Chapter 3

Bifurcation analysis of some forced Chen systems

3.1 Introduction

The Lorenz system \(\text{(Lorenz, 1963)}\) and forced Lorenz systems have been extensively investigated with applications in diverse disciplines like meteorology, lasers, electrical circuit etc. Many systems in different disciplines exhibit two-regime behaviour similar to the Lorenz attractors \(\text{(Mittal et. al., 2007)}\). For example, in meteorology, the Indian Summer Monsoon Rainfall has been usefully classified into active and passive regimes of high and low rainfalls. When forcing is introduced in the Lorenz system, the structure of the attractor does not change; only the probability for the system to be found in a particular regime changes \(\text{(Dwivedi et. al., 2006; Dwivedi and Mittal, 2007; Mittal et. al., 2005, 2007)}\). The forced Lorenz model has been used by \textit{Palmer (1994)} to provide a conceptual model to explain the variation of Indian Summer Monsoon Rainfall with El Nino index.

Bifurcation analysis of a non-linear system is useful for understanding the behavior of the system in different parameter ranges. Mittal et al. \(\text{(2005)}\) presented detailed bifurcation analysis of forced Lorenz systems. Forcing terms of these systems were so chosen as to make the mathematical analysis simpler.

In recent times, the Chen system \(\text{(Chen and Ueta, 1999; Ueta and Chen, 2000)}\) and forced Chen system has been the subject of several studies. The attractors of the Chen systems exhibit a slightly more complex behaviour as compared to the Lorenz
systems. Although they have not yet seen many applications, from the wide applicability of the Lorenz and Forced Lorenz systems, one can anticipate that they too will find diverse applications, as they will be able to model behaviour, which is slightly more complex than two-regime behaviour.

The Lu system (Lu and Chen, 2002) connects the Lorenz and the Chen systems, and represents transition between them. Different types of chaotic attractors for the Chen system were classified by Shukla et. al., (2010) into mainly four types: Lorenz type (L), Transition type (T), Chen type (C) and Transverse 8 type (S). The Lu system also undergoes similar transitions from L to T to C to S as the parameter c is increased. Shukla et. al., (2010) found that return maps can be used to characterize the transition between different types of chaotic attractors more reliably than the generalized competitive modes (GCMs) proposed by Yu et. al., (2007).

In Sec 3.2, we present detailed bifurcation analysis of analogous forced Chen systems. We consider a class of forced Chen systems, with forcing different from that investigated by Lu et. al.,(2002b). We have chosen forcing so that we can obtain mathematical expressions for equilibrium points as functions of the system parameters and analyze their stability easily.

In Sec 3.3, we describe the use of return maps (Mehta et. al., 2003; Mittal et. al., 2005, 2007) to look for quantitative signatures of transition between different types of chaotic attractors. In addition to the type of one-sided attractor reported by Lu et. al., (2002b), we find two other types of one-sided attractors. The return maps for these different types of one-sided attractors have different characteristics, but are related to those for the corresponding unforced systems.
3.2 Local Bifurcation Analysis of forced Chen System

The Chen system is described by the equations

\[
\begin{align*}
\frac{dx}{dt} &= a(y-x), \\
\frac{dy}{dt} &= (c-a)x - xz + cy, \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\] (3.1)

The standard values of the 'a', 'b', 'and c' are 35, 3, and 28, respectively.

We consider the forced Chen system:

\[
\begin{align*}
\frac{dx}{dt} &= a(y-x) + F_x, \\
\frac{dy}{dt} &= (c-a)x - xz + cy + F_y, \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\] (3.2)

Forcing term has not been added to the third equation, as that is equivalent to changing the coefficient of \(x\) in the second equation.

The fixed points of equation (3.2) satisfy

\[
y = x - \frac{F_x}{a}, \quad (3.3)
\]

\[
z = \frac{1}{b} \left[ x^2 - \left( \frac{F_x}{a} \right) x \right]
\]
The roots of equation (3.4) are obtained for the following cases:

**Case-3.1:** \( [F_x = aF] \) and \( [F_y = cF] \)

**Case-3.2:** \( [F_x = aF] \) and \( [F_y = (a-c)F] \)

**Case-3.1:**

In case –3.1 the roots of equation (3.4) are:

\[
x = 0 \text{ and } x_\pm = \frac{F \pm \sqrt{F^2 - 4(ab - 2bc)}}{2}
\]  
(3.5)

Thus the fixed points of equation (3.2) are \( O(0, -F, 0) \) and \( P_z = (x_z, y_z, z_z) \)

\[
y_\pm = \left(-F \pm \sqrt{F^2 - 4(ab - 2bc)}\right)/2
\]  
(3.6)

\[
z_\pm = (2c - a)
\]

The points \( P_z \) exist only if \( F_c^2 = F^2 - 4(ab - 2bc) > 0 \). In Fig. 3.1(a), the curve \( L \) plots \( F_c \) as a function of \( c \). Thus, there is only one equilibrium point, \( O \), in region I of the figure.

The Jacobian matrix at the point \( O \) is:

\[
J_0 = \begin{pmatrix}
-a & a & 0 \\
-c & c & 0 \\
-F & 0 & -b
\end{pmatrix}
\]  
(3.7)
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The Jacobian matrix at the point $P_{\pm}$ is:

$$J_{p_{\pm}} = \begin{pmatrix} -a & a & 0 \\ -c & c & -x_{\pm} \\ y_{\pm} & x_{\pm} & -b \end{pmatrix} \quad (3.8)$$

**Stability of $O$:**

The characteristic equation for the point $O(0, -F, 0)$ can be written as

$$(\lambda + b)\{\lambda^2 + (a - c)\lambda + a(a - 2c)\} = 0 \quad (3.9)$$

All the Eigen values have negative real parts if $(a - c) > 0$ and $(a - 2c) > 0$. Hence $O$ is locally stable for $c < a/2$ and unstable for $c > a/2$ independent of $F$.

**Stability of $P_{\pm}$:**

The characteristic equation for the point $P_{\pm} = (x_{\pm}, y_{\pm}, z_{\pm})$ can be written as

$$\lambda^3 + (a + b - c)\lambda^2 + (ab - bc + x_{\pm}^2)\lambda + a(x_{\pm}^2 + x_{\pm}y_{\pm}) = 0 \quad (3.10)$$

Now from Hurwitz criteria, necessary and sufficient condition for all roots of eqn. (3.10) to have negative real parts is:

$$(a + b - c) > 0 \quad (3.11)$$

$$(ab - bc + x_{\pm}^2) > 0 \quad (3.12)$$

$$a(x_{\pm}^2 + x_{\pm}y_{\pm}) > 0 \quad (3.13)$$

$$(a + b - c)(ab - bc + x_{\pm}^2) - a(x_{\pm}^2 + x_{\pm}y_{\pm}) > 0 \quad (3.14)$$
Condition (3.11) is satisfied for \( c < (a + b) = 38 \).

**Unforced Case:**

For the unforced case, \( F = 0 \), \( x_0 = \sqrt{b(2c-a)} = y_0 \). The points \( P_+ \) and \( P_- \) exist for \( c > \frac{a}{2} \). When these points exist, it is easy to verify that conditions (3.12) and (3.13) are satisfied. The inequality (3.14) reduces to

\[
c^2 + (3a-b)c - 2a^2 < 0
\]

This can be written as

\[
(c - c_1)(c - c_2) < 0
\]

Where

\[
c_{1,2} = \frac{-3a+b \pm \sqrt{(3a-b)^2 + 8a^2}}{2}
\]

\( c_1 = -122.07, c_2 = 20.07 \)

The inequality (3.16) is satisfied for \( c_1 < c < c_2 \). Thus for the unforced case the points \( P_+ \) and \( P_- \) are locally stable only for \( \frac{a}{2} < c < c_2 \).

**Forced Case:**

**Stability of \( P_+ \):**

In the presence of forcing, at the point \( P_+ \), condition (3.13) reduces to

\[
\sqrt{F^2 - 4b(a - 2c)} \left( \sqrt{F^2 - 4b(a - 2c) + F} \right) > 0
\]

If \( c > a/2 \), this inequality is satisfied for all values of \( F \).

If \( c < a/2 \), this inequality is satisfied for positive \( F \) but not for negative \( F \).
Condition (3.14) reduces to

\[ b\{(a + b - c)(a - c) - a(2c - a)\} + (b - c)x_+^2 > 0 \]  \hspace{1cm} (3.19)

For \( c < b \) this inequality is always satisfied, whereas for \( c > b \), it reduces to

\[ x_+^2 < \frac{b\{(a + b - c)(a - c) - a(2c - a)\}}{(c - b)} \]  \hspace{1cm} (3.20)

Substituting the value of \( x_+ \) from equation (3.5), this reduces to

\[ F\sqrt{F^2 - 4b(a - 2c)} < -F^2 + \frac{2b(c - c_1)(c_2 - c)}{(c - b)} \]  \hspace{1cm} (3.21)

where \( c_{1,2} \) are as in eqn (3.17).

For \( F > 0 \), since the left hand side of inequality (3.21) is always positive, the inequality cannot hold if the right hand side is negative. Therefore a necessary, but not sufficient condition for inequality (3.21) to hold is

\[ F^2 < \frac{2b(c - c_1)(c_2 - c)}{(c - b)} \]  \hspace{1cm} (3.22)

When inequality (3.22) holds, the inequality obtained by squaring both sides of eqn (3.21) will also hold. Thus, we can obtain,

\[ \{(c - c_1)(c_2 - c) - (a - 2c)(c - b)\}F^2 < \frac{b(c - c_1)^2(c_2 - c)^2}{(c - b)} \]  \hspace{1cm} (3.23)

The coefficient of \( F^2 \) in the above inequality is equal to \( c^3 - c(4a + b) + a(2a + b) \)

which can be written as \( (c - r_1)(c - r_2) \), where \( r_{1,2} = \{(4a + b) \pm \sqrt{8a^2 + 4ab + b^2}\} / 2 \).

We see that \( r_1 = 20.93 \) and \( r_2 = 122.07 \).

If \( c > c_2 \), the inequality (3.22) cannot hold. For \( c < c_2 \), the coefficient of \( F^2 \) in (3.23) is positive because \( c_2 < r_1 \), therefore
In Fig. 3.1(a) curve $M$ plots $F_c$ as a function of $c$.

For inequality (3.21) to hold, it is necessary for both (3.22) and (3.24) to hold. For $b < c < c_2$, right hand side of (3.22) is more than right hand side of (3.24), therefore, if (3.24) is satisfied, (3.22) is automatically satisfied.

For $F < 0$, as already shown $P_+$ is unstable if $c < a/2$ because inequality (3.13) is violated. If $\frac{2}{5} < c < c_2$ then inequality (3.21) is satisfied because $LHS < -F^2 < RHS$. If $c_2 < c < r_1$, then inequality (3.21) holds for $F < -|F_c|$.

We conclude that the point $P_+$, when it exists, is stable in the range $0 < c < b$ for positive $F$; in the range $b < c < \frac{2}{5}$ for positive $F < F_c$; in the range $\frac{2}{5} < c < c_2$ for all $F < F_c$; and in the range $c_2 < c < r_1$ for $F < -|F_c|$.

**Stability of $P_-$**

At the point $P_-$, equation (3.13) reduces to

$$F^2 - 4b(a - 2c) - F\sqrt{F^2 - 4b(a - 2c)} > 0 \quad (3.25)$$

This can be written as

$$\sqrt{F^2 - 4b(a - 2c)} > F \quad (3.26)$$

For $F > 0$, if $c < a/2$, this inequality is never satisfied whereas if $c > a/2$, it is always satisfied. For $F < 0$, this inequality is always satisfied.

Condition (3.14) reduces to

$$b\{(a + b - c)(a - c) - a(2c - a)\} + (b - c)x^2 > 0 \quad (3.27)$$

For $c < b$ this inequality is always satisfied, whereas for $c > b$, it reduces to
\[ x_-^2 < \frac{b\{(a+b-c)(a-c)-a(2c-a)\}}{(c-b)} \]  
(3.28)

Substituting the value of \( x_- \) from equation (3.5), this reduces to

\[ F\sqrt{F^2-4b(a-2c)} > F^2 - \frac{2b(c-c_1)(c_2-c)}{(c-b)} \]  
(3.29)

Where \( c_{1,2} \) are as in eqn (3.17).

For \( F > 0 \), we have already seen that \( P_- \) is unstable if \( c < \frac{a}{2} \) as inequality (3.26) is violated. If \( \frac{a}{2} < c < c_2 \), the inequality (3.29) is satisfied as \( LHS > F^2 > RHS \). If \( c_2 < c \)

\[ F\sqrt{F^2-4b(a-2c)} > F^2 + \frac{2b(c-c_1)(c_2-c)}{(c-b)} \]  
(3.30)

As both sides of the inequality are positive, the inequality is satisfied if and only if the squares on both sides satisfy the inequality. This leads to

\[ (c-r_1)(c-r_2)F^2 > \frac{b}{(c-b)(c-c_1)}(c-c_2)^2 \]  
(3.31)

If \( c_2 < c < r_1 \) this implies \( F > F_c \). If \( r_1 < c < r_2 \), inequality (3.31) cannot be satisfied.

Similarly it can be shown that for \( F < 0 \), in the range \( b < c < c_2 \), inequality (3.29) leads to \( F > -|F_c| \). If \( c > c_2 \), this inequality is never satisfied.

We conclude that \( P_- \), when it exists, is stable in the range \( 0 < c < b \) for all negative \( F \); in the range \( b < c < \frac{a}{2} \) for negative \( F > -|F_c| \); in the range \( \frac{a}{2} < c < c_2 \) for all \( F > -|F_c| \) and in the range \( c_2 < c < r_1 \) for \( F > F_c \).
Fig. 3.1(a), shows the bifurcation structure for the stability of the equilibrium points. In region I, the only equilibrium point $O$ is stable. On crossing the curve L, from region I to region II or to region III, a limit point bifurcation takes place, and a node-saddle pair is born. In region II, $P_+$ is a node and $P_-$ is a saddle whereas in region III, it is the other way round. In region IV, $O$ is stable, whereas $P_+$ and $P_-$ are unstable. In region V, $O$ is unstable, whereas $P_+$ and $P_-$ are stable. In region VI, $O$ and $P_+$ are unstable whereas $P_-$ is stable. As the vertical line T is crossed in going from region II (III) to region V, or in going from region IV to region VI (VII), $O$ and $P_- (P_+)$ collide and exchange their stability; a transcritical bifurcation takes place. Region I and region V touch at the intersection point of L and T. A pitchfork bifurcation takes place at this point as $O$ loses stability, giving rise to two stable equilibrium points $P_+$ and $P_-$. In region VII, $O$ and $P_-$ are unstable whereas $P_+$ is stable. In region VIII, a strange attractor coexists with locally stable fixed points $P_+$ and $P_-$. Fig. 3.1(b) is an enlargement of the boxed region of Fig. 3.1(a). In region IX, all the equilibrium points are unstable.
Figure 3.1(a) Bifurcation structure of forced Chen system for case 3.1, i.e. $F_x = aF$; and $F_y = cF$.

Figure 3.1(b) The zoomed portion of the boxed region of Fig. 3.1(a).
Case-3.2: In this case, the roots of equation (3.4) are:

\[ x = F \quad \text{and} \quad x_\pm = \pm \sqrt{b(2c-a)} \quad (3.32) \]

Thus the fixed points of eqn (3.2) are: \( R(F,0,0) \) and \( S_\pm = \{x_\pm, y_\pm, z_\pm\} \), where

\[ y_\pm = \pm \sqrt{b(2c-a)} - F \quad (3.33) \]

\[ z = (2c-a) - (\pm F \sqrt{(2c-a)/b}) \quad (3.34) \]

The Jacobian matrix at the point \( R \) is:

\[ j_R = \begin{pmatrix} -a & a & 0 \\ c-a & c & -F \\ 0 & F & -b \end{pmatrix} \quad (3.35) \]

The Jacobian matrix at the point \( S_\pm = \{x_\pm, y_\pm, z_\pm\} \) is:

\[ j_{s_\pm} = \begin{pmatrix} -a & a & 0 \\ c-a-z_\pm & c & -x_\pm \\ y_\pm & x_\pm & -b \end{pmatrix} \quad (3.36) \]

**Stability of \( R \)**

The characteristic equation for the point \( R(F,0,0) \) is:

\[ \lambda^3 + (a+b-c)\lambda^2 + (a^2 + ab - 2ac - bc + F^2)\lambda + (a^2b + AF^2 - 2abc) = 0 \quad (3.37) \]

By the Hurwitz criteria, necessary and sufficient conditions for all roots of eqn. (3.37) to have negative real parts are:
\[(a + b - c) > 0 \quad (3.38)\]

\[(a^2 + ab - 2ac - bc + F^2) > 0 \quad (3.39)\]

\[(a^2b + aF^2 - 2abc) > 0 \quad (3.40)\]

\[(a + b - c)(a^2 + ab - 2ac - bc + F^2) - (a^2b + aF^2 - 2abc) > 0 \quad (3.41)\]

Inequality (3.39) is a consequence of the inequalities (3.38), (3.40) and (3.41).

Inequality (3.40) reduces to

\[F^2 > b(2c - a) \quad (3.42)\]

For \(c < a/2\), this inequality is always satisfied

Inequality (3.41) reduces to

\[(2a + b)(c - c_3)(c - c_4) + (b - c)F^2 > 0 \quad (3.43)\]

Where \(c_3 = (a^2 + ab + b^2) / (2a + b) = 18.34\) and \(c_4 = a = 35\).

For \(c < b\), inequality (3.43) is always satisfied. For \(b < c < c_3\), it reduces to

\[F^2 < \frac{(2a + b)(c - c_3)(c - c_4)}{(c - b)} = F_{c_3}^2 \quad (3.44)\]

Whereas for \(c_3 < c < c_4\), it is never satisfied. The curve \(T\) in Figs 3.2 (a), (b) is a plot of \(F_c\) as a function of \(c\).

Thus for \(0 < c < b\), \(R\) is stable for all \(F\). For \(b < c < a/2\), \(R\) is stable for all \(F\) that satisfy inequality (3.44). For \(a/2 < c < c_3\), it is stable for \(F\) lying between two values determined by inequalities (3.42) and (3.44). For \(c_3 < c < c_4\), it is unstable.
Stability of $S_+ S_-$

The characteristic equation for the point $S_\pm = \{x_\pm, y_\pm, z_\pm\}$ is:

$$\begin{align*}
\lambda^3 + (a + b - c)\lambda^2 + (a^2 + ab - 2ac - bc + az_\pm + x^2_\pm)\lambda + (a^2b - 2abc + abz_\pm + ax_\pm y_\pm + ax^2_\pm) &= 0 \\
(3.45)
\end{align*}$$

By the Hurwitz criteria, necessary and sufficient conditions for all roots of eqn. (3.44) to have negative real parts are:

$$
(a + b - c) > 0 \quad (3.46)
$$

$$
(a^2 + ab - 2ac - bc + az_\pm + x^2_\pm) > 0 \quad (3.47)
$$

$$
(a^2b - 2abc + abz_\pm + ax_\pm y_\pm + ax^2_\pm) > 0 \quad (3.48)
$$

$$
(a + b - c)(a^2 + ab - 2ac - bc + az_\pm + ax_\pm y_\pm + ax^2_\pm) - (a^2b - 2abc + abz_\pm + ax_\pm y_\pm + ax^2_\pm) > 0 \\
(3.49)
$$

The inequality (3.47) is a consequence of inequalities (3.46), (3.48) and (3.49).

At the point $S_\pm$, inequality (3.48) reduces to

$$
F < \sqrt{b(2c - a)} = F_c \quad (3.50)
$$

In Fig.3.2 (b), the curve $H_1$ plots $F_c$ as a function of $c$.

Whereas inequality (3.49) reduces to

$$
-b(c - c_1)(c - c_2) - a(a - b - c)\sqrt{\frac{(2c - a)}{b}}F > 0 \\
(3.51)
$$

Where $c_1$ and $c_2$ are as in equation (3.17).

For $c < c_2$ this gives,

$$
F < \frac{b^{\frac{\gamma}{2}}(c - c_1)(c_2 - c)}{a(a - b - c)\sqrt{2c - a}} = F_c \\
(3.52)
$$
In Fig. 2(b) the curve $H_2$ plots $F_c$ for $c_1 < c < c_2$.

For $c_2 < c < a - b$, inequality (3.51) is never satisfied, whereas for $c > a - b$, it gives,

$$F > \frac{b^{\frac{3}{2}}(c - c_1)(c - c_2)}{a(c - a + b)\sqrt{2c - a}} = F_c$$  \hspace{1cm} (3.53)

In Fig. 3.2(b) the curve $H_3$ plots $F_c$ for $c_2 < c < a - b$.

Thus for $a/2 < c < c_2$, $S_+$ is stable if $F$ satisfies both the inequalities (3.50) and (3.52). In the range $(a - b) < c < (a + b)$, the right hand side of (3.50) is less than that of (3.53), therefore both the inequalities (3.50) and (3.53) cannot be satisfied.

For $c > a + b$, condition (3.46) is violated. Thus for $c > c_2$, $S_+$ is unstable for every positive $F$.

At the point $S_-$, the inequality (3.48) reduces to

$$\sqrt{b(2c-a)} + F > 0$$  \hspace{1cm} (3.54)

This is always satisfied, whereas inequality (3.49) reduces to

$$-b(c - c_1)(c - c_2) + (a - b - c)\sqrt{(2c-a)/b}F > 0$$  \hspace{1cm} (3.55)

For $c_1 < c < c_2$ this inequality is always satisfied. For $c_2 < c < a - b$, it gives

$$F > \frac{b^{\frac{3}{2}}(c - c_1)(c_2 - c)}{a(a - b - c)\sqrt{2c - a}}$$  \hspace{1cm} (3.56)

For $c > a - b$, inequality (3.55) is never satisfied. Thus for $a/2 < c < c_2$, $S_-$ is stable for all positive $F$ and for $c > c_2$ it is stable for $F$ satisfying inequality (3.56).
Fig. 3.2(a) shows the bifurcations structure based on the stability of equilibrium points for case 2. Fig. 3.2(b) is a magnification of the boxed region of Fig. 3.2 (a). On crossing the curve T from region I to region II or from VI (VII) to III (IV) a Hopf bifurcation \cite{Kuznetsov1998} takes place as the equilibrium point R loses stability. On crossing the line L from region I to VI, or from I to VII, or from II to III, or from II to IV, a saddle-node bifurcation takes place as a pair of equilibrium points S+, S- emerge. One of S+, S- is stable, while the other is unstable. On crossing the curve H1, from region V to VI (VII), a transcritical bifurcation \cite{Kuznetsov1998, Sparrow1982} takes place as R and S+ (S-) exchange stability. On crossing the curve H2, from region V to III (IV), a Hopf - bifurcation takes place as S+ (S-) loses stability. On crossing the curve H3, from region III (IV) to VIII, a Hopf - bifurcation takes place as S+ (S-) loses stability.

\begin{align*}
F_x &= Fa \\
F_y &= -cF
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig32a.png}
\caption{Fig. 3.2(a)}
\end{figure}

\textbf{Figure 3.2(a)} Bifurcation structure of forced Chen system for case 3.2, i.e. $F_x = aF$ ; and $F_y = (a-c)F$. 
3.3 Chaotic Attractors and their Return maps

We explored the nature of the chaotic attractors for both the cases. An interesting feature found in the second case, but not in the first, is the existence of one-sided attractors. Below we present the results for the Case 3.2.

For the case - 3.2, \( F_x = aF \) and \( F_y = (a-c) F \), keeping the values of \( a = 35 \) and \( b = 3 \) fixed, we examined the chaotic attractors for different values of \( c \) and \( F \). The visual appearances of the attractors for well-separated values of \( c \) are clearly different. The visual appearances of the attractors for well-separated values of \( c \) are clearly different. The attractor is evidently L type for \( c = 20.8 \), T type for \( c = 23.0 \), C type for \( c = 26.0 \) and S type for \( c = 28.3 \) (Figs. 2.1(b), 2.1(d), 3.8(a), 3.9(a)). For the forced case, one-sided attractors are found for certain value of \( c \) and \( F \). We find a one sided attractor for \( c = 28.3, F = 0.95 \) (Fig. 3.9(c)). This is similar to the type of
one-sided attractor reported by *Lu et. al.*, (2002b). But we find two different types of one-sided attractors for $c = 23.0$, $F = 0.57$ (Fig. 3.5(a)) and for $c = 26.0$, $F = 0.8$ (Fig 3.8(c)).

In the present study we consider the return map, which maps maximum of the absolute value of $x$-variable to the next maximum. We find characteristically different signatures in the return maps, which can be used to classify the different types of chaotic attractors.

The return map for the unforced L type attractor, $c = 20.4$ (Fig. 3.3(a)) shows a single cusp (Fig. 3.3(b)). When forcing, $F = 0.02$ is introduced at $c = 20.4$, this cusp splits into a double cusp (Fig. 3.3(d)). When value of $c = 20.8$ (Fig. 2.1(b)) also shows a single cusp (Fig. 2.2(b)). Shukla *et. al.*, (2010) have shown that as the value of $c$ is increased, at $c = 20.9$ (Fig. 2.2(c)), a point $C$ appears to the left of the cusp. Although at this stage it is not evident from visual inspection (Fig. 2.1(c)) of the attractor, they demonstrated that the appearance of this point is a signature for transition to the T type attractor. When forcing, $F = 0.02$ is introduced at $c = 20.9$, the cusp splits into a double cusp (Fig. 3.4(b)).

For $F = 0$, as $c$ is increased further, the point on the left of the cusp grows and the T type nature of the attractor begins to become evident. With further increase in $c$, at $c = 23.0$, the return map grows into a cusp to the left of the original cusp (Fig. 2.2(e)) and the attractor (Fig. 2.1(d)) is evidently T type. At $c = 23.5$, the right branch of this cusp approaches the left branch of the original cusp (Fig 2.2(f)). When forcing, $F = 0.04$ is introduced at $c = 23.5$, both the cusps split into double cusps (Fig. 3.6(b)).

As $c$ is increased further, the right branch of this cusp approaches the left branch of the original cusp. At $c = 23.6$, three new point $B’1$, $C’1$, and $D’1$ (Fig. 2.1(g))
appear which do not lie on the two cusps in the return map, indicating the emergence of a different behaviour in the attractor. It was demonstrated that this is the point of transition to C type attractor, although at this stage it is not evident from visual inspection of the attractor (Fig. 2.1(f)). When forcing, F = 0.04 is introduced at c = 23.6, the two cusps split into double cusps (Fig. 3.7(b)).

As c is increased the attractor becomes clearly C type, as is evident at c = 26.0 (Fig. 3.8(a)), while the return map (Fig. 3.8(b)) becomes increasingly complex. At a critical value, c = 28.3, the C type attractor makes a transition to S type attractor (Fig. 3.9(a)) and the corresponding return map (Fig. 3.9(b)) also shows a sudden change; a complex return map suddenly changes to a smooth looking tilted curve.

For some values of c, we find one-sided attractors for a fixed value of F. These attractors also show distinct types of return map behaviour, which are related to the return maps for the corresponding one-sided attractors.

Fig. 2.1(d) shows the attractor for the unforced case with c = 23.0. This is a T type attractor and the corresponding one-dimensional return map (Fig. 2.2(e)) is single-valued. A one-sided attractor (Fig. 3.5(a)) is obtained when this system is forced with F = 0.57. This return map, as shown in Fig. 3.5(b), is also single-valued.

Fig. 3.8(a) shows the attractor for the unforced case with c = 26. This is a C type attractor and the corresponding return map (Fig. 3.8(b)) is multiple valued. A one sided attractor (Fig. 3.8(c)) is obtained for F = 0.80 with return map shown in Fig. 3.8(d). This return map is also multiple valued.

Fig. 3.9(a) shows the attractor for the unforced case with c = 28.3. This is an S type attractor. The corresponding one-dimensional return map is a smooth looking tilted curve (Fig. 3.9(b)). A one-sided attractor (Fig. 3.9(c)) is obtained when this
system is forced with $F = 0.95$. The corresponding return map (Fig. 3.9(d)) has similar tilted curves.

Figure 3.3 (a, b) Type L attractor and its corresponding return map for $c = 20.4$ and $F = 0$.

Figure 3.3 (c, d) Type L attractor and its corresponding return map for $c = 20.4$ and $F = 0.02$. 
Figure 3.4(a, b) Transition from attractor (Type L) to the attractor (Type T) and its corresponding return map for $c = 20.9$, $F = 0.02$.

Figure 3.5(a, b) Type T One-sided attractor and its respective return map obtained for $c = 23$ and $F = 0.57$. 
Figure 3.6(a, b) Attractor (Type T) and its corresponding return map for $c = 23.5$ and $F = 0.04$.

Figure 3.7(a, b) Transition from attractor (Type T) to the attractor (Type C) and its respective return map for $c = 23.6$ and $F = 0.04$. 

Fig. 3.6 (a) 

Fig. 3.6(b) 

Fig. 3.7 (a) 

Fig. 3.7 (b)
Figure 3.8(a, b) Attractor (Type C) and its respective return map for $c = 26$ and $F = 0$.

Figure 3.8(c, d) Type C one-sided attractor and its respective return map for $c = 26$ and $F = 0.80$. 
Figure 3.9(a, b) Attractor (Type S) and its respective return map for $c = 28.3$ and $F = 0$.

Figure 3.9(c, d) Type S one-sided attractor and its respective return map for $c = 28.3$ and $F = 0.95$. 
3.4 Conclusions:

The bifurcation behaviour of a forced Chen system is analyzed. Two-parameter bifurcation behaviour of forced Chen systems, as the forcing parameter $F$ and the system parameter $c$ are varied, is analyzed. The unforced Chen attractor is known to have distinct types of chaotic attractors. A one dimensional return map, which maps maxima of the absolute value of $x$-variable of the system to the next maxima, can be used to classify these attractors and determine the precise point of transition from one type of chaotic attractor to another type. Some authors proposed that single cusp obtained from return map for Lorenz attractor splits into double cusp in the presence of forcing in the Lorenz system. We have found that return map of Lorenz, Lu and Chen system split into double cusp in the presence of forcing in the Chen system. We also found three distinct types of one-sided attractors in a class of forced Chen systems. The return maps for these three types of one-sided chaotic attractors also show different characteristic signatures, which are related to those for the corresponding unforced cases.