Chapter 5

MOTIONS IN FINSLER SPACES

1. Introduction ........................................... 99
2. Lie Derivative in a Finsler Space .................... 101
3. Conformal Motion in a Recurrent Finsler Space .... 102
4. Theorems on Conformal Motion in a Recurrent Finsler Space ........................................... 104
5. Special Projective Motion in a Special Projective Recurrent Finsler Space ........................................... 106
6. Concrete From of Special Projective Motion ........ 109
7. Projective Motion in a Symmetric Finsler Space .... 113
Reference .................................................... 117
1. INTRODUCTION:

In $F_n$, we assume the differential equation of paths \[9\] in the form

\begin{equation}
\frac{\dot{x}^i + 2G^i(x, \dot{x})}{\dot{x}^i} = \frac{\ddot{x}^k + 2G^k(x, \dot{x})}{\dot{x}^k}.
\end{equation}

These equations remain unchanged if we replace the function $G^i(x, \dot{x})$ by new functions $\tilde{G}^i(x, \dot{x})$ which are defined by

\begin{equation}
\tilde{G}^i(x, \dot{x}) = G^i(x, \dot{x}) = P(x, \dot{x})\dot{x}^i.
\end{equation}

where $F(x, \dot{x})$ is an arbitrary scalar function, positively homogeneous of the first degree in its directional arguments $\dot{x}^i$. The relation (1.2) is the most general modification of the function $G^i(x, \dot{x})$ which leaves (1.1) unchanged.

The tensor defined by

\begin{equation}
W^i_k = H^i_k - H\delta^i_k - \frac{1}{n+1}(\dot{\partial}_i H^i_k - \dot{\partial}_k H)\dot{x}^i,
\end{equation}

is invariant under the projective change (1.2) and it is, therefore, regarded as the projective deviation tensor.

Projective curvature tensor is introduced with the help of (1.3). We have

\begin{equation}
W^j_{hk} = \frac{2}{3} \dot{\partial}_h W^j_k
\end{equation}

and

\begin{equation}
W^j_{ikh} = \partial_i W^j_{hk} = \frac{1}{3}(\dot{\partial}^2_{ih} W^i_k - \dot{\partial}^2_{ik} W^i_h).
\end{equation}

The expression for $W^i_{hk}(x, \dot{x})$ in terms of $H^i_j$ is given by

\begin{equation}
W^j_{hk}(x, \dot{x}) = H^j_{hk} + \frac{\dot{x}^j}{n+1} H^i_{ijk} + \frac{\delta^i_k}{n^2-1}(nH_k - \dot{x}^i H_{ki}) - \frac{\delta^i_k}{n^2-1}(nH_h + \dot{x}^i H_{hi}).
\end{equation}

The tensor $W^j_{ikh}(x, \dot{x})$ is obtained differentiating (1.6) with respect to $\dot{x}^i$. Thus
\[ W_{ihk}^j (x, \dot{x}) = H_{ihk} + \frac{s_i}{n+1} (H_{hk} - H_{kh}) + (\dot{\delta}_i H_{hk} - \dot{\delta}_i H_{kh}) + \]
\[ + \frac{s_i}{n^2 - 1} (nH_{ik} + H_{ki} + \dot{x}^r \dot{\delta}_i H_{kr}) - \]
\[ - \frac{s_i}{n^2 - 1} (nH_{ih} + H_{hi} + \dot{x}^r \dot{\delta}_i H_{hr}) . \]

This is the generalization of a tensor introduced by Weyl. The projective deviation tensor satisfies the following identities

\[ (1.8) \]
\( (a) \) \( W_{ik}^j \dot{x}^k = 0, \)
\( (b) \) \( \dot{\delta}_k W_{ik}^j \dot{x}^h = -W_{ih}^j, \)
\( (c) \) \( \dot{\delta}_i W_{ik}^j = 0, \)
\( (d) \) \( W_{hk}^j \dot{x}^h = W_{ik}^j, \)
\( (e) \) \( W_{ihk}^j \dot{x}^i = W_{hk}^j, \)
\( (f) \) \( W_{ihk}^j \dot{x}^i \dot{x}^h = W_{ik}^j, \)
\( (g) \) \( W_{hr}^r = 0, \)
\( (h) \) \( W_{ihr}^r = 0, \)
\( (i) \) \( W_{ihk}^j = -W_{ikh}^j, \)
\( (j) \) \( W_{ikh}^j = -W_{hk}^j, \)
\( (k) \) \( W_{[ihk]}^j = 0, \)
\( (l) \) \( \dot{\delta}_j W_{ik}^j \dot{x}^i = 2W_{ik}^j, \)
\( (m) \) \( W_{l}^j = 0, \)
\( (n) \) \( W_{ihj}^i = 0. \)

The quantities defined by

\[ (1.9) \]
\( \Pi_{k\ell}^i = G_{k\ell}^i - \frac{1}{n+1} [\delta_{\ell}^j G_{rk}^i + \delta_{k}^j G_{r\ell}^i + \dot{x}^j G_{rtk}^r], \)

are invariant under the change (1.2), there quantities are known as projective connection coefficients because of the fact that they give rise to a projective covariant derivative in the same manner as \( G_{hk}^i \). It is given by

\[ (1.10) \]
\( X_{[(x)]}^i = \partial_k X^i - (\dot{\delta}_m X^i) \Pi_{rk}^m \dot{x}^r + X^h \Pi_{hk}^i . \)

It is verified that

\[ (1.11) \]
\( 2T_{j[[(h)]][(k)]}^i = -\dot{\delta}_r T_{sj}^j Q_{shk}^s \dot{x}^s + T_{j}^s Q_{shk}^i - T_{j}^s Q_{jhk}^s, \)

where

\[ (1.12) \]
\( Q_{hjk}^i (x, \dot{x}) \equiv 2[\partial_{[k} \Pi_{j]}^i - \Pi_{r[k]}^i \Pi_{r]}^j + \Pi_{h[k]}^j \Pi_{r]}^i - \Pi_{r[k]}^i \Pi_{r]}^j] \)

is called the projective entity and satisfies the following relations [4]
(1.13) \( Q^{i}_{hjk} = -Q^{i}_{hkj}, \)

(1.14) \( Q^{i}_{hji} \overset{\text{def}}{=} Q_{hj}, \)

(1.15) \( Q^{i}_{jkh} = H^{i}_{jkh} + \frac{1}{n+1} \left( \delta^{i}_{j} H^{r}_{rhk} + \dot{x}^{i} \dot{\delta}_{j} H^{r}_{rhk} \right) + \right. \\
\left. + \frac{2}{(n+1)^{2}} \left\{ (n+1)G^{r}_{rj[k]} \delta^{i}_{h} + \delta^{i}_{[h} \dot{\delta}_{k]} \left( G^{r}_{rj} G^{s}_{s} \right) \right\} \right\}

and

(1.16) \( W^{i}_{hjk} = Q^{i}_{hjk} - \frac{2}{(n^{2}-1)} \left\{ (n+1)Q_{j[k} - H_{j[k} - H_{(k}j) + (n-1)\dot{\delta}_{j} \dot{\delta}_{k} H - \dot{x}^{s} \dot{\delta}_{j} H^{s}_{rs[k]} \delta^{i}_{h} \right\}.

2. LIE DERIVATIVE IN A FINSLER SPACE:

In an \( F_{n} \), let us consider an infinitesimal point transformation

(2.1) \( \bar{x}^{i} = \dot{x}^{i} + v^{i}(x)dt, \)

where \( v^{i}(x) \) is a covariant vector field defined over the domain of the space under consideration and \( dt \) is an infinitesimal constant. We assume that under this transformation, an element \( \dot{x}^{i} \) representing a direction attached to each point \( x^{i} \) is transformed to

(2.2) \( \ddot{x}^{i} = \dot{x}^{a} \frac{\partial x^{i}}{\partial x^{a}}. \)

Using the covariant derivative given by Berwald, we obtain the Lie derivative, of a tensor \( T^{i}_{j} \) and connection coefficient \( G^{i}_{jk} \) [12] as

(2.3) \( E_{v} T^{i}_{j}(x, \dot{x}) = T^{i}_{j(h)} v^{h} - T^{h}_{j(h)} v^{i} + T^{i}_{h} v^{h}_{(j)} + \left( \dot{\delta}_{h} T^{i}_{j} \right) v^{h}_{(s)} \dot{x}^{s} \)

and

(2.4) \( E_{v} G^{i}_{jk}(x, \dot{x}) = v^{i}_{(j)(k)} + H^{i}_{hjk} v^{h} + G^{i}_{hjk} v^{h}_{(s)} \dot{x}^{s}, \)

respectively.

In view of projective covariant derivative (1.10) and infinitesimal point transformation (2.1), the Lie derivatives of \( T^{i}_{j} \) and \( \Pi^{i}_{jk} \) [8] are given

(2.5) \( E_{v} T^{i}_{j} = T^{i}_{j(h)} v^{h} + \dot{\delta}_{h} T^{i}_{j} v^{h}_{(s)} \dot{x}^{s} + T^{i}_{s} v^{s}_{(j)} - T^{i}_{s} v^{s}_{(s)} \)
and

\[(2.6) \ \mathcal{E}_v \Pi^i_{jk} = v^i_{(j)(k)} + Q^i_{hjk} v^h + \Pi^i_{s(jk)} v^s_{(r)} \dot{x}^r,\]

respectively.

The commutation formulae involving the Berwald’s covariant derivative and Lie-derivative are given by

\[(2.7) \ \mathcal{E}_v (T^i_{j(k)}) = (\mathcal{E}_v T^i_j)_{(k)} \]
\[= T^h \mathcal{E}_v G^i_{kh} - T^h \mathcal{E}_v G^i_{jk} - (\partial^i_h T^j_k) \mathcal{E}_v G^h_{ks} \dot{x}^s\]

and

\[(2.8) \ \left( \mathcal{E}_v G^i_{jh} \right)_{(j)} - \left( \mathcal{E}_v G^i_{kh} \right)_{(j)} = \mathcal{E}_v H^i_{hjk} + 2 \mathcal{E}_v G^r_{l[k} G^i_{j]rh} \dot{x}^l.\]

We also have

\[(2.9) \ \mathcal{E}_v (\partial^i \xi T^j_k) - \partial^i \xi \mathcal{E}_v T^j_k = 0.\]

The commutation formulae involving the projective covariant derivative (1.10) and Lie-derivative are given by

\[(2.10) \ \mathcal{E}_v \left( T^i_{j(s)} \right) = (\mathcal{E}_v T^i_j)_{(s)} \]
\[= (\mathcal{E}_v \Pi^i_{sh}) T^h_j - (\mathcal{E}_v \Pi^h_{sj}) T^i_j - (\mathcal{E}_v \Pi^h_{sm}) \dot{x}^m \partial^i_h T^j_k\]

and

\[(2.11) \ \left( \mathcal{E}_v \Pi^i_{jh} \right)_{(j)} - \left( \mathcal{E}_v \Pi^i_{kh} \right)_{(j)} \]
\[= \mathcal{E}_v Q^i_{hjk} + 2 \dot{x}^s \Pi^i_{rh[j} \mathcal{E}_v \Pi^r_{k]s}.\]

3. **CONFORMAL MOTION IN A RECURRENT FINSLER SPACE:**

**DEFINITION (3.1):**

A Finsler space $F_n$ is said to be recurrent Finsler space of first order [6], if the curvature tensor field $H^i_{jkh}$ satisfies the relation

\[(3.1) \ H^i_{jkh(m)} = K_m H^i_{jkh}(K_m \neq 0),\]

where $K_m(x, \dot{x})$ is a homogeneous vector of degree zero in its directional arguments. We shall denote such a Finsler space by $F^*_n$. 

102
Transvecting (3.1) successively by $\dot{x}^i$ and thereafter using the relation [1-(13.15)], we get

\begin{equation}
(3.2) \quad H_{kh(m)}^i = K_m H_{kh}^i
\end{equation}

and

\begin{equation}
(3.3) \quad H_{h(m)}^i = K_m H_h^i
\end{equation}

respectively.

Contracting (3.2) with respect to the indices $i$ and $h$, we get

\begin{equation}
(3.4) \quad H_{(m)} = K_m H.
\end{equation}

**DEFINITION (3.2):**

If the point transformation on (2.1) implies that the magnitude of the vectors defined in the same tangent space are proportional and the angle between the two directions is also the same with respect to the metrics then it is said to possess conformal property in $F_n$.

**DEFINITION (3.3):**

A necessary and sufficient condition in order that the infinitesimal point transformation (2.1) may define a conformal motion is given by

\begin{equation}
(3.5) \quad \mathcal{E}_v G_{jk}^i = -\sigma_m B_{jk}^{im}
\end{equation}

where

\begin{equation}
(3.6) \quad B_{jk}^{im} \triangleq \partial_j \dot{x}^k B^{im},
\end{equation}

\begin{equation}
(3.7) \quad g_{\mathcal{B}(\mathcal{X},\mathcal{X})}^{im} = \frac{1}{2} F^2 g^{im} - \dot{x}^i \dot{x}^m
\end{equation}

and

\begin{equation}
(3.8) \quad \sigma_m \triangleq \frac{\partial \sigma}{\partial x^m}.
\end{equation}

Thus, for a conformal motion we have the following relations:

\begin{equation}
(3.9) \quad \mathcal{E}_v G^i = -\sigma_m B^{im}
\end{equation}

and
\[ (3.10) \quad \mathcal{E}_v G^i_{rk} = -\sigma_m B^i_{rk}, \]
where $B^i_{rk}$ is a symmetric function with respect to its indices and satisfies the following identities:

\[ (3.11) \quad B^i_{rk} \dot{x}^r = 0, \]
\[ (3.12) \quad B^i_{rk} \dot{x}^i = B^i_k \]
and

\[ (3.13) \quad B^i_{rk} \dot{x}^h = 2B^i_{rk}. \]

We also have the following relations:

\[ (3.14) \quad \mathcal{E}_v W^i_k (x, \dot{x}) \]
\[ = \frac{(n-2)}{(n-1)} \left[ \sigma_r \{ B^{ir}_{k(j)} \dot{x}^j - 2B^{ir}_{(k)} + 2B^{is} G^r_{sk} - B^i_s G^s_r \} - ight. \]
\[ -2B^{ir} \sigma_r(k) + B^{ir}_k \sigma_r(j) \dot{x}^j - \dot{x}^i \{ \sigma_r (B^{sr}_{ks(j)} \dot{x}^j + B^{ir}_k) - \}
\[ -2B^{sr}_{s(k)} + 2B^{sp} G^r_{pk} + 2B^{sp} G^r_{pks} - G^r_p B^{sp}_{ks} - \]
\[ -G^{sr}_{ps} B^{sp}_k - 2B^{sr}_{ks} \sigma_r(k) + B^{sr}_{ks} \sigma_r(j) \dot{x}^j + B^{ir}_k \sigma_r(j) \} \right]. \]

\[ (3.15) \quad W^i_k (m) = K_m W^i_k - \dot{x}^i (B^i_k \dot{\delta}_r K_m - H^r_k G^r_{krm} + \]
\[ + H^r_s G^s_{krm} - H^r_k \dot{\delta}_r K_m) / (n + 1). \]

We shall now discuss the existence of conformal motion (2.1) satisfying (3.5) in an $F_n$.

4. **THEOREM ON CONFORMAL MOTION IN A RECURRENT FINSLER SPACE:**

In [8], it has been shown that

\[ (4.1) \quad \mathcal{E}_v \left( H^i_k - H \delta^i_k \right) \]
\[ = \frac{(n-2)}{(n-1)} \left[ \sigma_m \{ B^{im}_{k(j)} \dot{x}^j - 2(B^{im}_{(k)} - B^{ir} G^m_{rk}) - B^{ir}_k G^m_r \} - \right. \]
\[ -2B^{im} \sigma_m(k) + B^{im}_k \sigma_m(j) \dot{x}^j \right], \]
and
\[(4.3) \quad (E_v W_k^i)_{(m)} + \sigma_s \left[ W_r^i B_{km}^{rs} - W_k^r B_{rm}^{is} + \left( \partial_r W_k^i \right) B_{m}^{rs} - W_k^i \partial_r K_m - \frac{(n-2)}{(n-1)} K_m \sigma_r \left( B_{k(j)}^{ir} \dot{x}^j - 2(B_{(k)}^{ir}) B_{m}^{rs} \right) - B_k^{is} G_{s}^{r} - 2B_k^{ir} \sigma_{r(k)} + B_k^{ir} \left( B_{(j)}^{sr} \dot{x}^j \right) - \frac{\dot{x}^i}{n+1} \left( \sigma_r \left( B_{ks}^{sr} \dot{x}^j \right) + B_k^{ij} \right) + \frac{\dot{x}^i}{n+1} \left( E_v \dot{\partial}_r K_m \right) \left( H_r^r - H_k^r \right) + \dot{\partial}_r K_m \right) \frac{(n-2)}{(n-1)} \left( \sigma_s \left( B_{k(j)}^{rs} \dot{x}^j \right) - 2B_{(k)}^{rs} + 2B_{(k)}^{rp} G_{p}^{s} - B_k^{rp} G_{s}^{p} \right) - 2B_{s(k)}^{rs} \sigma_{r(k)} + B_{(k)}^{rs} \sigma_{r(j)} \dot{x}^j + B_k^{ij} \sigma_{r(j)} \right) + \frac{\dot{x}^i}{n+1} \left( E_v \dot{\partial}_r K_m \right) \left( H_r^r - H_k^r \right) + \dot{\partial}_r K_m \right) \frac{(n-2)}{(n-1)} \left( \sigma_s \left( B_{k(j)}^{rs} \dot{x}^j \right) - 2B_{(k)}^{rs} + 2B_{(k)}^{rp} G_{p}^{s} - B_k^{rp} G_{s}^{p} \right) - 2B_{s(k)}^{rs} \sigma_{r(k)} + B_{(k)}^{rs} \sigma_{r(j)} \dot{x}^j + B_k^{ij} \sigma_{r(j)} \right) + \frac{\dot{x}^i}{n+1} \left( E_v \dot{\partial}_r K_m \right) \left( H_r^r - H_k^r \right) + \dot{\partial}_r K_m \right) \frac{(n-2)}{(n-1)} \left( \sigma_s \left( B_{k(j)}^{rs} \dot{x}^j \right) - 2B_{(k)}^{rs} + 2B_{(k)}^{rp} G_{p}^{s} - B_k^{rp} G_{s}^{p} \right) - 2B_{s(k)}^{rs} \sigma_{r(k)} + B_{(k)}^{rs} \sigma_{r(j)} \dot{x}^j + B_k^{ij} \sigma_{r(j)} \right) = 0.\]

Contracting (4.3) with respect to the indices \(i\) and \(m\) and then multiplying it by \(\dot{x}^i\) and then after subtracting the equation thus obtained from (4.3) transversed by \(\dot{x}^m\) and using (1.8), we get

\[(4.4) \quad (E_v W_k^i)_{(m)} \dot{x}^m - (E_v W_k^m)_{(m)} \dot{x}^i + \sigma_m \left[ (W_r^i B_{km}^{rs} - W_k^r B_{rm}^{is} + \left( \partial_r W_k^i \right) B_{m}^{rs} \right) - \frac{(n-2)}{(n-1)} \sigma_r \left( B_{k(j)}^{ir} \dot{x}^j - 2(B_{(k)}^{ir}) B_{m}^{rs} \right) - B_k^{is} G_{s}^{r} - 2B_k^{ir} \sigma_{r(k)} + B_k^{ir} \left( B_{(j)}^{sr} \dot{x}^j \right) - \frac{\dot{x}^i}{n+1} \left( \sigma_r \left( B_{ks}^{sr} \dot{x}^j \right) + B_k^{ij} \right) + \frac{\dot{x}^i}{n+1} \left( E_v \dot{\partial}_r K_m \right) \left( H_r^r - H_k^r \right) + \dot{\partial}_r K_m \right) \frac{(n-2)}{(n-1)} \left( \sigma_s \left( B_{k(j)}^{rs} \dot{x}^j \right) - 2B_{(k)}^{rs} + 2B_{(k)}^{rp} G_{p}^{s} - B_k^{rp} G_{s}^{p} \right) - 2B_{s(k)}^{rs} \sigma_{r(k)} + B_{(k)}^{rs} \sigma_{r(j)} \dot{x}^j + B_k^{ij} \sigma_{r(j)} \right) + \frac{\dot{x}^i}{n+1} \left( E_v \dot{\partial}_r K_m \right) \left( H_r^r - H_k^r \right) + \dot{\partial}_r K_m \right) \frac{(n-2)}{(n-1)} \left( \sigma_s \left( B_{k(j)}^{rs} \dot{x}^j \right) - 2B_{(k)}^{rs} + 2B_{(k)}^{rp} G_{p}^{s} - B_k^{rp} G_{s}^{p} \right) - 2B_{s(k)}^{rs} \sigma_{r(k)} + B_{(k)}^{rs} \sigma_{r(j)} \dot{x}^j + B_k^{ij} \sigma_{r(j)} \right) = 0.\]
Transvecting (4.4) by $\dot{x}^k$ and then using equations (1.8), (3.11), (3.12) and (3.13), we get

\begin{equation}
(n - 1)\sigma_s \{ 2W_i^i B^{rs} + 2(\partial_r W_k^i)B^{rs} \dot{x}^k - \dot{x}^i (W_r^m B^{rs}_m + \partial_r W_k^m B^{rs}_m \dot{x}^k) \} + \left. + (n - 2)K_m^r (B_{(k)}^{ir} \dot{x}^m + B_{(k)}^{mr} \dot{x}^i) \sigma_r \dot{x}^k = 0. \right.
\end{equation}

Therefore, we can state:

**THEOREM (4.1):**

If an $F_n^*$ admits a conformal motion then the equation (4.5) always hold.

In order that the conformal motion under consideration becomes an affine motion ($E_v G_j^i = 0$) then with the help of (3.5), (3.6), (3.7) and (3.8) we shall have

\begin{equation}
B_{jk}^s \sigma_s = 0,
\end{equation}

with the help of (4.6), we shall have the following forms

\begin{equation}
B_{j}^s \sigma_s = 0, \quad B_{is}^s \sigma_s = 0.
\end{equation}

Therefore, with the help of equations (4.5), (4.6) and (4.7), we get

\begin{equation}
K_m^r \{ \sigma_r (B_{(k)}^{ir} \dot{x}^m + B_{(k)}^{mr} \dot{x}^i) \} \dot{x}^k = 0.
\end{equation}

Therefore, we can state:

**THEOREM (4.2):**

The necessary condition in order that a conformal motion admitted in an $F_n^*$ may become an affine motion is given by (4.8).

5. SPECIAL PROJECTIVE MOTION IN A SPECIAL PROJECTIVE RECURRENT FINSLER SPACE:

**DEFINITION (5.1):**

A Finsler space is said to be a special projective recurrent Finsler space [2], if the projective entity $Q_{hjk}^i (x, \dot{x})$ satisfies the relation
(5.1) \[ Q^i_{hjk(s)} = \mu_s Q^i_{hjk} (\mu_s \neq 0), \]

where \(\mu_s(x)\) is any vector field homogeneous of degree zero in its directional arguments. Such a space shall be denoted by \(SPR - F_n\) and \(\mu_s(x)\) is called the special projective recurrence vector. In view of the equation (1.15) and the definition (5.1), we observe that the Berwald’s curvature tensor will also satisfy the above definition of recurrency.

**DEFINITION (5.2):**

When an infinitesimal point transformation (2.1) transforms the system of geodesics into the same system then (2.1) is called an infinitesimal special projective motion. The necessary and sufficient condition in order that (2.1) be a special projective motion in \(SPR - F_n\) is that the Lie-derivative of \(\Pi^i_{jk}\) with respect to (2.1) has the form

(5.2) \[ \mathcal{L}_v \Pi^i_{jk} = \delta^i_j \Psi_k + \delta^i_k \Psi_j, \]

for a certain non-zero covariant vector \(\Psi_j(x)\) [11]. Let us introduce a quantity \(B^o_{hj}\) by the following rule

(5.3) \[ B^o_{hj} \overset{\text{def}}{=} - \frac{1}{(n^2-1)} \left( nQ_{hj} + Q_{jh} \right). \]

By virtue of (1.16), the projective curvature tensor \(W^i_{hjk}\) can also be written as

(5.4) \[ W^i_{hjk} = Q^i_{hjk} + E^i_{hjk}, \]

where

(5.5) \[ E^i_{hjk} = - \frac{2}{(n^2-1)} \left\{ (n+1)Q_{j[k} - H_{j[k} - H_{[k(j)} + (n-1)\hat{\delta}_j \hat{\delta}_{[k} H - \hat{x}^s \hat{\delta}_j H^r_{rs[k]} \right\} \delta^i_h. \]

We shall also introduce a curvature tensor \(W^o_{hjk}(x, \dot{x})\) which will be of use in the later discussions by the following rule:
\( W^\circ_{hjk} = B^\circ_{hj}(k) - B^\circ_{hk}(j) \).

In view of (5.2) and (2.11), we get
\[
\begin{align*}
(5.7) \quad \mathcal{E}_v Q_{hjk}^i &= \delta^i_j \Psi_{h,(k)} - \delta^i_k \Psi_{h,(j)} + \delta^i_h \Psi_{j,(k)} - \\
&- \delta^i_h \Psi_{k,(j)},
\end{align*}
\]

where, we have taken into account the following facts
\[
(5.8) \quad (a) \quad \Pi_{hjk}^i \dot{x}^h = 0, \quad (b) \quad \Psi_s \dot{x}^s = 0.
\]

Contracting (5.7) with respect to the indices \( i \) and \( k \) and thereafter using the equation (1.14), we get
\[
(5.9) \quad \mathcal{E}_v Q_{hj} = \Psi_{j,(h)} - n \Psi_{h,(j)}.
\]

From (5.3), we get
\[
(5.10) \quad \mathcal{E}_v B^\circ_{hj} = \Psi_{h,(j)}.
\]

In view of equations (2.5) and (2.6), we get
\[
(5.11) \quad \mathcal{E}_v \Pi_{jk}^i = v_{(j)(k)}^i + Q_{jkh}^i v^h + \Pi_{sjk}^i v_s^r \dot{x}^r
\]

and
\[
(5.12) \quad \mathcal{E}_v B^\circ_{jk} = B^\circ_{jkh} v^h + B^\circ_{hk} v^h_{(j)} + B^\circ_{jkh} v^h_{(k)} + \\
+ \dot{\delta}^i_h B^\circ_{jk} v^h_{(s)} \dot{x}^s.
\]

The first set of integrability condition of (5.2) and (5.10) is composed of

\[
(5.13) \quad (a) \quad \mathcal{E}_v W_{hjk}^i = 0, \quad (b) \quad \mathcal{E}_v W_{hjk}^o = -\Psi_s W_{hjk}^s.
\]

In an affinely connected \( SPR - F_n \) [9], let us try to discuss the existence of special projective motion (2.1) satisfying (5.2). For this purpose, at first, we have to assume the condition (5.13). In what follows, we shall find an important property of \( W_{hjk}^i \) holding in \( SPR - F_n \) admitting a special projective motion given by (2.1).

With the help of equations (1.14) and (5.1), we get
\[
(5.14) \quad Q_{hj,(s)}^i = \mu_s Q_{hj}.
\]
Taking the covariant derivative of (5.3) with respect to $x^8$ and noting (5.14) and (5.3) itself, we get

\begin{equation}
B_{h j}^0(s) = \mu_s B_{h j}^0.
\end{equation}

By virtue of (5.1), (5.5) and (5.14), we deduce that

\begin{equation}
E_{h j k}^i(s) = \mu_s E_{h j k}^i.
\end{equation}

In this way, from equation (5.1), (5.4) and (5.16), we get at least an essential property of $W_{h j k}^i$ in the form

\begin{equation}
W_{h j k}^i(s) = \mu_s W_{h j k}^i.
\end{equation}

6. CONCRETE FORM OF SPECIAL PROJECTIVE MOTION:

Taking the Lie-derivative of both sides of (5.17) and thereafter using (5.13), we get

\begin{equation}
E_v \left(W_{h j k}^i(s)\right) = (E_v \mu_s) W_{h j k}^i.
\end{equation}

Applying the commutation formula (2.10) to the projective curvature tensor $W_{h j k}^i$ we get

\begin{equation}
E_v \left(W_{h j k}^i(s)\right) - (E_v W_{h j k}^i)_i(s) = (E_v \Pi^i_r s) W_{h j k}^r - (E_v \Pi^r_i s) W_{h j k}^r - (E_v \Pi^r_i s) W_{h j k}^r - (E_v \Pi^r_i s) x^r \partial_m W_{h j k}^i.
\end{equation}

Using (5.2) and (5.8a) in (6.2), we get

\begin{equation}
E_v \left(W_{h j k}^i(s)\right) - (E_v W_{h j k}^i)_i(s) = -2 \Psi_s W_{h j k}^i - \Psi_s W_{s j k}^i - \Psi_s W_{h s k}^i - \Psi_s W_{h j k}^i - \Psi_s W_{s j k}^i - \Psi_s W_{h s k}^i - \Psi_s W_{h j k}^i.
\end{equation}

Using (6.1) in (6.4), we get
(6.5) \((2\Psi_s + \varepsilon_v \mu_s) W_{hjk}^i = \delta^i_s \Psi_r W_{hjk}^r - \Psi_h W_{sjk}^i - \Psi_j W_{hsk}^i - \Psi_k W_{hjs}^i.\)

Contracting (6.5) with respect to the indices \(i\) and \(s\) and thereafter using the equation (1.8i), we get

\[
(6.6) \ (2\Psi_s + \mu_s) W_{hjk}^s = n \Psi_r W_{hjk}^r - \Psi_h W_{sjk}^s + \Psi_j W_{hks}^s - \Psi_k W_{hjs}^s.
\]

Using (1.8h) and (1.8n) in (6.6), we get

\[
(6.7) \ (E_v \mu_s) W_{hjk}^s = (n - 2) \Psi_s W_{hjk}^s.
\]

Using (5.13b) in (6.7), we get

\[
(6.8) \ (E_v \mu_s) W_{hjk}^s = -(n - 2) E_v W_{hjk}^o.
\]

Transvecting (6.5) by \(\Psi_i\) and then summing over \(i\), we get

\[
(6.9) \ (2\Psi_s + \varepsilon_v \mu_s) \Psi_i W_{hjk}^i = \Psi_s \Psi_r W_{hjk}^r - \Psi_h \Psi_i W_{sjk}^i - \Psi_j \Psi_i W_{hsk}^i - \Psi_k \Psi_i W_{hjs}^i.
\]

In order to avoid getting a special form of special projective motion in a \(SPR - F_n\), we assume here and hereafter that \(\Psi_i W_{hjk}^i\) does not vanish i.e. \(E_v W_{hjk}^s \neq 0\). In fact, if we have the condition \(\Psi_i W_{hjk}^i = 0\), the vector \(\Psi_i\) becomes restricted by this condition and hence the motion is specialized.

The formula (6.9) takes the form

\[
(6.10) \ (E_v \mu_s) \Psi_i W_{hjk}^i = -\Psi_s \Psi_i W_{hjk}^i - \Psi_h \Psi_i W_{sjk}^i - \Psi_j \Psi_i W_{hsk}^i - \Psi_k \Psi_i W_{hjs}^i.
\]

For \(n \geq 3\), the equation (6.7) yields

\[
(6.11) \ \Psi_s W_{hjk}^s = \frac{1}{n-2} (E_v \mu_s) W_{hjk}^s.
\]

Using (6.10) and (6.11), we get
\[(6.10) \quad (E_v \mu_s)(E_v \mu_i) W^i_{hjk} \]

\[= -\psi_s (E_v \mu_i) W^i_{hjk} - \psi_h (E_v \mu_i) W^i_{sjk} - \]

\[\psi_j (E_v \mu_i) W^i_{hsk} - \psi_k (E_v \mu_i) W^i_{hjs}.\]

or

\[(6.13) \quad (E_v \mu_i) [(E_v \mu_s) W^i_{hjk} + \psi_s W^i_{hjk} + \]

\[+ \psi_h W^i_{sjk} + \psi_j W^i_{hsk} + \psi_k W^i_{hjs}] = 0.\]

Using (6.5), we get

\[(6.14) \quad \psi_h W^i_{sjk} + \psi_j W^i_{hsk} + \psi_k W^i_{hjs} \]

\[= \delta_s^i \psi_r W^r_{hjk} - (2 \psi_s + E_v \mu_s) W^i_{hjk}.\]

Using (6.13) and (6.14), we get

\[(6.15) \quad \psi_s W^i_{hjk} (E_v \mu_i) = (E_v \mu_s) \psi_r W^r_{hjk}.\]

Using (5.13b) and (6.8) in (6.15), we get

\[(6.16) \quad [(n - 2) \psi_s - E_v \mu_s] E_v W^\phi_{hjk} = 0.\]

But \(E_v W^\phi_{hjk} \neq 0\) and hence from (6.16), we get

\[(6.17) \quad \psi_s = \frac{1}{(n-2)} (E_v \mu_s), \quad (n \geq 3).\]

Therefore, we can state:

**THEOREM (6.1):**

If an \(SPR - F_n(n \geq 3)\) admits an infinitesimal special projective motion then the motion should be of the form

\[(6.18) \quad \tilde{x}^i = x^i + \nu^i(x) dt,\]

\[E_v \Pi^i_{jk} = \delta^i_j \psi_k + \delta^i_k \psi_j,\]

\[\psi_k = \frac{1}{(n-2)} E_v \mu_k.\]

Next, let us examine the case when \((E_v \mu_h)\) denotes a parallel vector field, i.e.

\[(6.19) \quad (E_v \mu_h)_{(ij)} = 0.\]

In view of (6.7) and (5.10), we get
(6.20) \( E_v B^o_{hj} = \frac{1}{(n-2)} (E_v \mu_h)(j) \).

By virtue of the formula (6.19), the equation (6.20) takes the form

(6.21) \( E_v B^o_{hj} = 0 \).

Using (5.3) in (6.21), we get

(6.22) \( E_v E^i_{hjk} = 0 \).

We now operate (5.4) by \( E_v \) and thereafter use (6.22) and get

(6.23) \( E_v W^i_{hjk} = E_v Q^i_{hjk} \).

In the case of the present theory we know that \( E_v W^i_{hjk} = 0 \), therefore from (6.23), we get

(6.24) \( E_v Q^i_{hjk} = 0 \).

But this gives us the parallel property of \( E_v \mu_k \). In the following lines, we shall try to prove it.

The equations (5.13a) and (6.24) give us (6.22), through which we can find (6.21). In view of (6.21) the formulae (5.10) takes the form

(6.25) \( \Psi_h(j) = 0 \).

With the help of equations (6.17) and (6.25), we get

(6.26) \( (E_v \mu_h)(j) = 0 \).

Therefore, we can state:

**Theorem (6.2):**

In order that \( E_v \mu_k \) may behave like a parallel vector in \( SPR - F_n \) admitting special projective motion it is necessary that \( E_v Q^i_{hjk} = 0 \).

In order that special projective motion becomes special projective affine motion, it is necessary and sufficient that \( \Psi_h = 0 \) or \( E_v \mu_h = 0 \), consequently we can state:
THEOREM (6.3):

In a $\text{SPR} - F_n$, in order that a special projective motion may become a special projective affine motion, it is necessary and sufficient that $\mathcal{L}_v \mu_h = 0$.

7. PROJECTIVE MOTION IN A SYMMETRIC FINSLER SPACE:

If the infinitesimal point transformation (2.1) transforms the system of geodesics into the same system then such a transformation is said to define an infinitesimal projective transformation.

DEFINITION (7.1):

The necessary and sufficient condition in order that infinitesimal point transformation (2.1) admits a projective motion is that the Lie-derivative of $G^i_{jk}[1]$ satisfies

\( \mathcal{L}_v G^i_{jk} = \delta^i_j p_k + \delta^i_k p_j - g_{jk} g^{ir} d_r, \)

where $p_j(x, \dot{x})$ and $d_r(x, \dot{x})$ are non-zero covariant vectors and they satisfy the following conditions:

\( \begin{align*}
    (a) & \quad \dot{p}_h = \dot{h} p, \\
    (b) & \quad \dot{h} p_k = p_{hk}, \\
    (c) & \quad p_{hk} \dot{x}^h = 0, \\
    (d) & \quad p_j \dot{x}^j = p(x, \dot{x}), \\
    (e) & \quad d_h = \dot{h} d, \\
    (f) & \quad \dot{h} d_k = d_{hk}, \\
    (g) & \quad d_{hk} \dot{x}^h = 0, \\
    (h) & \quad d_j \dot{x}^j = d(x, \dot{x}).
\end{align*} \)

Differentiating (7.1) partially with respect to $\dot{x}^h$ and using equations (2.9), (7.2b) and (7.2f), we get

\( \mathcal{L}_v Q^i_{hk} = \delta^i_j p_{hk} + \delta^i_k p_{hj} - 2c_{hjk} g^{i\ell} d_\ell - g_{jk} \dot{h} g^{i\ell} d_\ell - g_{jk} g^{i\ell} d_{h\ell}. \)

Similarly with the help of (7.1) and (7.2d), we get

\( \mathcal{L}_v G^i_k = p_k \dot{x}^i + p \delta^i_k - g_{jk} g^{ir} d_r \dot{x}^j. \)
DEFINITION (7.2):

If the Berwald’s covariant derivative of $H^i_{hjk}$ [6] satisfies the relation

$$(7.5) \quad H^i_{hjk(m)} = 0,$$

then such a Finsler space is called a symmetric Finsler space. Transvecting (7.5) successively by $\dot{x}'s$ and thereafter noting the properties of the Berwald curvature tensor, we get

$$(7.6) \quad H^i_{jk(r)} = 0$$

and

$$(7.7) \quad H^i_{k(r)} = 0.$$

Applying the commutation formula (2.7) to the deviation tensor field $H^i_{j}$ and using equation (7.1) and (7.7), we get

$$(7.8) \quad (\mathcal{E}_\nu H^k_j)_{(k)} = H^i_{kj}p_j + 2H^i_{j}p_k - H^k_{ij}p_h\delta^i_k + p\hat{\delta}_k H^i_j +$$

$$+d_r\{H^h_j g_{kh} g^{hr} - H^i_j g_{kj} g^{hr} -$$

$$+ (\hat{\delta}_h H^i_j) g_{km} g^{hr} \dot{x}^m\}.$$ 

Contracting (7.8) with respect to the indices $i$ and $j$ and using equation (4.3b) and the fact that the operations of contraction and Lie-derivation are commutative, we get

$$(7.9) \quad (\mathcal{E}_\nu H)_{(k)} = 2Hp_k + \frac{1}{(n-1)}\{p(\hat{\delta}_j H^i_i) -$$

$$- d_r (\hat{\delta}_h H^i_i) g_{km} g^{hr} \dot{x}^m\}.$$ 

Therefore, we can state:

THEOREM (7.1):

If a symmetric Finsler space admits a projective motion of the type (7.1) then the relation (7.9) always holds.

In view of the commutation formula (2.8) and the equation (3.26) and (3.27), the Lie-derivative of $H^i_{hjk}$ is given by
(7.10) \( E_v H^i_{jk} = 2\{\delta^i_h p_{[j(k)]} + p_{h[(k]}\delta^i_j] - g^{it}d_{\ell[(k)}g_{j]\ell} + g^{st}d_{\ell}G^i_{sh[j}g_{k]r}\hat{x}^r}\). 

Transvecting (7.10) by \( \hat{x}^h \hat{x}^j \) and thereafter using the fact that \( E_v \hat{x}^i = 0 \), we get

(7.11) \( E_v H^i_k = \hat{x}^i\{2p_{(k)} - pk_{(j)}\hat{x}^j\} - \delta^i_j p_{(j)}\hat{x}^j - \hat{x}^h \hat{x}^j g^{it}\{d_{\ell(k)}g_{jn} - d_{\ell(j)}g_{kh}\} \).

Contracting (8.4) with respect to the indices \( i \) and \( j \), we get

(7.12) \( E_v H = -\hat{x}^h \left[ p_{(h)} - \frac{1}{(n-1)}\{d_{(h)} - g^{it}d_{(i)}g_{jh}\hat{x}^j\} \right] \).

Differentiating (7.12) covariately with respect to \( x^k \) and equating the equation thus obtained with (7.9), we get

(7.13) \( (n - 1)\{2Hp_k + p_{(h)}\hat{x}^h\} + p(\hat{d}_kH^i_k) - d_r(\hat{d}_hH^i_k)g_{km}g^{hr}\hat{x}^m - \hat{x}^h\{d_{(h)} - g^{it}d_{(i)}g_{jh}\hat{x}^j\} = 0 \).

Therefore, we can state:

**THEOREM (7.2):**

If a symmetric Finsler space admits a projective motion of the type (7.1) then the equation (7.13) holds.

In view of equation (2.3) and (7.6), we get

(7.14) \( E_v H^i_{jk} (m) = 2H^i_{h[k}\nu^h_{(j)}(m) - H^h_{jk}\nu^i_{(h)}(m) + +H^i_{hjk}\nu^h_{(s)}(m)\hat{x}^s \).

Eliminating \( \nu^i_{(j)(k)} \) with the help of (2.4) and (7.14), we get

(7.15) \( E_v H^i_{jk} (m) = (H^i_{hk}E_v G^h_{jm} - H^i_{hj}E_v G^h_{km} - H^h_{jk}E_v G^i_{hm} + +H^i_{hjk}E_v G^h_{sm}\hat{x}^s \) + \( \nu^s H^i_{hk}H^h_{sjm} - -H^i_{hj}H^h_{skm} - H^h_{jk}H^i_{shm} + H^i_{hjk}H^h_{sm} \) + \( \nu^s_{(r)}\hat{x}^r (H^i_{hk}G^r_{jms} - H^i_{hj}G^r_{kms} - H^i_{jk}G^r_{hms}) \).
Using (7.5), (7.6) and the relevant commutation formula for $H^i_{jk}$ in chapter 1, we get

\[(7.16) \quad H^i_{jk} G^i_{hms} - 2H^i_{s[k} G^s_{j]hm} = 0\]

and

\[(7.17) \quad H^i_{jk} H^i_{s mh} - H^i_{hk} H^h_{smj} - H^i_{jh} H^h_{smk} - H^i_{hjk} H^h_{sm} = 0,\]

respectively.

Using the equations (7.1), (7.15), (7.16) and (7.17), we get

\[(7.18) \quad (\mathcal{E}_v H^i_{jk})_{(m)} = p H^i_{mjk} - H^i_{jk} p_h g^i_m + 2(H^i_{jk} p_m +
+ H^i_{m[k} p_{j]} + g^{hs} d_s H^i_{h[j} g_{k]m}) +
+ H^i_{jk} g_{hm} g^i_s d_s - g_{sm} g^{hn} d_n H^i_{hjk} \dot{x}^s.
\]

Contracting (7.18) with respect to the indices $i$ and $k$, we get

\[(7.19) \quad (\mathcal{E}_v H^i_j)_{(m)} = 2H^i_j p_m + H^i_m p_j + p H^i_{mj} +
+ g^{hr} d^r (H^i_{hmj} - H^i_h g_{jm}) +
+ H^i_{ji} g^{is} g_{hm} d_s - H^i_{hj} g_{sm} g^{hn} d_n \dot{x}^s,
\]

where

\[(7.20) \quad H^i_{hj} = H^i_{hj} g_i.\]

Therefore, we can state:

**THEOREM (7.3):**

In symmetric Finsler space admitting a projective motion characterized by (7.1) the equation (7.19) always holds.
### Reference

| [9]  | Rund, H. | The differential geometry of Finsler |


*****