k \leq m\};
\{v_{i,j}^{k}v_{i-1,j}^{k} : 2 \leq i \leq n; 3 \leq j \leq n; 1 \leq k \leq m\};
\{v_{i,j}^{k}v_{i,j-2}^{k-1} : 1 \leq i \leq n; 1 \leq j \leq n - 2; 1 \leq k \leq m - 1\};
\{v_{i,j}^{k}v_{i,j+2}^{k} : 1 \leq i \leq n; 3 \leq j \leq n; 1 \leq k \leq m - 1\};
\{v_{i,j}^{k}v_{i-2,j}^{k-1} : 1 \leq i \leq n - 2; 1 \leq j \leq n; 1 \leq k \leq m - 1\};
\{v_{i,j}^{k}v_{i-2,j}^{k-1} : 3 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m - 1\};
\{v_{i,j}^{k}v_{i,j+2}^{k-1} : 1 \leq i \leq n; 1 \leq j \leq n - 2; 2 \leq k \leq m\};
\{v_{i,j}^{k}v_{i,j-2}^{k} : 1 \leq i \leq n; 3 \leq j \leq n; 2 \leq k \leq m\};
\{v_{i,j}^{k}v_{i-2,j}^{k-1} : 1 \leq i \leq n - 2; 1 \leq j \leq n; 2 \leq k \leq m\};
\{v_{i,j}^{k}v_{i-2,j}^{k-1} : 3 \leq i \leq n; 1 \leq j \leq n; 2 \leq k \leq m\};
\{v_{i,j}^{k}v_{i,j+1}^{k+2} : 1 \leq i \leq n; 1 \leq j \leq n - 1; 1 \leq k \leq m - 2\};
\{v_{i,j}^{k}v_{i+1,j}^{k+2} : 1 \leq i \leq n; 2 \leq j \leq n; 1 \leq k \leq m - 2\};
\{v_{i,j}^{k}v_{i,j-1}^{k+2} : 1 \leq i \leq n - 1; 1 \leq j \leq n; 1 \leq k \leq m - 2\};
\{v_{i,j}^{k}v_{i+1,j}^{k+2} : 2 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m - 2\};
\{v_{i,j}^{k}v_{i-1,j}^{k+2} : 1 \leq i \leq n; 1 \leq j \leq n - 1; 3 \leq k \leq m\};
\[
\begin{align*}
\{v_{i,j}^{k} : 1 \leq i \leq n; 2 \leq j \leq n; 3 \leq k \leq m\}; \\
\{v_{i,j}^{k} : 1 \leq i \leq n-1; 1 \leq j \leq n; 3 \leq k \leq m\}; \\
\{v_{i,j}^{k} : 2 \leq i \leq n; 1 \leq j \leq n; 3 \leq k \leq m\}; \\
\{v_{i,j}^{k} : 1 \leq i \leq n; 1 \leq j \leq n; 1 \leq k \leq m-3\}; \\
\{v_{i,j}^{k} : 1 \leq i \leq n; 1 \leq j \leq n; 4 \leq k \leq m\}.
\end{align*}
\]

All the vertices at difference three have distinct labeling. Thus in this case we have \(|f(v_i) - f(v_j)| \geq 1\) for all \(i\) and \(j\). By Algorithm 2.7.4, the maximum labeling number is 46. Hence the graph \((P_n \times P_n)^M\) has \(L(3, 2, 1)\) labeling and span \(\lambda((P_n \times P_n)^M) = 46\).

In the following algorithm, we give the construction and \(L(3, 2, 1)\) labeling for the graph \(K_n^M\). The complete graph \(K_n\) is placed in \(M\) stores and the \(n\) vertices of \(K_n\) linked to the corresponding \(n\) vertices of the \(K_n\) in succeeding and preceding layers. The \(L(3, 2, 1)\) labeling of \(K_n^M\) is discussed in three cases. In the first case, \(L(3, 2, 1)\) labeling for the graph \(K_n^M\) with \(M < 4\) is given. In the second case, \(L(3, 2, 1)\) labeling for the graph \(K_n^M\) with \(M \geq 4\) is given under four sub cases \(M = 1 \text{ (mod 4)},\ M = 2 \text{ (mod 4)},\ M = 3 \text{ (mod 4)}\) and \(M = 0 \text{ (mod 4)}\).

**Algorithm 2.7.6**

**Input:** Number of vertices \(mn\) of \(K_n^M\)

**Output:** \(L(3, 2, 1)\) labeling of vertices of the graph \(K_n^M\)

begin

\[
V(K_n^M) = \left\{ \bigcup_{j=1}^{m} V(K_n^j) : V(K_n^j) = \{v_1^j, v_2^j, \ldots, v_n^j\} \right\}
\]

end
\[ E(K_n^M) = \bigcup_{j=1}^{m} E(K_n^j) \cup E' \] where \( E(K_n^j) = \{v_i^jv_k^j : 1 \leq i, k \leq n \text{ and } i \neq k\} \)
\[ E' = \{v_i^k v_i^{k-1} : 1 \leq i \leq n; 1 \leq k \leq m - 1\} \]

if \((m < 4)\)

for \(j = 1\) to \(m\)

for \(i = 1\) to \(n\)

\[ f(v_i^j) = 3i - 2 + (3n - 1)(j - 1); \]

if \((m \geq 4)\) and \((m \equiv 1(\text{mod} 4))\)

for \(j = 1\) to \(\frac{m-1}{4}\)

for \(i = 1\) to \(n\)

\[ f(v_i^{4j-3}) = f(v_i^m) = 3i - 2; \]
\[ f(v_i^{4j-2}) = 3i - 2 + (3n - 1); \]
\[ f(v_i^{4j-1}) = 3i - 2 - 2(3n - 1); \]
\[ f(v_i^{4j}) = 3i - 2 + 3(3n - 1); \]

else if \((m \geq 4)\) and \((m \equiv 2(\text{mod} 4))\)

for \(j = 1\) to \(\frac{m-2}{4}\)

for \(i = 1\) to \(n\)

\[ f(v_i^{4j-3}) = f(v_i^{m-1}) = 3i - 2; \]
\[ f(v_i^{4j-2}) = f(v_i^m) = 3i - 2 + (3n - 1); \]
\[ f(v_i^{4,j-1}) = 3i - 2 + 2(3n - 1); \]
\[ f(v_i^{4,j}) = 3i - 2 + 3(3n - 1); \]

else if \((m \geq 4)\) and \((m \equiv 3(\text{mod}4))\)

for \(j = 1\) to \(\frac{m-3}{4}\)

for \(i = 1\) to \(n\)

\{

\[ f(v_i^{4,j-3}) = f(v_i^{m-2}) = 3i - 2; \]
\[ f(v_i^{4,j-2}) = f(v_i^{m-1}) = 3i - 2 + (3n - 1); \]
\[ f(v_i^{4,j-1}) = f(v_i^m) = 3i - 2 + 2(3n - 1); \]
\[ f(v_i^{4,j}) = 3i - 2 + 3(3n - 1); \]
\}

elseif \((m \geq 4)\) and \((m \equiv 0(\text{mod}4))\)

for \(j = 1\) to \(\frac{m}{4}\)

for \(i = 1\) to \(n\)

\{

\[ f(v_i^{4,j-3}) = 3i - 2; \]
\[ f(v_i^{4,j-2}) = 3i - 2 + (3n - 1); \]
\[ f(v_i^{4,j-1}) = 3i - 2 + 2(3n - 1); \]
\[ f(v_i^{4,j}) = 3i - 2 + 3(3n - 1); \]
\}

end.
Theorem 2.7.7. The $K_n \times P_m$ graph $K_n^M$ has $L(3, 2, 1)$ labeling with span 

$$
\lambda(K_n^M) = \begin{cases} 
12\Delta_M - 45 & \text{if } M = 2 \\
12\Delta_M - 31 & \text{if } M = 3 \\
12\Delta_M - 17 & \text{if } M \geq 4.
\end{cases}
$$

Proof: Consider the graph $K_n^M$ with the vertex set, edge set and the function 
$f : V(K_n^M) \to \{1, 2, \ldots, 12\Delta_M - 17\}$ be defined as in Algorithm 2.7.6. One can easily verify that $\lambda = 12\Delta_M - 45$ for $m = 2$ and $\lambda = 12\Delta_M - 31$ for $m = 3$ using above Algorithm 2.7.6. For $m \geq 4$, the theorem is proved in possible three cases.

Case 1. If $d(v_i, v_j) = 1$ then to prove $|f(v_i) - f(v_j)| \geq 3$.

$d(v_i, v_j) = 1$ occur for the edges 

$$
\{v_i^jv_k^j : 1 \leq i, k \leq n \text{ and } i \neq k; 1 \leq j \leq m\};\{v_i^jv_{i+1}^j : 1 \leq i \leq n; 1 \leq j \leq m - 1\}.
$$

By Algorithm 2.7.6, in all cases,

We get $|f(v_i^j) - f(v_k^j)| = 3$ for $1 \leq i, k \leq n; i \neq k$ and $1 \leq j \leq m$.

Thus for all $i$ and $j$, we have $|f(v_i) - f(v_j)| \geq 3$.

Case 2. If $d(v_i, v_j) = 2$ then to prove $|f(v_i) - f(v_j)| \geq 2$.

$d(v_i, v_j) = 2$ occur for the edges 

$$
\{v_i^jv_k^{j+1} : 1 \leq i, k \leq n \text{ and } i \neq k; 1 \leq j \leq m - 1\};
\{v_i^jv_k^{j+1} : 1 \leq i, k \leq n \text{ and } i \neq k; 2 \leq j \leq m\}
$$

$$
\{v_i^jv_{i+1}^{j+1} : 1 \leq i \leq n; 1 \leq j \leq m - 2\} \text{ and } \{v_i^jv_{i+2}^{j+2} : 1 \leq i \leq n; 3 \leq j \leq m\}.
$$

By Algorithm 2.7.6, in all cases,
We get $|f(v_i^i) - f(v_k^{i+1})| \geq 2$ for $1 \leq i, k \leq n; i \neq k$ and $1 \leq j \leq m - 1$.

$|f(v_i^i) - f(v_k^{i-1})| \geq 2$ for $1 \leq i, k \leq n; i \neq k$ and $2 \leq j \leq m$.

$|f(v_i^i) - f(v_k^{i+2})| \geq 2$ for $1 \leq i \leq n$ and $1 \leq j \leq m - 2$.

$|f(v_i^i) - f(v_k^{i+2})| \geq 2$ for $1 \leq i \leq n$ and $3 \leq j \leq m$.

Thus for all $i$ and $j$, we have $|f(v_i) - f(v_j)| \geq 2$.

**Case 3.** If $d(v_i, v_j) = 3$ then to prove $|f(v_i) - f(v_j)| \geq 1$.

$d(v_i, v_j) = 3$ occur for the edges

$\{v_i^i v_i^{i+2} : 1 \leq i \leq n; 1 \leq j \leq m - 3\}; \{v_i^i v_i^{i-3} : 1 \leq i \leq n; 4 \leq j \leq m\}$.

all the vertices at difference three have distinct labeling. Thus in this case we have $|f(v_i) - f(v_j)| \geq 1$ for all $i$ and $j$. By Algorithm 2.7.6, the vertices $v_n^m$ has the maximum label, that is $f(v_n^m) = 12\Delta_m - 17$ for $m \geq 4$. Hence the graph $K_n^M$ has $L(3, 2, 1)$ labeling and

$$
\lambda(K_n^M) = \begin{cases} 
12\Delta_m - 45 & \text{if } M = 2 \\
12\Delta_m - 31 & \text{if } M = 3 \\
12\Delta_m - 17 & \text{if } M \geq 4.
\end{cases}
$$

**Observation 2.7.8.** The maximum degree for the $G \times P_m$ graph $G^M$ is

$$
\Delta_M = \begin{cases} 
\Delta + 1 & \text{if } M = 2 \\
\Delta + 2 & \text{if } M \geq 3
\end{cases} \leq |V| + 1,
$$

(2.3)

where $\Delta$ is the maximum degree of the simple graph $G$ and $|V|$ is the number of vertices in the simple graph $G$.

Jean Clipperton et al (2006) has given upper bound span of $L(3, 2, 1)$ labeling for any graph $G$ with maximum degree $\Delta$ to be
\[ \lambda(G) \leq \Delta^3 + \Delta^2 + 3\Delta \]  

(2.4)

By (2.3), \( \Delta = \Delta_M - 2 \). Substituting \( \Delta \) in (2.4) we get,

\[ \lambda(G^M) \leq \Delta^3 + \Delta^2 + 3\Delta = (\Delta_M - 2)^3 + (\Delta_M - 2)^2 + 3(\Delta_M - 2) \]

\[ = \Delta_M^3 - 5\Delta_M^2 + 11\Delta_M - 10. \]

Hence the upper bound span of \( L(3, 2, 1) \) labeling for \( G^M \) graphs in terms of maximum degree \( \Delta_M \) is

\[ \lambda(G^M) \leq \Delta_M^3 - 5\Delta_M^2 + 11\Delta_M - 10. \]