CHAPTER
THREE
RECURRENCE IN SPECIAL
FINSLER SPACES

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1. **INTRODUCTION:**

A generalization of the concept of recurrence was first introduced by Moor [17] who treated the recurrence of the curvature tensor $R_{ijk}^h$ and gave some interesting results. Since then the theory of recurrent Finsler spaces has been explored by several authors. In 1971, Matsumoto [8] introduced the notion of $C^h$-recurrent Finsler spaces using the $h$-covariant derivative of the torsion tensor $C_{ijk}$. The purpose of the present chapter is to define $S^h$ and $T^h$ recurrent Finsler spaces using the $h$-covariant differentiation with respect to Cartan’s connection of $V$-curvature tensor $S_{hijk}$ and $T$-tensor $T_{hijk}$. We have obtained conditions in order that some Finsler spaces of special kind as introduced by Matsumoto and others may become Finsler spaces of these kinds.

We consider an $n$-dimensional Finsler space $F_n$ referred to a local coordinate system $x^i$, whose metric function $L(x,y)$ satisfies all the conditions usually imposed upon such a metric function. Now, we introduce some special Finsler spaces which will be used in the present chapter.

A Finsler space $F_n(n > 2)$ with the non-zero length $C$ of the torsion tensor $C^i$ is called semi-$C$-reducible [15], if

\[(1.1) \quad C_{ijk} = \frac{p}{n+1} (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{q}{C^2} C_i C_j C_k,\]

where $p$ and $q = 1 - p$ do not vanish and $h_{ij}$ is given by [I-(7.38)].

A semi $C$-reducible space is of the first kind or of the second kind, according as $p \neq \frac{n+1}{2}$ or $p = \frac{n+1}{2}$ [15].
A non-Riemannian Finsler space $F_n$ is called $C^h$-recurrent [8], if the $(h)hv$-torsion tensor $C_{ijk}$ satisfies

\[(1.2) \quad C_{ijk|\ell} = k_\ell C_{ijk}\]

where $k_\ell$ is a covariant vector field.

The Berwald's $V$-connection $G^i_{jk}$ defined by [I-(8.1)] is not, in general, independent of directional element $y^i$. A Finsler space in which $G^i_{jk}$ is independent of $y^i$ is called a Berwald's space. This space is characterized by the condition [71]

\[(1.3) \quad C_{ijk|\ell} = 0.\]

It follows from (1.4) that each Berwald's space is a Landsberg space.

2. PROPERTIES OF RECURRENCE:

For carrying out further studies under this heading, we give the following definitions.

**DEFINITION (2.1):**

A tensor field $T_{ij}$ is called $h$-recurrent, if

\[T_{ijk} = \lambda_k T_{ij},\]

where $\lambda_k$ is a covariant vector field.

**DEFINITION (2.2):**

A vector field $T_i$ is called $h$-recurrent, if for a covariant vector, field $k_h$, we have

\[T_{ih} = K_{ih} T_i.\]

On contracting (1.2) by $g^{ik}$, we get

\[(2.1) \quad C_{ih} = k_h C_i.\]
Thus, we can state

**THEOREM (2.1):**

In a $C^h$-recurrent Finsler space the covariant vector field $C_i$ is $h$-recurrent.

In general, the condition (2.1) is only a necessary condition in a $C^h$-recurrent Finsler space. However, there are some special Finsler spaces where this condition is necessary as well as sufficient in order that the space may be $C^h$-recurrent. We discuss such special Finsler spaces as under:

In 1972, Matsumoto [9] introduced the notion of $C$-reducible Finsler space. A non-Riemannian Finsler space $F_n (n > 2)$ is called $C$-reducible if the $(h)hv$-torsion tensor $C_{ijk}$ of $F_n$ is written in the form

\[ (2.2) \quad C_{ijk} = \frac{1}{n+1} \left( h_{ij} C_k + h_{jk} C_i + h_{ki} C_j \right), \]

where $C_i$ is a torsion vector defined by $C_i = C_{ik}$. Taking the covariant derivative of (2.2) with respect to $x^1$ and using [I-(7.37)], we get

\[ (2.3) \quad C_{ijk|\ell} = \frac{1}{n+1} \left( h_{ij} C_{k|\ell} + h_{jk} C_{i|\ell} + h_{ki} C_{j|\ell} \right). \]

At this stage if we assume that (2.1) holds, then equation (2.3) with the help of (2.2) gives that the space is $C^h$-recurrent. Therefore in consequence of theorem (2.1), we have the following:

**THEOREM (2.2):**

A necessary and sufficient condition in order that a $C$-reducible Finsler space is $C^h$-recurrent is that the vector field $C_i$ is $h$-recurrent.
The notion of C-2 like Finsler space has been introduced by Matsumoto and Numata [14]. Such a Finsler space is characterized by a special form of torsion tensor $C_{ijk}$ given by

\begin{equation}
C_{ijk} = \frac{1}{C^2} C_i C_j C_k \quad \text{with} \quad C^2 = 0.
\end{equation}

The $h$-covariant derivative of (2.4) with respect to $x^\ell$ gives

\begin{equation}
C^2 C_{ijk|\ell} + 2C_{ijk} C^p C_{\rho|\ell} = C_i C_j C_k C_{\rho|\ell} + C_j C_k C_{i|\ell} + C_k C_i C_{j|\ell}.
\end{equation}

At this stage, we now consider $C_{\rho|\ell} = \psi C_k$, the equation (2.4) and (2.5) show that the C-2 like Finsler space is $C^h$-recurrent. Thus, in view of theorem (2.1), we can state:

**THEOREM (2.3):**

A C2-like Finsler space in $C^h$-recurrent if the vector field $C_i$ is $h$-recurrent.

Further, in the case of semi-$C$-reducible Finsler space of second kind the equation (1.1) reduces to

\begin{equation}
C^2 C_{ijk} = \frac{C^2}{2} G(\ell^k \rho_i \rho_j) \left( h_{ij} C_k \right) + \frac{1-n}{2} C_i C_j C_k,
\end{equation}

where the symbol $G(\ell^k \rho_i \rho_j)$ stands for cyclic permutation of the indices $i, j, k$ and summation. The $h$-covariant differentiation of (2.6) with respect to $x^\ell$ and application of [I-(7.37a)] gives

\begin{equation}
C^2 C_{ijk|\ell} + 2C_{ijk} C^p C_{\rho|\ell} = C^p C_{\rho|\ell} G(\ell^k \rho_i \rho_j) \left( h_{ij} C_k \right) +
\end{equation}

\begin{equation}
+ \frac{C^2}{2} G(\ell^k \rho_i \rho_j) \left( h_{ij} C_k \right) + \frac{1-n}{2} G(\ell^k \rho_i \rho_j) \left( C_i C_j C_k \right).
\end{equation}
If we now assume that $C_{i|\ell} = k_{\ell} C_{i}$, the equation (2.5) and (2.7) give that the space under consideration in $C^h$-recurrent. Therefore, in virtue of theorem (2.1), we can state:

**THEOREM (2.4):**

A semi-$C$-reducible Finsler space of second kind is $C^h$-recurrent if and only if $C_{i}$ is $h$-recurrent.

We now give the following definition which is similar to the definition of $C^h$-recurrent Finsler space as under:

**DEFINITION (2.3):**

A non-Riemannian Finsler space is called $S^h$-or $T^h$-recurrent according as the $v$-curvature tensor $S_{hijk}$ or the $T$-tensor $T_{hijk}$ of the space satisfies the relation

\begin{equation}
S_{hijk|\ell} = \psi_{\ell} S_{hijk},
\end{equation}

or

\begin{equation}
P_{ijk|\ell} = \psi_{\ell} P_{ijk},
\end{equation}

\begin{equation}
T_{hijk|\ell} = \psi_{\ell} T_{hijk},
\end{equation}

respectively, where $\psi_{\ell}$ is a covariant vector field.

It has been shown by Mastumoto [8] that if $F_{n}$ is $C^h$-recurrent with recurrence vector $k_{\ell}$, then $S_{hijk}$ satisfies

\begin{equation}
S_{hijk|\rho} = 2k_{\rho} S_{hijk}.
\end{equation}

This relation and (2.8) give

**THEOREM (2.5):**

A $C^h$-recurrent Finsler space is $S^h$-recurrent.

In a $C$-reducible Finsler space [9] the $v$-curvature tensor $S_{hijk}$ has the following form:
(2.11) $S_{ilm} = (n+1)^{-2} \left( h_{im} C_{rl} + h_{rl} C_{im} - h_{i\ell} C_{rm} - h_{rm} C_{i\ell} \right)$,

with $C_{ij}$ given by

(2.12) $C_{ij} = 2^{-1} C^2 h_{ij} + C_i C_j$.

On taking the $h$-covariant derivatives of (2.11) and (2.12) with respect to $x^p$, we get

(2.13) $S_{ijk\ell|p} = (n+1)^{-2} \left( h_{i\ell} C_{j\ell|p} + h_{jk} C_{i\ell|p} - h_{ik} C_{j\ell|p} - h_{j\ell} C_{ik|p} \right)$

and (2.14) $C_{ij|p} = C^r C_{r|p} h_{ij} + C_{i|p} C_j + C_i C_{j|p}$,

respectively.

At this stage, we now suppose that a $C$-reducible Finsler space is $S^b$-recurrent. Then from the relations (2.11), (2.8) and (2.13), we have

(2.15) $h_{i\ell} C_{j|p} + h_{jk} C_{i|p} - C_{i\ell|p} - h_{ik} - h_{j\ell} C_{ik|p}$

$\psi_p \left( h_{i\ell} C_{j|p} + h_{jk} C_{i|p} - h_{ik} C_{j|p} - h_{j\ell} C_{ik|p} \right)$,

where after contraction with $g^{i\ell}$ yields

(2.16) $(n-3) C_{j|p} + g^{i\ell} C_{i|p} h_{jk} = \psi_p \left[ (n-3) C_{j|p} + h_{jk} g^{i\ell} C_{i\ell} \right]$.

We now contract (2.16) with $g^{jk}$ and use (2.12) and (2.14) and get

(2.17) $C^r C_{r|p} = \frac{C^2}{2} \psi_p$.

In view of (2.12), (2.14) and (2.17), (2.16) reduces to

(2.18) $C_{i|p} C_j + C_i C_{j|p} = \psi_p C_i C_j$ \quad for \quad $n > 3$.

We now contract (2.18) with $C^j$ which after making use of (2.17) gives

(2.19) $C_{i|p} = \frac{1}{2} \psi_p C_i$. 


where we have used the fact that $C^2 \neq 0$. Conversely the relations (2.11), (2.12), (2.13), (2.14) and (2.19) give the equation (2.8). Hence, we have

**THEOREM (2.6):**

A $C$-reducible Finsler space $F_n(n > 3)$ is $S^h$-recurrent if and only if the vector field $C^i$ is $h$-recurrent.

Theorems (2.2), (2.5) and (2.6) now yield.

**THEOREM (2.7):**

The necessary and sufficient condition in order that a $C$-reducible Finsler space $F_n(n > 3)$ is $S^h$-recurrent is that it is $C^h$-recurrent.

From theorem (2.5), we observe that in general, $S^h$-recurrence is only a necessary condition for a Finsler space to be $C^h$-recurrent. However, the theorem (2.7) shows that in a $C$-reducible Finsler space $F_n(n > 3)$ is $S^h$-recurrent is necessary as well as sufficient condition for the space to be $C^h$-recurrent. An $S3$-like Finsler space is characterized by

$$ (2.20) \quad S_{ijkl} = \frac{S}{(n-1)(n-2)}(h_{ik}h_{jl} - h_{il}h_{jk}). $$

In such a Finsler space, the $h$-covariant derivative of (2.20) after using equation [I-(7.37)], gives

$$ (2.21) \quad S_{ijkl|h} = \frac{1}{(n-1)(n-2)}S_{ijh}(h_{ik}h_{jl} - h_{il}h_{jk}), $$

which in view of (2.20) reduces to

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(2.22) \[ S_{ij\ell|\mu} = \frac{S^h}{S} S_{ijkl}, \]

provided that \( S \neq 0 \). Further from (2.20) and the fact that a Finsler space is called flat Finsler space if the \( V \)-curvature tensor \( S_{hijk} \) of the space satisfies

(2.23) \[ S_{hijk} = 0. \]

Thus, it can easily be verified that an \( S3 \)-like Finsler space is flat if and only if \( S = 0 \).

Therefore, we have

**THEOREM (2.8):**

A non-flat \( S3 \)-like Finsler space is \( S^h \)-recurrent.

We now consider an \( S4 \)-like Finsler space which is characterized by

(2.24) \[ S_{hijk} = \frac{S}{(n-2)(n-3)} (h_{ij} h_{hk} - h_{ik} h_{ij}) + \]

\[ + \frac{1}{(n-3)} (h_{ij} S_{ik} + h_{ik} S_{ij} - h_{ik} S_{ij} - h_{ij} S_{ik}). \]

The \( h \)-covariant derivative of (2.24) with respect to \( x^\ell \) after making use of equation [I-(7.37a)] gives

(2.25) \[ S_{hijk|\ell} = \frac{S^\ell}{(n-2)(n-3)} (h_{ij} h_{hk} - h_{ik} h_{ij}) + \]

\[ + \frac{1}{(n-3)} (h_{ij} S_{ik|\ell} + h_{ik} S_{ij|\ell} - h_{ik} S_{ij|\ell} - h_{ij} S_{ik|\ell}). \]

At this stage, we now suppose that

(2.26) \[ S_{ij} |_{\ell} = k_{\ell} S_{ij}, \]
for a covariant vector field $K_{\ell}$. The contraction of (2.26) with $g^{ij}$
gives $S_{\nu} = k_{\nu}S$ and in view of (2.24) the relation (2.25) reduces to
$S_{\alpha\beta\gamma\delta} = k_{\alpha}S_{\beta\gamma\delta}$. Conversely (2.26) is a necessary condition for a
Finsler space to be $S^{h}$-recurrent with recurrence vector field $K_{\ell}$.
Thus, we can state

**THEOREM (2.9):**

For an $S^{4}$-like Finsler to be $S^{h}$-recurrent it is necessary and
sufficient that the tensor field $S_{ij}$ is $h$-recurrent.

3. **$T^{h}$-RECURRENT FINSLER SPACES:**

In a 2-dimensional Finsler space $F_{2}$, the $T$-tensor $T_{hijk}$ can be
expressed in the following two forms

\begin{equation}
T_{hijk} = \phi \left( h_{hi}h_{jk} + h_{hj}h_{ki} + h_{hk}h_{ij} \right)
\tag{3.1}
\end{equation}

and

\begin{equation}
T_{hijk} = h_{hi}A_{jk} + h_{hj}A_{ik} + h_{hk}A_{ij} + h_{ij}A_{hk} + h_{hk}A_{ij} + h_{ij}A_{hk} + h_{jk}A_{i},
\tag{3.2}
\end{equation}

where $\phi$ is a scalar and $A_{ij}$ is a symmetric indicatory tensor field.

Ikeda [3] studied the theory of $n$-dimensional Finsler spaces with
$T$-tensor in the form (3.1). The $T$-tensor of a $C$-reducible Finsler-
space is of the form (3.1) [10].

The form of $T$-tensor is a special case of form (3.2) with
$A_{ij} = \frac{1}{2}\phi h_{ij}$. Thus, the $T$-tensor of a $C$-reducible Finsler space is of
the special form (3.2). From (3.1), it can be verified that $T_{hijk} = 0$ if
and only if $\phi = 0$. For the studies under this heading we assume $\phi$
to be non-vanishing. Differentiating (3.1) covariantly with respect
to $x^{\ell}$ and using [I-(7.37)], we get

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(3.3) \[ T_{ijk|\ell} = \phi_{|\ell} \left( h_{ji}h_{jk} + h_{ij}h_{ki} + h_{ik}h_{ij} \right). \]

By virtue of (3.1) and (3.3), we get

(3.4) \[ T_{ijk|\ell} \frac{\phi}{\phi} T_{ijk}. \]

(3.4) in view of the relation (2.9) and the following definition.

**DEFINITION (3.1):**

A Finsler space is said to satisfy the $T$-condition if the $T$-tensor of the vanishes [11] gives

**THEOREM (3.1):**

A C-reducible Finsler space is $T^h$-recurrent, provided that it does not satisfy the $T$-condition.

We can also state the following in analogy of the above theorem.

**THEOREM (3.2):**

A two dimensional Finsler space is $T^h$-recurrent, provided that it does not satisfy the $T$-condition.

Now, we propose to discuss the $T^h$-recurrence, in a Finsler space $F_n(n \geq 2)$ with $T$-tensor of the form given by (3.2)

The $h$-covariant derivative of (3.2) gives

(3.5) \[ T_{ijk|\ell} = h_{hi}A_{jk|\ell} + h_{ij}A_{hk|\ell} + h_{ij}A_{ik|\ell} + h_{jk}A_{ik|\ell} + h_{hk}A_{ij|\ell} + h_{ki}A_{jh|\ell}. \]

We now write $A_{jk|\ell} = \psi_{|\ell}A_{jk}$ in (3.5) and use (3.2) again and get (2.9) conversely, we now suppose that a Finsler space with its $T$-tensor in the form (3.2) is $T^h$-recurrent. Then substituting from (3.2) and (3.5) into (2.9), contracting the result thus obtained by $g^{hi}$ and using the fact that $\ell^iA_{ijk} = 0$, we get
\[(3.6) \quad (n+3) A_{jk|\ell} + h_{jk} g^{ih} A_{ih|\ell} = \psi_{\ell} \left[ (n+3) A_{jk} + h_{jk} g^{ih} A_{ih} \right].\]

Contraction of (3.5) with respect to \( g^{jk} \) gives

\[(3.7) \quad g^{ih} A_{ih|\ell} = \psi_{\ell} g^{ih} A_{ih}.\]

Thus, from (3.7) we have that \( A_{jk} \) is \( h \)-recurrent. Hence, we can state

**THEOREM (3.3):**

A Finsler space with its \( T \)-tensor in the form (3.2) is \( T^h \)-recurrent if and only if the tensor field \( A_{ij} \) is \( h \)-recurrent.

4. **\( P^h \)-RECURRENT FINSLER SPACES:**

In 1976, Izumi [4] introduced the notion of \( P^* \)-Finsler space. A Finsler space \( F_n (n > 2) \) with the non-zero length \( C \) of the torsion \( C^i \) is called a \( P^* \)-Finsler space, if the \((\nu)hv\)-torsion tensor of the space is written in the form

\[(4.1) \quad P_{ijk} = \lambda C_{ijk},\]

where \( \lambda (x, y) \) is a scalar function given by

\[(4.2) \quad \begin{align*}
(\text{a}) \quad \lambda &= \frac{1}{C^2} P_i C^i, \\
(\text{b}) \quad C^i &= g^{ij} C_j, \\
(\text{c}) \quad C_i &= C^j_{ij}, \\
(\text{d}) \quad P_i &= P^r_{ij} = C^r_{ij}. 
\end{align*}\]

A Finsler space in which \( g_{ij} (k) = 0 \) is called a Landsberg space and such a space is characterized by the condition

\[(4.3) \quad P_{ijk} = C_{ijk|\nu} = 0.\]

Equations (4.1) and (4.3) together give
THEOREM (4.1):

A $P^*$-Finsler space with scalar coefficient $\lambda$ is a Landsberg space if and only if $\lambda = 0$.

As an immediate consequence of (4.1) and (1.2) we can state

THEOREM (4.2):

A $C^h$-recurrent Finsler space with recurrence vector field $k_\ell$ is a Landsberg space if and only if $k_\omega = 0$.

It can be seen in [8] that a $C^h$-recurrent Finsler space with recurrence vector field $k_i (k_\omega \neq 0)$ satisfies the relation

(4.4) $P_{ijk|\ell} = \left( k_\ell + \frac{k_\omega k_\ell}{k_\omega} \right) P_{ijk}.$

(4.4) in view of theorem (4.2) and relation (2.8a) gives

THEOREM (4.3):

A non-Landsberg $C^h$-recurrent Finsler space is $P^h$-recurrent.

Now, we assume that a $P^*$-recurrent then from (4.1) and (2.8a) we can get

(4.5) $\lambda k_i C_{ijk} + \lambda C_{ijk|\ell} = \psi_{\ell} \lambda C_{ijk}.$

Which can alternatively be expressed in the following more convenient form

(4.6) $C_{ijk|\ell} = \left( \psi_{\ell} - \frac{\lambda k_i}{\lambda} \right) C_{ijk} (\lambda \neq 0).$

Therefore, theorems (4.3) and (4.1) enable us to state the following theorem.
THEOREM (4.4):

A non-Landsberg $P^*$-Finsler space is $P^h$-recurrent if and only if it is $C^h$-recurrent.

According to Matsumoto and Shimada [16] a Finsler space $F_n (n>2)$ is called $P$-reducible, if the $(v')hv$-torsion tensor $P_{ijk}$ of $F_n$ is written in the form

$$ (4.7) \quad P_{ijk} = \frac{1}{(n+1)} \left( P_i h_{jk} + P_j h_{ki} + P_k h_{ij} \right), $$

where $P_i$ is given by (4.2d).

In order to study the $P^h$-recurrence in a $P$-reducible Finsler space, we take the $h$-covariant derivative of (4.7) and this gives

$$ (4.8) \quad P_{ijk;\ell} = \frac{1}{(n+1)} \left( h_{jk} P_{i;\ell} + h_{ki} P_{j;\ell} + h_{ij} P_{k;\ell} \right). $$

If we now assume that $P_{i;\ell} = \psi_{\ell} P_i$, then (2.8a) (4.8) and (4.7) clearly tell that the space under consideration in $P^h$-recurrent. Conversely, the equation (2.8a) shows that $P_{i;\ell} = \psi_{\ell} P_i$ is a necessary condition for a Finsler space to be $P^h$-recurrent with recurrence vector field $\psi_{\ell}$ and as such we can state:

THEOREM (4.5):

A $P$-reducible Finsler space is $P^h$-recurrent if and only if the vector field $P_i$ is $h$-recurrent.

Since a $C$-reducible Finsler space is $P$-reducible [16] hence with the help of above theorem, we can also state:

THEOREM (4.6):

A $C$-reducible Finsler space is $P^h$-recurrent if and only if the vector field $P_i$ is $h$-recurrent.
# REFERENCES


[16] Matsumoto, M. and and Finsler spaces with curvature tensor P_{hijk} and S_{hijk} satisfying


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