CHAPTER IV

A (Q, R) MODEL FOR FUZZIFIED DETERIORATION UNDER COBWEB PHENOMENON AND PERMISSIBLE DELAY IN PAYMENT

There always exists a significant time lag between production and consumption of the inventory items and in this lag producers face the risk of decision making not only about levels of production and consumption but deterioration of the items also because their assessment about the future price of the inventory items is based on the backward trend of the price in the market. In order to provide a scientific foundation to aforesaid decision making about optimal quantity and price of the items, a (Q, R) model is here studied for the fuzzified deterioration occurring in time lag between production and consumption of items.

The chapter deals with the traditional cobweb phenomenon and an attractive policy of permissible delay in the payment. A computational algorithm has been developed to solve the problem in order to attain the optimal quantity and its price for the model. The chapter also focuses on comparative study of aforesaid model under crisp and fuzzy environments with the help of illustrative examples to simply gain the better perspectives of the model from the application point of view.
A (Q, R) MODEL FOR FUZZIFIED DETERIORATION
UNDER COBWEB PHENOMENON AND PERMISSIBLE
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4.0 INTRODUCTION

The most important factor of an inventory model which cannot be
neglected by researchers engaged in this field is deterioration of an
inventory item. It can be defined as decay, evaporation, obsolescence,
loss of production quantity due to faulty machine, aged machines and
manufacturing defects etc. Vegetables, meat, fertilizers, gasoline,
different types of oils, medicines, milk, machines etc. are examples of
deteriorating items. Inventory models for deteriorating items have been
studied by several researchers in recent decades.

Ghare and Schrader [12] developed an inventory model for an item
with an exponentially decaying inventory. An inventory model for items
with a variable rate of deterioration was discussed by Covert and Philip
[10] who used two-parameter Weibull distribution for the deterioration
time. Philip [29] adopted three-parameter distribution for deterioration
time. Mishra [20] formulated an inventory model with a variable rate of
deterioration along with a finite rate of production. Very recently, Mishra
and Mishra [21] computed the optimum quantity and the price of EOQ
for deteriorating items under perfect competition using marginal revenue
and marginal cost approach. Several researchers like Cohen [9], Kang
and Kim [15], Aggarwal and Jaggi [3], Wee [32], Giri and Choudhuri [13] and Chang et.al. [7] developed economic production lot size models with different assumptions on the patterns of the deteriorating rate. These researches show that most of our inventory models have been developed on the assumption that the deterioration rate is constant or it is dependent on time. For solving the optimum production quantity models, we oftenly consider the demand rate, production rate and deterioration rate as constant or dependent on time in the crisp model. But, in the real life situations, the economic production quantity deviates from the exact value if the variables are not crisp but uncertain in nature. Hence, these variables should be treated as fuzzy variables because fuzzy decision making is a powerful paradigm for dealing with risk ridden problem of decision and management in the environment of uncertainty, vide for example, Yao and Lee [19], Zimmerman [35], Kaufmann and Gupta [16] and Mahata and Goswami [25]. Wu [17] employed continuous review of (Q, R) model and also allowed permissible delay in payment in deterministic environment assuming exponential demand during lead time. The cobweb model is based on a time lag between production and consumption of the inventory problem. Agricultural markets are thought to be a situation where the cobweb model might apply; since, there is a lag between planting and harvesting. Suppose that as a result of bad weather, farmers go to market with an unusually small crop of tomatoes (say). This shortage, equivalent to a leftward shift in the market’s production curve, results in high prices. If farmers expect these high price
conditions to continue, then in the following year, they will raise their production of items of tomatoes, relative to other crops. Therefore, when they go to market, the supply will be high (because production is high), resulting in low prices. If, they then expect low prices to continue they will decrease their production of tomatoes for the next year, resulting in high prices again. This kind of risk regarding high and low productions of inventory can be managed by handling the problem with the help of some systematic study of market economy which is based on rational expectations of price of the inventory. The cobweb model is nothing but serves as one of the best ways of why understanding formation of expectation is so important for economic dynamics so that planning and controlling of production is made to reach the equilibrium price of the inventory. Nicholson [27] defined the equilibrium price for which the quantity demanded is precisely equal to the quantity supplied. At such price, there is no incentive for either demand makers or suppliers to alter their behaviours. As per arguments of Ostaszewski [28], if demand exceeds supply or supply exceeds demand, then the market will not be in equilibrium. Before the market reaches equilibrium, market participants have to decide how much they will supply and demand. Other related literature can also be reviewed, vide for example, Walras [31], Marshall [23], Muth [24], Nerlove [26] and Aggarwal and Jaggi [2], Brock [5], Bergman [4] and Chapman et.al. [6], Chen and Chen [8], Frankenberger and Lu [11], Jacobson and Obermiller [14], Lichtestein et.al. [18], Marshall [23] Mishra et.al. [22] and Winer [33], Yao and Lee
[34]. In deriving the optimum quantity formula, it is always assumed that the supplier must be paid for the items as soon as the items are received. In practice, however, supplier offers their customers a certain credit period without interest during the permissible delay time period. Allowing a delay in payment is a form of a price discount for the customers. Such delay of payment is some kind of encouragement to the customers to order large quantities because a delay of payment indirectly reduces the purchase cost. Previous researchers have attempted to study inventory models with deterioration and permissible delay in payment but in only deterministic environment and without cobweb theory.

This chapter deals with the (Q, R) model with fuzzified deterioration occurring during the time lag in between production and consumption of the given inventory which follows the traditional cobweb phenomenon and considers an attractive effect of permissible delay in payment to lure the customers to buy more. A system of non-linear equations has been developed by making use of Zadeh’s extension principle and has been solved with the help of a computing algorithm to obtain optimum quantity and its price as the most powerful performance measures of the model. Advantages of the present problem lie in the fact that performance measures such as price of inventory and optimum quantity are computed in the fuzzy environment and, in turn, these are compared with crisp environment to gain the better insight for planning and controlling the production of the inventory.

4.1 Description of the model
Here, we study a \((Q, R)\) model which follows the cobweb trend between production and consumption of the inventory in the market of agricultural goods. Since, there exists a time lag between production and consumption of inventory items in which there occurs an uncertain spoilage or deterioration of inventory items, for example, infections in food items etc. For controlling this kind of uncertainty, we have two ways; one is probabilistic and another is fuzzy approach. Since, fuzzy approach is more versatile approach than probabilistic approach; we apply fuzzy approach to solve the problem of aforesaid uncertainty. The present problem lays down emphasis on a continuous review \((Q, R)\) model with back order cost which prescribes ordering \(Q\) items whenever the inventory level reaches \(R\). The objective of choosing \(Q\) and \(R\) in such a way so as to minimize the holding, ordering and back order costs for the average inventory. Moreover, the model considers that if any demand, when out of stock, is back ordered then the back order penalty is proportional to the number of items back ordered.

The following basic assumptions are used:

i. Demand and supply both are linear functions of price.

ii. Replenishment is non-instantaneous.

iii. Demand during lead time follows exponential distribution.

iv. Deterioration of item is in fuzzy nature and follows triangular membership function.

v. There exists a fixed reorder level in different cycles.
vi. Permissible delay is allowed in payment till credit period and thereafter interest is charged from the customer. We use the following notations and assumptions throughout this chapter:

\[ S_t = \text{current product quantity level at time } t \]
\[ Q_d = \text{demand quantity at time } t \]
\[ Q_{S_{t-1}} = \text{supplied quantity at time } t \text{ but it is decided by producers at time } t-1 \]
\[ \theta = \text{rate of deterioration during production as well as during waiting time.} \]
\[ D_t = \text{the current demand quantity} \]
\[ A = \text{fixed cost} \]
\[ D = \text{annual demand} \]
\[ p_t = \text{price per item at time } t \]
\[ \overline{P}_t = \text{price of an item in fuzzy environment at time } t \]
\[ h = \text{holding cost of per unit of item per year excluding the interest charges for financing the stock} \]
\[ \mu = \text{mean demand during lead time} \]
\[ \xi = \text{unit shortage cost} \]
\[ x = \text{demand during the lead-time} \]
\[ F(x) = \text{cumulative distribution function of the demand during lead time} \]
\[ t_c = \text{the credit period} \]
\[ I_d = \text{the discount derived by the buyer during the credit period} \]
\( \frac{Q}{2} \) = the average cycle stock

\( R \) = the reorder level

\((R - \mu)\) = the safety stock which is held throughout the cycle

\( \theta_1 \) = lower value of deterioration

\( \theta_0 \) = middle value of deterioration

\( \theta_2 \) = upper value of deterioration

We can now easily denote the holding costs for the safety stock and the cycle stock as \((h + p_t I_c)(R - \mu)\) and \(\frac{hQ^2}{2D}\) respectively. Here, we assume that \(t_c\) is less than the reorder interval and production is started at time \(t\); however, the final quantity of product does not come out until time \(t + 1\). Due to this time lag, producer has to use price level of current time at time \(t\) to decide the quantity produced at time \(t + 1\). Therefore, the current supply \((S_t)\) is the current product quantity level at time \(t\) \((Q_{s,t-1})\) which is decided by producers at time \(t - 1\) while considering the market price at time \(t - 1\). The model further assumes that the current supply at time \(t\) \((S_t)\) is the function of \(p_{t-1}\), which is a price level at time \(t - 1\). This implies that \(S_t : Q_{s,t-1} = f(p_{t-1})\) and demand at time \(t\) \((D_t)\) is the current demand quantity \(Q_{d_t}\) which is the function of \(p_t\), price level at time \(t\). Here, it is worthwhile to mention that quantity produced in between time interval \(t - 1\) to \(t\) be supplied in the market to fulfill the demand at time \(t\).

### 4.2 Mathematical Analysis in Crisp Environment
We assume that lead-time demand follows an exponential distribution with mean $\mu$ as $f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, x \geq 0, \mu > 0$ and for this condition we obtain $F(x) = \int_{0}^{x} f(x) dx = 1 - e^{-\frac{x}{\mu}}, x \geq 0, \mu > 0$.

Here, we define the following average annual cost functions for the (Q, R) model in the crisp and fuzzy environments respectively as

\[
C(Q, R) = \frac{AD}{Q} + p_r D + h\left(\frac{Q}{2} + R - \mu\right) + \frac{Q}{2} \pi n(R)
\]

\[
\overline{C}(Q, R) = \frac{AD}{Q} + \frac{Q}{2} + h\left(\frac{Q}{2} + R - \mu\right) + \frac{Q}{2} \pi n(R)
\] (4.2.1) (4.2.2)

Where $p_r$ is a function of deterioration in the fuzzy environment defined in the next section by the equation (4.2.4) and also have been finally expressed by the equation (4.2.7) and $n(R) = \int_{R}^{\infty} (x - R) dF(x)$.

It is further to mention that whenever deterioration of the items takes place, we can obviously express that $S_i - \theta = D_i \Rightarrow Q_i^* - \theta = Q_i^d$.

We further assume that $t_c$ is less than reorder interval, which means that credit period cannot be longer than the time at which another order is placed. This is in agreement with the usual practice prevalent in the market situation. Hence, the interest derived from the sale of the items during the credit period is $\frac{Dp_r t_c^2}{2}$ which is worked out on purchase price. The quantity of back orders of the previous cycle, which are cleared in the beginning of the current cycle, earns an interest of
\( n(R)p_t t_c I_d \) during the credit period. Thus, the total interest derived during the credit period is \( D\ p_t t_c^2 I_d^2 + n(R)p_t t_c I_d \). The on-hand inventory at \( t_c \) is \( (Q - D t_c) \) and on average it takes \( \frac{(Q - D t_c)}{D} \) units of time to consume this stock. The interest charges applicable to this portion of cycle stock become \( \frac{p_t t_c (Q - D t_c)^2}{2D} \). Hence, the total cost per cycle is given by

\[
A + p_t Q + \frac{hQ^2}{2D} + (h + p_t t_c)(R - \mu)\frac{Q}{D} + \pi n(R) - \frac{D p_t t_c^2 I_d}{2} - n(R)p_t t_c I_d
\]

\[+ \frac{\xi D}{Q} n(R) - \frac{D^2 p_t t_c^2 I_d}{2Q} - n(R)p_t t_c I_d \ \frac{D}{Q} + \frac{(Q - D t_c)^2 p_t t_c}{2D} \]

The average cycle length is \( \frac{D}{Q} \) and hence the total average cost (TAC) is

\[
TAC = \frac{AD}{Q} + p_t D + \frac{hQ}{2} + (h + p_t t_c)(R - \mu) - \frac{D^2 p_t t_c^2 I_d}{2Q}
\]

\[-n(R)p_t t_c I_d \ \frac{D}{Q} + \frac{\xi D}{Q} n(R) + \frac{(Q - D t_c)^2 p_t t_c}{2Q} \tag{4.2.3} \]

We have the following expressions for the cobweb model with deterioration rate \( \theta \), \( D: Q_t^d = \alpha + a p_t \), \( S: Q_{S-1}^t = \beta + b p_{t-1} \) which obviously shows that

\( Q_{t-1}^c = \theta + Q_{t-1}^d \), where \( \alpha, \beta > 0 \) and \( \frac{b}{a} < 1 \).

This further implies that \( p_t = \frac{b}{a} p_{t-1} + \frac{\beta - \alpha}{a} \left( \frac{\theta}{a} \right) \)

and then \( p_t = \left( \frac{b}{a} \right) p_0 + \left( \frac{\beta - \alpha}{b-a} \right) \left( \frac{b-a}{a} \right) - \left( \frac{\theta}{a} \right) \)
This further shows that

\[
p_t = \left( \frac{b}{a} \right)^\gamma \left( p_0 + \frac{\beta - \alpha}{b-a} \right) - \left( \frac{\beta - \alpha}{b-a} \right) \left( \frac{1}{a} \right) \left( \frac{b-a}{a} \right)^\gamma
\]

\[M_0(\theta_1, \theta_0, \theta_2) = \frac{\theta_1 + \theta_0 + \theta_2}{3}
\]

We further observe that \( \Phi = \frac{p_t - \theta_1}{f_2} \geq 0 \), for \( f_2 \neq 0, p_t \geq f_1 \); This implies that \( f_1 \leq p_{u_1} \leq p_{u_0} \leq p_{u_2} \); where \( p_{u_1}, p_{u_0}, and p_{u_2} \) are lower, middle and upper price level at time \( t \) in fuzzy environment.

In view of above, membership function for price at time \( t \) is expressed as

\[
\mu_{\Theta_0}(p_t) = \begin{cases} 
\frac{p_t - \theta_1 - \theta_2 f_2}{(\theta_0 - \theta_1) f_2} & \text{for } p_{u_1} \leq p_t \leq p_{u_0} \\
\frac{f_1 + \theta_2 f_2 - p_t}{(\theta_2 - \theta_0) f_2} & \text{for } p_{u_0} \leq p_t \leq p_{u_2} \\
0 & \text{elsewhere}
\end{cases}
\]

As we know that the extension principle of the zadeh is very important tool in the fuzzy set theory for providing procedure to fuzzily a crisp function which is given below. Let \( f : X \to Y \) be a crisp function and \( F(X) \) (respectively \( F(Y) \)) be the set of all fuzzy sets (called fuzzy power set) of \( X \) (respectively \( Y \)). The function \( f : X \to Y \) induces two functions \( f : F(X) \to F(Y) \) and \( f^{-1} : F(Y) \to F(X) \), and the extension principle of Zadeh gives formulas to compute the membership function of fuzzy sets.
\( f(A) \) in \( Y \) (respectively \( f^{-1}(B) \) in \( X \)) in terms of membership function of fuzzy set \( A \) in \( X \) (respectively \( B \) in \( Y \)).

**Definition** (Zadeh’s extension principle, vide [16]): In terms of the notations introduced above, extension principle of Zadeh states that

(i) \( \mu_{f(A)}(y) = \text{Sup}_{x \in X, f(x) = y} \mu_A(x) \) for all \( A \in F(X) \), and

(ii) \( \mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \), \( \forall B \in F(Y) \).

Hence, from above theorem we get

\[
\mu_{G_1(\Phi)}(p_i) = \text{Sup}_{\phi \in \Phi^{-1}(p_i)} \mu_\phi(\theta)
\]

After using this definition, we have the quantities as follows: \( \Pi = \int_{-\infty}^{\infty} \mu_{G_1(\Phi)}(p_i) dp_i \) and \( \Pi_0 = \int_{-\infty}^{\infty} p_i \mu_{G_1(\Phi)}(p_i) dp_i \); the centroid for \( \mu_{G_1(\Phi)}(p_i) \) is given by \( \frac{\Pi_0}{\Pi} \) which is the estimate of total price in fuzzified environment. Therefore, \( \Pi \) and \( \Pi_0 \) can be written as

\[
\Pi_0 = \frac{1}{(\theta_2 - \theta_0)f_2} \int_{p_u}^{p_u} f_2 \{p_i - f_1 + \theta_2 - f_2\} dp_i + \frac{1}{(\theta_2 - \theta_0)f_2} \int_{p_u}^{p_u} f_2 \{f_1 + \theta_2 - f_2 - p_i\} dp_i \tag{4.2.5}
\]

\[
\Pi = \frac{1}{(\theta_2 - \theta_0)f_2} \int_{p_u}^{p_u} f_2 \{p_i - f_1 - \theta_2 f_2\} dp_i + \frac{1}{(\theta_2 - \theta_0)f_2} \int_{p_u}^{p_u} f_2 \{f_1 + \theta_2 f_2 - p_i\} dp_i
\]

\[
(\theta_2 - \theta_0) \left( \frac{p_u^2}{2} - p_u f_1 - \theta_1 f_2 p_u - \frac{p^2_u}{2} + p_u f_2 + \theta_1 f_2 p_u \right)
\]

\[
+ (\theta_2 - \theta_0) \left( p_2 f_1 + \theta_2 f_2 - \frac{p^2_2}{2} - p_u f_1 - \theta_2 f_2 p_u + \frac{p^2_u}{2} \right)
\]

\[
\Pi = \frac{(\theta_2 - \theta_0)(\theta_2 - \theta_0)f_2^2}{(\theta_2 - \theta_0)(\theta_2 - \theta_0)f_2^2} \tag{4.2.6}
\]

Page 143 of 224
After evaluating integrals of \( \Pi_0 \) and \( \Pi \) from above equations. We obtain the centroid of \( \mu_{c_i}(p_t) \) as

\[
\overline{p_t} = \frac{\Pi_0}{\Pi}, \text{ which finally turns out to be}
\]

\[
\frac{(\theta_2 - \theta_0)}{p_t} = \frac{(\theta_2 - \theta_0)}{p_t} \left( \frac{p_{0t}^2}{3} - \frac{p_{0t}^2}{2} f_1 - \theta_1 f_2 - \frac{p_{0t}^2}{3} + \frac{p_{lt}^2}{2} f_1 + \theta_1 f_2 - \frac{p_{lt}^2}{2} \right) + \frac{(\theta_0 - \theta_1)}{p_t} \left( \frac{p_{0t}^2}{2} f_1 + \theta_2 f_2 - \frac{p_{0t}^2}{3} - \frac{p_{lt}^2}{2} f_1 - \theta_2 f_2 + \frac{p_{lt}^2}{2} + \frac{p_{lt}^3}{3} \right)
\]

(4.2.7)

To find the optimal solution of \( \overline{p_t} \), we take first order partial derivative with respect to \( \theta_1 \) and \( \theta_2 \) for fixed values of \( \theta_0 \) and \( t \), we get the two nonlinear equations of \( \theta_2 \) and \( \theta_1 \) respectively as follows:

\[
u = (a \theta_1 + b) \theta_2^2 + c \theta_2 + d \theta_1 + e \theta_1 \theta_2 + A = 0 \tag{4.2.9}
\]

\[
u = (\alpha \theta_2 + \beta) \theta_1^2 + \gamma \theta_1 + \delta \theta_2 + \lambda \theta_2 \theta_1 + \omega = 0 \tag{4.2.10}
\]

Where \( a, b, c, e, A \) and \( \alpha, \beta, \gamma, \delta, \lambda, \omega \) are constants for fixed value of \( \theta_0 \) and \( t \).

In order to solve a system of non-linear equations involving (4.2.9) and (4.2.10) to obtain the optimal deterioration parameters \( \theta_1^* \) and \( \theta_2^* \). We use following computational procedure:
\[ \Delta = \begin{vmatrix} \frac{\partial u}{\partial \theta_1} & \frac{\partial u}{\partial \theta_2} \\ \frac{\partial v}{\partial \theta_1} & \frac{\partial v}{\partial \theta_2} \end{vmatrix} = \begin{vmatrix} a\theta_1^2 + d + \varepsilon \theta_2 & 2(\alpha \theta_1 + \beta)\theta_2 + c + \varepsilon \theta_1 \\ 2(\alpha \theta_2 + \beta)\theta_1 + \gamma + \lambda \theta_2 & a\theta_1^2 + \delta + \lambda \theta_1 \end{vmatrix} \]

\[ \Delta_1 = \begin{vmatrix} -u & \frac{\partial u}{\partial \theta_2} \\ -v & \frac{\partial v}{\partial \theta_2} \end{vmatrix} = \begin{vmatrix} -\left(\alpha \theta_1 + b\right)\theta_2^2 - c \theta_2 - d \theta_1 - \varepsilon \theta_1 \theta_2 - A & 2(\alpha \theta_1 + b)\theta_2 + c + \varepsilon \theta_1 \\ -\left(\alpha \theta_2 + \beta\right)\theta_1^2 - \gamma \theta_1 - \delta \theta_2 - \lambda \theta_1 \theta_2 - \omega & a\theta_1^2 + \delta + \lambda \theta_1 \end{vmatrix} \]

\[ \Delta_2 = \begin{vmatrix} \frac{\partial u}{\partial \theta_1} & -u \\ \frac{\partial v}{\partial \theta_1} & -v \end{vmatrix} = \begin{vmatrix} a\theta_2^2 + d + \varepsilon \theta_2 & -\left(\alpha \theta_1 + b\right)\theta_2^2 - c \theta_2 - d \theta_1 - \varepsilon \theta_1 \theta_2 - A \\ 2(\alpha \theta_2 + \beta)\theta_1 + \gamma + \lambda \theta_2 & \left(\alpha \theta_2 + \beta\right)\theta_1^2 - \gamma \theta_1 - \delta \theta_2 - \lambda \theta_1 \theta_2 - \omega \end{vmatrix} \]

These values are required for using the fast converging N-R method (see for example Jeffery [15]) which is employed to solve the above system of non-linear equations by developing the following algorithm.

**4.3 Computing Algorithm**

We use the following algorithm with C++ language to compute the optimal results.

Step 1: Begin.
Step 2: Input data.
Step 3: Compute low price.
Step 4: Compute middle price.
Step 5: Compute high price.
Step 6: Compute the coefficient of first non-linear equation.
Step 7: Compute the coefficient of second non-linear equation.
Step 8: Do.
Step 9: W — Ratio of first function derivative and that function.
Step 10: R — Ratio of second function derivative and that function.
Step 11: Compute optimal value of theta one.
Step 12: Compute optimal value of theta two.
Step 13: Compute optimal value of price in fuzzy environment.
Step 14: While (error <= 0.00000009)
Step 15: Compute the optimum quantity in fuzzy environment.
Step 16: End.

Remarks: Computing time is higher due to large number of parameters in computing programmes in fuzzy environment and in this environment programme covers more memory space as compared to crisp one. It has also been observed that number of iterations in computing the result in fuzzy environment is more than t the number of iterations in crisp environment.

4.4 Optimal Solution

For fixed value of $\theta_0$, we have to find $(\theta'_1, \theta'_2, t')$ such that the centroid of the fuzzy total price is optimal denoted as $\overline{p_t}$.

i.e. $\overline{p_t} \min_{\theta_1, \theta_2} \frac{\Pi_0(\theta_1, \theta_2, t)}{\Pi(\theta_1, \theta_2, t)} = \frac{\Pi_0(\theta'_1, \theta'_2, t')}{\Pi(\theta'_1, \theta'_2, t')}$. After obtaining the optimal value of price ($p'_t$), we substitute it in the following expressions to yield the optimal value of $C^*(Q, R)$ and $TAC^*(Q, R)$ as


\[
\overline{C}^*(Q, R) = \frac{AD}{Q} + \frac{p_t}{D} + h \left( \frac{Q}{2} + R - \mu \right) + \frac{D \xi}{Q} n(R)
\]

(4.4.1)

\[
\overline{TAC}^*(Q, R) = \frac{AD}{Q} + \frac{p_t}{D} + \frac{hQ}{2} + (h + p_t l_c)(R - \mu) - \frac{D^2 p_t t_c l_d}{2Q}
\]

\[- n(R) p_t t_d l_d \frac{D}{Q} + \frac{\xi D}{Q} n(R) + \frac{(Q - D t_c) p_t l_c}{2Q}
\]

Differentiating \( \overline{TAC}^*(Q, R) \) with respect to \( Q \) and \( R \), we get

\[
\frac{\partial \overline{TAC}^*(Q, R)}{\partial Q} = -\frac{AD}{Q^2} + h \frac{\xi D n(R)}{2Q^2} + \frac{D^2 p_t t_c l_d}{2Q^2}
\]

\[+ \frac{p_t t_c l_d n(R) D}{Q^2} + \frac{p_t l_c p_t l_c D^2 t_c^2}{2Q^2}
\]

(4.4.2)

and

\[
\frac{\partial \overline{TAC}^*(Q, R)}{\partial R} = (h + p_t l_c) - \frac{\xi D}{Q} \left[ 1 - (1 - e^{-\mu}) \right] + \frac{p_t t_c l_d D}{Q} \left[ 1 - (1 - e^{-\mu}) \right]
\]

(4.4.3)

\[= (h + p_t l_c) - \frac{\xi D e^{-\mu} \mu}{Q} + p_t t_c l_d \frac{De^{-\mu}}{Q}, \text{ which implies that}
\]

\[(h + p_t l_c) = \frac{De^{-\mu}}{Q} (\xi - p_t t_c l_d) \Rightarrow e^{-\mu} = \frac{(h + p_t l_c) Q}{(\xi - p_t t_c l_d) D}, \text{ Further, we have}
\]

\[n(R) = \int_{R}^{\infty} (x - R) f(x) dx = \int_{R}^{\infty} (x - R) - \frac{1}{\mu} e^{-\mu} \int_{R}^{\infty} e^{-\mu} dx = \int_{R}^{\infty} e^{-\mu} dx = (x - R) e^{-\mu} = \mu e^{-\mu}
\]

That is, to minimize \( \overline{TAC}^*(Q, R) \) the equations (4.4.2) and (4.4.3) yield the following optimal quantity in fuzzy environment

\[
\frac{Q^*}{h + p_t l_c} = \sqrt{2D \left[ A + \left( \xi - p_t t_c l_d \right) n(R) + 0.5D t_c^2 p_t (l_c - l_d) \right]}
\]

Page 147 of 224
which finally leads to

\[
\overline{Q^*} = \sqrt{\frac{2D}{h+\overline{p}_tI_c} \left[ A + (\xi - \overline{p}_t I_c I_d) \mu e^{\mu t} + 0.5D \overline{r}_c^2 \overline{p}_t (I_c - I_d) \right]} - \frac{R}{\mu}
\]

and \(1 - F(R) = \frac{Q}{D} \frac{h + \overline{p}_t I_c}{(\xi - \overline{p}_t I_c I_d)}\)

\(\overline{TAC}^*(Q,R)\) is also observed to be a convex function for \((Q,R)\) when \(\xi - \overline{p}_t I_c I_d \geq 0\).

4.5 Sensitivity analysis of the Model

A sensitivity analysis is a powerful means to judge the economic viability of the model based on various parameters of the performance measures involved in the study of the model. It is the process of varying model-parameters over a reasonable range and observing the relative change in model response. Mainly, notable are the observed changes in optimal price and quantity of inventory, rate of deterioration and time horizon etc. The aim of the sensitivity analysis is to demonstrate the variability of the model based on the simulations or hypothetical data-input (this paper prefers a set of hypothetical data-input in order to run our search programme). It also focuses on the sensitivity of one model parameter relative to other parameters with the help of numerical demonstration. Sensitivity analyses are significantly beneficial in determining the direction of future data-input and its analysis. Parameters
for which the model is relatively sensitive (vide various tables of parameters and their observations under this section) would require researcher’s attention, as compared to the parameters for which the model is relatively insensitive. But, it is equally important to assess the possible reasons for this insensitivity of the parameters under the study of this model. Values of various parameters are computed and given in the tables which ultimately form the basis for observations related to the sensitivity analysis of the model under consideration.

**Table: 4.5.1**

Table for optimal value of deterioration and also price of an item in fuzzy environment

<table>
<thead>
<tr>
<th>$(\theta_1, \theta_2, \theta_3)$</th>
<th>$t$ (years)</th>
<th>$(a, b)$</th>
<th>$P_0$</th>
<th>$(\alpha, \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.000005, 0.0001, 0.001)</td>
<td>3</td>
<td>(0.0002, 0.0001)</td>
<td>1000</td>
<td>(0.0003, 0.0002)</td>
</tr>
<tr>
<td>(0.00005, 0.001, 0.01)</td>
<td>4</td>
<td>(0.002, 0.001)</td>
<td>1000</td>
<td>(0.003, 0.002)</td>
</tr>
<tr>
<td>(0.0005, 0.01, 0.1)</td>
<td>4</td>
<td>(0.00005, 0.00002)</td>
<td>1000</td>
<td>(0.0003, 0.0002)</td>
</tr>
<tr>
<td>(0.00000008, 0.00005, 0.001)</td>
<td>3</td>
<td>(0.00002, 0.00001)</td>
<td>1000</td>
<td>(0.0003, 0.0002)</td>
</tr>
<tr>
<td>(0.00000008, 0.00005, 0.001)</td>
<td>3</td>
<td>(0.00005, 0.00002)</td>
<td>1000</td>
<td>(0.0003, 0.0002)</td>
</tr>
<tr>
<td>(0.00001, 0.01, 0.1)</td>
<td>3</td>
<td>(0.0005, 0.0002)</td>
<td>1000</td>
<td>(0.0005, 0.0002)</td>
</tr>
<tr>
<td>(0.00002, 0.001, 0.02)</td>
<td>3</td>
<td>(0.00004, 0.00002)</td>
<td>1000</td>
<td>(0.000004, 0.000002)</td>
</tr>
<tr>
<td>(0.0002, 0.01, 0.2)</td>
<td>3</td>
<td>(0.0004, 0.0002)</td>
<td>1000</td>
<td>(0.0004, 0.0002)</td>
</tr>
<tr>
<td>(0.002, 0.01, 0.2)</td>
<td>3</td>
<td>(0.0004, 0.0002)</td>
<td>1000</td>
<td>(0.0004, 0.0002)</td>
</tr>
<tr>
<td>(0.01, 0.002, 0.2)</td>
<td>3</td>
<td>(0.0006, 0.0003)</td>
<td>2000</td>
<td>(0.0008, 0.00008)</td>
</tr>
</tbody>
</table>

$(\theta_1^*, \theta_2^*, \theta_3^*)$

\[
\frac{\bar{P}_i}{p_i}
\]

<table>
<thead>
<tr>
<th>$(\theta_1^<em>, \theta_2^</em>, \theta_3^*)$</th>
<th>$\frac{\bar{P}_i}{p_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.00000051, 0.0001, 0.001043)</td>
<td>0.51166</td>
</tr>
<tr>
<td>(0.000003945, 0.001, 0.012715)</td>
<td>1.22363</td>
</tr>
</tbody>
</table>
Table (4.5.1) shows the computation of optimal values of rate of deterioration and price in fuzzy environment. When lower value of deterioration increases then price will decrease. It is also observed that when the value of coefficients of price in fuzzy environment is slightly increased then the price will also decrease.

**Table: 4.5.2**

**Table for optimum quantity and corresponding price at different times in fuzzy deterioration environment**

D = 100 items, A = Rs.50, h = Rs. 100, \( \xi \) = Rs.50, R = 80, i = 50, \( I_c \) = 0.05, \( I_d \) = 0.02, \( p_0 \) = Rs.1000, \( a = 0.00005 \), \( b = 0.00003 \), \( \alpha = 0.00004 \), \( \beta = 0.00003 \), \( t_c = 2 \).

<table>
<thead>
<tr>
<th>( \theta_1 )</th>
<th>( \theta_0 )</th>
<th>( \theta_2 )</th>
<th>( t(\text{years}) )</th>
<th>( \bar{p}_r ) (Rs.)</th>
<th>( \bar{Q}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000002</td>
<td>0.00005</td>
<td>0.005</td>
<td>3</td>
<td>41.226</td>
<td>521.37</td>
</tr>
<tr>
<td>0.00002</td>
<td>0.0002</td>
<td>0.002</td>
<td>3</td>
<td>17.42</td>
<td>221.54</td>
</tr>
<tr>
<td>0.00002</td>
<td>0.0002</td>
<td>0.002</td>
<td>5</td>
<td>20.67</td>
<td>221.40</td>
</tr>
<tr>
<td>0.0000007</td>
<td>0.0005</td>
<td>0.009</td>
<td>5</td>
<td>92.32</td>
<td>218.19</td>
</tr>
</tbody>
</table>
From table (4.5.2), it is observed that for the given values of
deterioration and with increasing trend in the time horizon, optimal price
in fuzzy environment will increase but at the same time, there is a slight
decrement in optimal quantity of inventory too.

Table: 4.5.3

Table for optimum quantity and corresponding price with various
demands in fuzzy environment

A = Rs.50, h = Rs100, \( \xi = Rs.50 \), \( R = 80 \), \( I_c = 0.05 \), \( I_d = 0.02 \),
\( \theta_1 = 0.0000007, \theta_0 = 0.0002, \theta_2 = 0.002, p_0 = Rs.1000, a = 0.00005, b = 0.00003, \alpha = 0.00004, \beta = 0.00003, t_c = 2\) yrs.

<table>
<thead>
<tr>
<th>Annual demand (D)</th>
<th>( i, (\text{years}) )</th>
<th>( ar{p}, (Rs.) )</th>
<th>( ar{Q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>100</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>2000</td>
<td>900</td>
<td>5</td>
<td>20.20</td>
</tr>
<tr>
<td>2000</td>
<td>1000</td>
<td>5</td>
<td>20.20</td>
</tr>
<tr>
<td>4000</td>
<td>1000</td>
<td>7</td>
<td>21.37</td>
</tr>
<tr>
<td>4000</td>
<td>1000</td>
<td>9</td>
<td>21.80</td>
</tr>
</tbody>
</table>

Above table (4.5.3) shows that for increase in annual demand,
mean demand during lead time and time horizon the optimum price does
not fluctuate significantly but optimal quantity of inventory increases
increases significantly.

Table: 4.5.4

Table for optimum quantity with the different reorder level with
annual demand in fuzzy environment
\[ A = \text{Rs.}50, \ h = \text{Rs.}100, \ \xi = \text{Rs.}50, \ i = 100, \ I_c = 0.05, \ I_d = 0.02, \ \theta_1 = 0.0000007, \ \theta_0 = 0.0002, \ \theta_2 = 0.002, \ p_0 = \text{Rs.}1000, a = 0.000005, b = 0.000003, \ \alpha = 0.000004, \ \beta = 0.000003, \ t_c = 2 \text{ years}. \]

<table>
<thead>
<tr>
<th>Annual demand (D)</th>
<th>Reorder level (R)</th>
<th>( p_t (\text{Rs.}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>20.20</td>
<td>662.77</td>
</tr>
<tr>
<td>2000</td>
<td>20.20</td>
<td>937.29</td>
</tr>
<tr>
<td>500</td>
<td>16.92</td>
<td>7639.74</td>
</tr>
<tr>
<td>4000</td>
<td>16.92</td>
<td>10804.22</td>
</tr>
<tr>
<td>1000</td>
<td>16.92</td>
<td>131177.06</td>
</tr>
</tbody>
</table>

From this table (4.5.4), we can easily conclude that for the increasing trend in annual demand and reorder level, there is no significant change on optimal price but optimal quantity of the inventory increases considerably.

**Table: 4.5.5**

**Table for optimum quantity and price for different demand and supply pattern in fuzzy environment**

\[ D = 2000, \ A = \text{Rs.}50, \ h = \text{Rs.}100, \ \xi = \text{Rs.}50, \ R = 500, \ i = 100, \ I_c = 0.05, \ I_d = 0.02, \ \theta_1 = 0.0000007, \ \theta_0 = 0.0002, \ \theta_2 = 0.002, \ p_0 = \text{Rs.}1000, \ t_c = 2 \text{ years}. \]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( a )</th>
<th>( \beta )</th>
<th>( b )</th>
<th>( t )</th>
<th>( p_t )</th>
<th>( Q^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00004</td>
<td>0.00005</td>
<td>0.00003</td>
<td>0.00003</td>
<td>5</td>
<td>20.20</td>
<td>7647.73</td>
</tr>
<tr>
<td>0.00004</td>
<td>0.00005</td>
<td>0.00003</td>
<td>0.00003</td>
<td>7</td>
<td>21.379</td>
<td>7656.48</td>
</tr>
<tr>
<td>0.00003</td>
<td>0.00005</td>
<td>0.00003</td>
<td>0.00003</td>
<td>7</td>
<td>115.86</td>
<td>7498.77</td>
</tr>
<tr>
<td>0.00003</td>
<td>0.00005</td>
<td>0.00002</td>
<td>0.00004</td>
<td>5</td>
<td>14.52</td>
<td>7645.92</td>
</tr>
<tr>
<td>0.00003</td>
<td>0.00005</td>
<td>0.00001</td>
<td>0.00003</td>
<td>5</td>
<td>18.96</td>
<td>7649.66</td>
</tr>
</tbody>
</table>
We can easily draw the obvious observations from the table (4.5.5) that for the given amount of demand and supply patterns with increase in time, optimal price and optimal quantity slightly change.

Table: 4.5.6

<table>
<thead>
<tr>
<th>Table for optimum quantity and corresponding price with different interest rates in fuzzy environment</th>
</tr>
</thead>
<tbody>
<tr>
<td>D = 2000 items, A = Rs.50, h = Rs 100, μ = 50 items, ξ = Rs.50, R = 80 items, i = 50 items, p0 = Rs.1000, θ1 = 0.0000007, θ0 = 0.0002, θ2 = 0.002, a = 0.00005, b = 0.00003, α = 0.00004, β = 0.00003, tc = 2 years, t = 7;</td>
</tr>
<tr>
<td>Ic</td>
</tr>
<tr>
<td>----</td>
</tr>
<tr>
<td>0.05</td>
</tr>
<tr>
<td>0.07</td>
</tr>
<tr>
<td>0.07</td>
</tr>
<tr>
<td>0.05</td>
</tr>
<tr>
<td>0.09</td>
</tr>
</tbody>
</table>

From table (4.5.6), it is obvious to note that the given values of interest rates charged from buyer after the credit period and the discount derived by the buyer during the credit period both increase, optimal price as well as optimal quantity of inventory do not change significantly.

4.6 Comparative Illustrations of the Model

Here, we present numerical illustrations of the model under two different environments of crisp and fuzzy so that we can have a comparative study. We have been able to show that for the same amount
of the parameters used in both illustrations, except the different environments of deterioration, the optimal price in fuzzy environment is less than optimal price in crisp environment. Consequently, it also affects the optimal quantities in both of the environments, i.e. optimal quantity in crisp environment is less than the optimal quantity in fuzzy environment. Upon contemplating on these close observations; one can easily prefer the fuzzy environment as compared to crisp one. These things are proved in the following illustrations.

4.7 Illustration of the Crisp Model

Find the lot size of items and also the price of an item after the time horizon of years under the following information.

i. The deterioration rate is 0.0002 per item.

ii. Given that demand during lead time follows the exponential distribution with mean 50 items.

iii. Shortage cost is Rs.50 per item. Reorder level is 80 items when annual demand is 2000 items.

iv. Fixed cost per order is Rs.50. Initial price of item is Rs.1000. Holding cost per item is Rs 50.

v. Permissible delay in payment is allowed till the credit period.

vi. After the credit period 2 years 5% interest charged from buyer and 4% discount during the credit period given to the buyers.

Demand at time $t$ is such that $Q_t = 0.00004 + 0.00005p_t$; supply at time $t-1$ is such that $Q_{t-1} = 0.00003 + 0.0003p_{t-1}$, Where $p_t$ and $p_{t-1}$ are price at time $t$ and at $t-1$. 
**Solution:** Given that

\[ D = 2000 \text{ items}, \quad A = \text{Rs.50}, \quad p_0 = \text{Rs.1000}, \quad \xi = \text{Rs.50}, \quad t_c = 2, \quad I_d = 0, \quad 04, \]

\[ l_c = 0.05, \quad R = 80 \text{ items}, \quad \alpha = 0.00004, \quad \beta = 0.00003, \quad a = 0.00005, \quad b = 0.00003, \quad t = 5 \text{ years}, \quad \theta = 0.0002 \text{ per item}. \]

The price of an item at particular time \( t \) is given as

\[ p_t = \left( b \right)^t \left( p_0 + \frac{\beta - \alpha}{b - a} \right) - \left( \frac{\beta - \alpha}{b - a} \right) \left( \frac{\theta}{\alpha} \right) \left( 1 - \frac{b}{a} \right)^t \]

This implies that \( p_0 = \text{Rs.75.82} \)

The optimal quantity is given as

\[ Q^* = \sqrt{\frac{2D \left( A + (\xi - p_t l_c I_d) R + 0.5D l_c^2 p_t (l_c - l_d) \right)}{h + p_t l_c}} \]

After computing, we get the following result

Optimal \( Q^* = 526.60 \equiv 527 \) items.

**4.8 Illustration of the Fuzzy Model**

Find the lot size of an item and also the minimum price of an item after 5 years under the following information.

i. When the deterioration of items is in fuzzy nature triangular membership function for deterioration. In which lower, middle, upper values are given as 0.0000007, 0.0002, and 0.002 respectively.

ii. Given that demand during lead time follows the exponential distribution with mean 50 items.

iii. Shortage cost is Rs.50 per item. Reorder level is 80 items when annual
demand is 2000 items.

iv. Fixed cost per order is Rs.50. Initial price of item is Rs.1000. Holding cost per item is Rs 50. Permissible delay in payment is allowed till the credit period.

v. After the credit period 2 year 5% interest charged from buyer and 4% discount from sale amount during the credit period given to the buyers. In this case credit period is less than reorder interval.

vi. Demand at time t is such that \( Q_t' = 0.00004 t - 0.00005 p_t \), supply at time \( t-1 \) is such that \( O_{t-1} = 0.00003 + 0.00003 p_{t-1} \), where \( p_t \) and \( p_{t-1} \) are price at time \( t \) and at \( t-1 \) respectively.

Solution:

Given that

A = Rs.50, D = 2000 items, h = Rs 50, i = 50 items \( \xi = \) Rs.50 per item,

\( \theta_c = 2 \) years, \( p_0 = \) Rs.1000, R = 80 items \( I_d = 4\% = 0.04 \) per sale amount,

\( I_c = 5\% = 0.05 \) per stock value (In this chapter, we assume \( I_c > I_d \)).

To find out the minimum price after 5 years in fuzzy environment of deterioration

\[ \overline{p_t} = \min_{\theta_1, \theta_2} \frac{\pi_0(\theta_1, \theta_2, t)}{\pi(\theta_1, \theta_2, t)} = \frac{\pi_0(\theta_1', \theta_2', t)}{\pi(\theta_1', \theta_2', t)} \]

Upon computing, we obtain

\[ \overline{p_t} = \text{Rs.} 22.44 \] and

\[ \overline{Q} = \sqrt{\frac{2D(A + (\overline{\xi} - \overline{\theta_t}I_cI_d) \mu + 0.5D t_c^2 \overline{p_t}(I_c-I_d))}{h + \overline{p_t}I_c}} \]

\[ \overline{Q} = 1393.35 = 1393 \text{ items} \]

4.9 Conclusion
Deterioration of inventory in the study of inventory control and management attracts serious attention of professionals engaged in this field. Deterioration of inventory during time lag between its production and consumption occupies special place in the study of inventory control particularly when it follows the cobweb process in fuzzified environment. In this chapter, a (Q, R) model has been approached for investigation under two different environments of crisp and fuzzy to yield optimum price and optimum quantity for the model. Model is also subject to comparison to exhibit broader spectrum of applications. Here, it is worthwhile to mention as concluding remarks that the cobweb theory is an economic model that explains why prices might be subject to periodic fluctuations in certain types of markets. It describes cyclical supply and demand in a market where the amount produced must be chosen before prices are observed. Producer’s expectations about prices are assumed to be based on observations of previous prices. One reason to be skeptical about this model’s predictions is that it assumes producers are extremely shortsighted (because in the beginning, producers don’t have large data profile for sufficient number of years that can easily help to take decision about the future price of an item). While assuming that farmers look back at the most recent prices in order to forecast future prices might seem very reasonable, this backward-looking forecasting turns out to be crucial for the model’s fluctuations. When farmers expect high prices to continue, they produce too much and therefore end up with low prices,
and vice versa. At last, it is very interesting to note that our present model simplifies to be model of [3] after having following assumptions:
(i) There is no time lag in between production and consumption (it is quite far from real situations).
(ii) Cobweb phenomenon does not exist.
(iii) Our model operates in only crisp (certain) environment.

Future efforts may be focused on the following lines of research:
(i) Extending the model to solve the multi-echelon inventory problem.
(ii) Studying the model under various marketing structures including perfect competition and monopoly etc, (vide [23]).
REFERENCES


[34] Wee Hui-Ming., A replenishment policy for items with a price dependent demand and a varying rate of deterioration, Production Planning and Control, 8 (1997) 494 - 499.

