Chapter 1

Introduction

In this chapter, a brief survey of sequence spaces, fuzzy sequence spaces and an outline of the thesis are given.

1.1 A brief survey of Sequence Spaces

In several branches of analysis, the study of sequence spaces occupies a very prominent position. The theory of sequence spaces is a part of functional analysis, motivated by problems in Fourier series, power series and systems of equations with infinitely many variables. Apart from this, it is a powerful tool for obtaining positive results concerning Schauder bases and their associated types. Also it has made remarkable advances in recent times in enveloping summability theory via unified techniques effecting transformations from one sequence space into another. Therefore this study is made on this subject matter.
The theory of sequence spaces, topologized in a variety of ways has been developed in considerable detail, in particular by Kothe and Toeplitz [40]. General class of sequence space was introduced by K. Zeller [75]. Mazur and Orlicz have obtained some useful and important results [48].

The space of all entire functions is an important subject which unifies function theory and complex analysis. This theory is extended to matrix transformations involving entire sequences.

The space of entire functions was first formulated by V. Ganapathy Iyer [22]. Subsequently, he contributed his results on this space in a series of papers [23, 24, 25]. Louis De Brange [42] investigated properties of the Hilbert space of entire functions. D. Somasundaram [64] and M. K. Sen [63] continued the work on entire functions. Certain subspaces of the space of entire functions were propounded by J. P. J. Titus [71]. P. K. Kampthan [36] studied certain class of Frechet space.

The most general linear operator acting between sequence spaces is actually determined by an infinite matrix. In 1911, the celebrated German Mathematician Otto Toeplitz determined necessary and sufficient conditions on an infinite matrix which maps convergent sequences into convergent sequences.
The study on matrix transformations was made by many authors [9, 17, 32, 43, 45, 69]. The entire method of summation was investigated by H. I Brown [6] and further properties of entire method of summation were studied by K. Chandrasekhara Rao [11].

Today a lot of research is going on in Orlicz sequence spaces [1, 14, 41, 49], difference sequence spaces [26, 39, 46, 66] and statistically convergent sequence spaces [21, 70]. This dissertation focuses on entire sequence spaces.

1.2 A brief survey of Fuzzy Sequence Spaces

Among the various paradigmatic changes in science in this century, one such change concern the concept of uncertainty. According to modern view, uncertainty is considered essential to science, it is not only an unavoidable plague, but it has in fact a great utility.

It is generally agreed that an important point in the evolution of the modern concept of uncertainty is the publication of the seminal paper by Lotfi A. Zadeh [74]. Further, he introduced the theory whose objects are sets with boundaries that are not precise. Fuzzy theory has come a long way since it was introduced by Zadeh. Many research investigations by scientists and engineers all over the world have been made in the theory and applications of the subject.
Research on the theory of fuzzy sets has been growing steadily since the inception of the theory in the mid 1960s. Research on a broad variety of application has also been very active and impressive. This motivate us to study the fuzzy sequence spaces. In recent years, the fuzzy theory has emerged as an active area of research in many branches of mathematics and engineering.

The theory of fuzzy numbers is not only the foundation of fuzzy analysis but it also has important applications in fuzzy optimization and fuzzy decision making etc [27, 59]. Many authors have found interest in the study of the theory of fuzzy numbers [16, 19, 28, 33, 38, 56]. Because of the wide range in theory and applications, many results on fuzzy numbers have been achieved.

Dubois and Prade [19] introduced the notion of fuzzy real numbers. Goetschel and Voxman [28] have altered the definition given by Dubois and Prade. They defined a metric for the family of fuzzy numbers and studied the topological properties of it. Kaleva and Seikkala [34], Diamond and Kloeden [18] introduced fuzzy metric spaces in different ways. The theory and applications of fuzzy numbers often involves the functions from the set of fuzzy numbers into itself. In [47], Matloka developed sequences from fuzzy numbers. Bounded and convergent sequences of fuzzy numbers was introduced by Matloka [47].
Also he has shown that every convergent sequence of fuzzy numbers is bounded. Later on, sequences of fuzzy numbers have been discussed by Nandha [51]. He proved that the set of all convergent sequences of fuzzy numbers is a complete metric space. Savas [62] introduced the space \((m\Delta)\) of fuzzy numbers and shown that it is complete. Difference sequences of fuzzy numbers were developed and generalized by many authors [4, 15, 20, 57, 61].

Mursaleen [50] studied some new sequence spaces of fuzzy numbers. Statistical convergent and Statistical Cauchy sequences of fuzzy numbers were studied by Nuray and Savas [52]. Savas and Tripathy [60, 68] studied the concept of \(\lambda\)-summable sequences and absolute value of fuzzy real numbers. Tripathy [67] extended this concept to double sequence space of fuzzy real numbers. Lacunary statistical convergent sequences of fuzzy numbers were studied and generalized by Altin and Nuray [2, 3, 53]. It is proved by Talo and Basar [54] that the space \(bv_p(F)\) of sequences of \(p\)-bounded variation of fuzzy numbers is a complete metric space. Talo and Basar [55] determined the duals of classical sets of sequences of fuzzy numbers.

The fuzzy entire sequences were introduced by J. Kavikumar, Azme Bin Khamis [37]. Orlicz space of entire sequences of fuzzy numbers was introduced by N. Subramanian and Metin Basarir [65].
Nowadays a lot of research is going on in fuzzy sequence spaces.

1.3 Overview of the Thesis

There are seven chapters in this dissertation including the introduction. The second chapter is primarily a catalogue of definitions, notations and preliminary materials all of which will be used later in the work.

In chapter 3, the new sequence space $G_\lambda$ is introduced and its topological properties are studied.

$$G_\lambda = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \lambda_k^2|x_k|^2 < \infty \right\},$$

where $\lambda = (\lambda_k)$ is a fixed sequence of non-negative real numbers such that $\lambda_k \neq 1$ and $\frac{\lambda_{k+1}}{\lambda_k} \to 1$ as $k \to \infty$.

It is proved that $G_\lambda$ is a subspace of space of all entire sequences. Some results proved in this chapter are given below:

**Theorem 1.3.1.** If $G_\lambda$ and $G_\mu$ are two Banach spaces then $G_\lambda = G_\mu$ if and only if $k_1 \leq \frac{\lambda_n}{\mu_n} \leq k_2$, where $\lambda = (\lambda_n)$, $\mu = (\mu_n)$ are sequences as in the definition of $G_\lambda$ and $k_1, k_2$ are non-negative real numbers.
$G_\lambda$ is endowed with two topologies. One is the topology generated by the family of seminorms and is denoted by $\mathcal{S}_\eta$. The other is the topology generated by the norm on $G_\lambda$ and is denoted by $\mathcal{S}_\lambda$.

The following lemmas are related to these topologies.

**Lemma 1.3.2.** The $\mathcal{S}_\eta$ bounded subsets of $G_\lambda$ are just the $\mathcal{S}_\lambda$ bounded subsets of $G_\lambda$.

**Lemma 1.3.3.** If any $\mathcal{S}_\eta$-Cauchy net $(x^{(\alpha)})$ in $G_\lambda$ converges to some $x \in \omega$ then $x \in G_\lambda$.

Let $A = (a_{nk})(n, k = 1, 2, \ldots)$ be an infinite matrix whose elements are from $\mathbb{K}$, the field of real or complex numbers. Then, we have the following theorems for matrix transformations.

**Theorem 1.3.4.** $A \in (G_\lambda : c_0)$ if and only if

(i) $\sup \left\{ \left( \sum_{k=1}^{\infty} \frac{|a_{nk}|^2}{\lambda_k^2} \right)^{1/2} : n = 1, 2, \ldots \right\} \leq M < \infty$ for some $M > 0$ and

(ii) $\lim_{n \to \infty} a_{nk} = 0$ for each fixed $k$.

**Theorem 1.3.5.** $A \in (\ell : G_\lambda)$ if and only if for some constant $M > 0$,

$$\sup \left\{ \sum_{n=1}^{\infty} \lambda_n^2 |a_{nk}|^2 : k = 1, 2, \ldots \right\} \leq M < \infty$$ holds.
In chapter 4, we investigate generalized Kothe-Toeplitz duals which arise when the complex sequence \((a_k)\) is replaced by a sequence \((A_k)\) of linear operators. Thus if \(X\) and \(Y\) are Banach spaces, each \(A_k\) is a linear operator on \(X\) into \(Y\) and \(E\) is a non-empty set of sequences \(x = (x_k)\) with \((x_k) \in X\) then we define,

\[
E^\beta = \{(A_k) : \sum_{k=1}^{\infty} A_k x_k \text{ converge in the } Y\text{-norm for all } x \in X\}.
\]

The following proposition is an important tool used in this chapter.

**Proposition 1.3.6.** (i) If \((T_k)\) is a sequence in \(B(X,Y)\) and we write \(R_n = (T_n, T_{n+1}, \ldots)\) then

(a) \(\|T_m\| \leq \|R_n\|\) for all \(m \geq n\),

(b) \(\|R_{n+1}\| \leq \|R_n\|\), for all \(n \in \mathbb{N}\) and

(c) \(\left\| \sum_{k=n}^{n+p} T_k x_k \right\| \leq \|R_n\| \cdot \max\{\|x_k\| : n \leq k \leq n + p\}\).

(ii) If \((T_k)\) is a sequence in \(B(X,Y)\) then

\[
\sum_{k=1}^{\infty} \|T_k\| < \infty \text{ implies } \sup_k \|T_k\| < \infty.
\]

Here, we list out some important theorems proved in this chapter.

If \((X, \|\cdot\|)\) is any Banach space over \(\mathbb{C}\) then we define

\[
\Gamma(X) = \left\{ x = (x_k) : \lim_{k \to \infty} \|x_k\|^\frac{1}{2} = 0 \right\},
\]

\[
\Lambda(X) = \left\{ x = (x_k) : \sup_k \|x_k\|^\frac{1}{2} < \infty \right\}\text{ and}
\]

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\[ G_\lambda(X) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \lambda_k^2 \|x_k\|^2 < \infty \right\}, \] where \((\lambda_k)\) is a fixed sequence of real numbers and \(\frac{\lambda_{k+1}}{\lambda_k} \to 1\) as \(k \to \infty\).

**Proposition 1.3.7.** \((A_k) \in \Gamma^\beta(X)\) if and only if there exists \(m \in \mathbb{N}\) such that

(i) \(A_k \in B(X, Y)\) and

(ii) \(\sup_{k \geq m} \|A_k\|_1^1 < \infty\).

**Proposition 1.3.8.** \((A_k) \in \Lambda^\beta(X)\) if and only if

(i) \((A_k) \in \Gamma^\beta(X)\) and

(ii) \(\|R_n\|_1^n \to 0\) as \(n \to \infty\).

**Proposition 1.3.9.** \((A_k) \in G_\lambda^\beta(X)\) if and only if there exists \(m \in \mathbb{N}\) such that

(i) \((A_k) \in B(X, Y)\) for all \(k \geq m\) and

(ii) \(\sum_{k=m}^{\infty} \frac{1}{\lambda_k^2} \|A_k\|^2 < \infty\).

In chapter 5, the Analytical and Entire duals of some known sequence spaces are obtained. The concept of sectional analyticity is introduced and the relation between \(f\)-dual and \(\Lambda\)-dual is obtained. Some important results discussed in this chapter are listed below.
Let \( \Gamma = \{ x : x \in \omega, \lim_{n \to \infty} |x_n|^\frac{1}{n} = 0 \} \) and \\
\( \Lambda = \{ x : x \in \omega, \sup_{n} |x_n|^\frac{1}{n} \text{ exists} \} \).

**Theorem 1.3.10.** Suppose \( \Gamma \subseteq X \subseteq \Lambda \). Then \( X^\Lambda = \Lambda \).

**Lemma 1.3.11.** Let \( X \) be an FK space containing \( \Phi \). Let \( z \in \omega \). Then \( z \in A^+ \) if and only if \( z^{-1}X \supset \Gamma \).

**Theorem 1.3.12.** Let \( X \) be an FK space containing \( \Phi \). Then \( z \in X^{f\Lambda} \) if and only if \( z^{-1}X \supset \Gamma \).

**Theorem 1.3.13.** Let \( X \) be an FK space containing \( \Phi \). Let \( z \in \omega \). Then \( z \in A^+ \) if and only if \( z^{-1}X \) is semi conservative. In particular \( 1 \in A^+ \) if and only if \( X \) is semiconservative.

**Theorem 1.3.14.** Let \( X \) be any FK space containing \( \Phi \). Let \( z \in \omega \). Then \( z \in A \) if and only if \( z^{-1}X \) is variational semiconservative.

In chapter 6, some subspaces of \( \Gamma(F) \), the Entire sequence space of fuzzy numbers are discussed. The spaces \( \Gamma(F, \lambda), \Gamma(F, 1, d) \) and \( G_\lambda(F) \) are introduced. Some remarkable results in this chapter are given below.
**Theorem 1.3.15.** \(\Gamma(F)\) is a complete metric space with respect to the metric \(d\), defined by \(d(u, v) = \sup_k D(u_k, v_k)\).

**Theorem 1.3.16.** \(\Gamma(F, \lambda) = \Gamma(F)\) if and only if 
\[
\limsup \{D(\lambda_k, 0)\} < \infty.
\]

**Theorem 1.3.17.** \((\Gamma(F, \lambda), d_\lambda)\) is a complete metric space if and only if 
\[
\liminf \{D(\lambda_k, 0)\} > 0.
\]

**Theorem 1.3.18.** \(\Gamma(F, 1, d)\) is a complete metric space with respect to the metric \(d(u, v) = \sup_k D(u_k, v_k)\).

**Theorem 1.3.19.** \(G_\lambda(F)\) is a complete metric space with respect to the metric \(\rho\) defined by 
\[
\rho(u, v) = \left\{ \sum_{k=1}^{\infty} [D(\Lambda u_k, \Lambda v_k)]^2 \right\}^{\frac{1}{2}}.
\]

**Theorem 1.3.20.** The metric of \(G_\lambda(F)\) is monotone.

**Theorem 1.3.21.** \(G_\lambda(F)\) is a Banach space with the norm.

\[
\|u\| = \left[ \sum_k \|\Lambda u_k\|_F^2 \right]^{\frac{1}{2}}, \text{ where } u = (u_k) \in G_\lambda(F).
\] (1.1)
Theorem 1.3.22. The set \((E_k : k = 1, 2, 3, \ldots)\) is a Schauder basis for \(G_\lambda(F)\) under the norm defined in the above theorem and any \(u \in G_\lambda(F)\) has a unique representation of the form
\[
\sum_{k=1}^{\infty} u_k E_k. \tag{1.2}
\]

Theorem 1.3.23. \(A \in (z, G_\lambda(F))\) if and only if
\[
\phi_n = \left[ \sum D(\lambda_k a_{nk}, \bar{0}) \right]^2, \quad n = 1, 2, \ldots \text{ is bounded}. \tag{1.3}
\]

In chapter 7, we introduced \(h(F)\), the Hahn sequence space of fuzzy numbers. We have introduced the Cesaro space \(\sigma(\ell_\infty(F))\) of sequences of fuzzy numbers and we have proved that it is complete. Some of the important results we have proved in this chapter are listed below.

Theorem 1.3.24. \(h(F)\) and \(h_\infty(F)\) are complete metric spaces with the metrics \(dh\) and \(dh_\infty\) defined by
\[
dh(u, v) = \sum_{k=1}^{\infty} D[(Au)_k, (Av)_k] \quad \text{and}
\]
\[
dh_\infty(u, v) = \sup_k D[(Au)_k, (Av)_k] \quad \text{respectively, where} \quad u = (u_k) \quad \text{and} \quad v = (v_k) \quad \text{are the elements of the spaces} \quad h(F) \quad \text{and} \quad h_\infty(F).
\]

Proposition 1.3.25. The space \(h(F)\) is isomorphic to the space \(\ell(F)\).
Theorem 1.3.26. Let $d$ denote the set of all sequences of fuzzy numbers defined as follows:

$$d = \left\{ x = (x_k) \in w(F) : \sum_{k=1}^{\infty} k|x_k - x_{k-1}| < \infty \text{ and } x \in c_0(F) \right\}.$$ 

Then the set $d$ is identical with the set $h(F)$.

Theorem 1.3.27. Let $E$ and $E_1$ are sets of sequences of fuzzy numbers. Then (i) $E \subset E^{\beta\beta}$.

(ii) $E^{\beta\beta\beta} = E^\beta$.

(iii) If $E_1 \supset E$, then $E_1^\beta \subset E^\beta$.

The same results hold for $\gamma$-dual also.

Theorem 1.3.28. $\sigma(\ell_\infty(F))$ is a complete metric space with the metric $d_\sigma(u,v) = \sup_k D\left[(B_u)_k, (B_v)_k\right]$, where $u = (u_k)$ and $v = (v_k)$ are the elements of the space $\sigma(\ell_\infty(F))$.

Theorem 1.3.29. The $\gamma$-dual of $h(F)$ is $\sigma(\ell_\infty(F))$. 