Chapter 4

Bi-criteria two-level hierarchical transportation problem

This chapter discusses a two-level hierarchical transportation problem with two objective functions viz., sum of transportation times and sum of transportation costs of both the levels which are to be minimized simultaneously. In Chapter 3, this problem was discussed with the objective of minimizing the sum of transportation times only, however, the inclusion of another objective in its formulation puts the problem in the class of bi-objective optimization problems. Therefore, this problem is named as the bi-criteria two-level hierarchical transportation problem (BTHTP). Due to the conflicting nature of the objectives under consideration, it may not be possible to minimize them simultaneously, therefore, there is need to search for the Pareto optimal solutions which are the compromising or the trade-off solutions of the problem that involve loss of one objective in return for the gain in the other. These solutions are also known as the efficient solutions. To solve the problem BTHTP, we develop a polynomial time iterative algorithm which at each step, solves some specific restricted cost minimizing transportation problems (CMTPs) related to the original problem and finds the corresponding pairs of sum of transportation times and sum of transportation costs of both the levels. The restrictions imposed on
4.1 Mathematical formulation

Level-I and Level-II links in these problems, at a particular step of the algorithm, depend upon the sum of transportation times corresponding to the feasible pair obtained at the previous step. Further, the algorithm is developed in such a way that it expels all the dominated (non-efficient) pairs and records all the efficient pairs systematically, where an efficient pair is the pair of sum of transportation times and sum of transportation costs of both the levels corresponding to an efficient solution of the problem BTHTP. Here, we do not give an additional subjective preference to any objective, therefore, all the efficient pairs are considered equally good. The computational efficiency of the current algorithm is examined by observing the running time of the MATLAB program based on it, for randomly generated BTHTP instances of different sizes. It is observed that the proposed algorithm runs successfully for all these instances and therefore, can serve as an important tool for solving the related optimization problems.

4.1 Mathematical formulation

We formulate and solve the balanced case of two-level hierarchical transportation problem with the above mentioned objectives, however, the unbalanced case of this problem can be handled in a similar way as the unbalanced case of the problem THBTP (refer to Chapter 3). Let us denote the per unit cost of transportation for the \((i,j)^{th}\) source-destination link as \(c_{ij}\) and the time of transportation for the same as \(t_{ij}\). As mentioned in Chapter 1, unlike \(c_{ij}\), \(t_{ij}\) is independent of the quantity of the product transported from \(i^{th}\) source to \(j^{th}\) destination. It is assumed here that \(c_{ij} \geq 0, \forall (i,j) \in I \times J\).

Now, to define the problem BTHTP, we define the objectives and the solution sets of Level-I and the corresponding Level-II problems as follows:

**Level-I Problem:**

\[
\min_{X \in S_{L1}} \{ T_1(X), C_1(X) \}
\]
where

\[ T_1(X) = \max_{X \in S_{L_1}} \{ t_{ij}(x_{ij}) \} ; \quad C_1(X) = \sum_{(i,j) \in L_1} c_{ij} x_{ij} \]

and

\[
S_{L_1} = \left\{ X = \{ x_{ij} \}_{I \times J} \in \mathbb{R}^{mn} \mid \begin{array}{l}
\sum_{j \in J} x_{ij} \leq a_i, \forall i \in I \\
\sum_{i \in I} x_{ij} \leq b_j, \forall j \in J \\
x_{ij} \geq 0, \forall (i, j) \in L_1, \ x_{ij} = 0, \forall (i, j) \in L_2 \\
a_i, b_j, x_{ij} \text{ are integers and } \sum_{i \in I} \sum_{j \in J} x_{ij} \neq 0
\end{array} \right\}
\]

is the solution set of Level-I problem. Here, \( a_i, b_j \) and \( x_{ij} \) denote the availability of the product at the \( i^{th} \) source, demand of the same at the \( j^{th} \) destination and the amount to be transported from \( i^{th} \) source to the \( j^{th} \) destination in Level-I of the problem, respectively and \( t_{ij}(x_{ij}) = \begin{cases} t_{ij} & \text{if } x_{ij} > 0 \\ 0 & \text{otherwise} \end{cases} \).

**Level-II Problem corresponding to a feasible solution \( X \) of Level-I problem:**

\[
\min_{Y \in S_{L_2}(X)} \{ T_2(Y), C_2(Y) \}
\]

where

\[ T_2(Y) = \max_{Y \in S_{L_2}(X)} \{ t_{ij}(y_{ij}) \} ; \quad C_2(Y) = \sum_{(i,j) \in L_2} c_{ij} y_{ij} \]

and

\[
S_{L_2}(X) = \left\{ Y = \{ y_{ij} \}_{I \times J} \in \mathbb{R}^{mn} \mid \begin{array}{l}
\sum_{j \in J} y_{ij} = a_i - a'_i, \forall i \in I \\
\sum_{i \in I} y_{ij} = b_j - b'_j, \forall j \in J \\
y_{ij} \geq 0, \forall (i, j) \in L_2, \ y_{ij} = 0, \forall (i, j) \in L_1 \\
y_{ij} \text{ are integers and } \sum_{i \in I} \sum_{j \in J} y_{ij} \neq 0
\end{array} \right\}
\]
is the solution set of Level-II problem. Here, $a'_i = \sum_{j \in J} x_{ij}$ is the quantity transported from $i^{th}$ source and $b'_j = \sum_{i \in I} x_{ij}$ is the quantity transported to $j^{th}$ destination, in Level-I problem and $t_{ij}(y_{ij}) = \begin{cases} t_{ij} & \text{if } y_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$.

We study the problem BTHTP with the assumption that some quantity of the product has to be transported in both the levels.

Now, the problem BTHTP can be stated as

$$\min_{X \in S_{L_1}, Y \in S_{L_2}} \{ T_1(X) + T_2(Y), C_1(X) + C_2(Y) \}$$

\text{(BTHTP)}

A feasible solution $(X, Y) \in S_{L_1} \times S_{L_2}(X)$ of the problem BTHTP is obtained by solving an equivalent problem $P_0$ defined as

$$\min_{Z \in S} \{ S_T(Z), C(Z) \}$$

\text{(P$_0$)}

where $S$ is defined as

$$S = \left\{ Z = \{z_{ij}\}_{I \times J} \in \mathbb{R}^{mn} \mid \begin{array}{l} \sum_{j \in J} z_{ij} = a_i, \ \forall \ i \in I \\ \sum_{i \in I} z_{ij} = b_j, \ \forall \ j \in J \\ z_{ij} \geq 0 \text{ and integers, } \forall \ i \in I, j \in J \end{array} \right\}$$

Here, $C(Z)$ and $S_T(Z)$ denote the total transportation cost and the sum of transportation times of Level-I and Level-II respectively, corresponding to a feasible solution $Z = \{z_{ij}\}_{I \times J} \in S$ of the problem $P_0$. Therefore,

$$C(Z) = \sum_{(i,j) \in I \times J} c_{ij} z_{ij} ; \quad S_T(Z) = \max_{(i,j) \in L_1} \{t_{ij}(z_{ij})\} + \max_{(i,j) \in L_2} \{t_{ij}(z_{ij})\}$$

and $t_{ij}(z_{ij}) = \begin{cases} t_{ij} & \text{if } z_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$.

It should be noted here that the objective functions $C(Z)$ and $S_T(Z)$ represent linear
and concave function, respectively.

**Remark 4.1.1.** A feasible solution of the problem $P_0$, provides a corresponding feasible solution of the problem BTHTP.

Proof. Corresponding to a feasible solution $Z$ of the problem $P_0$, a feasible solution $(X, Y) \in S_{L_1} \times S_{L_2}(X)$ of the problem BTHTP is defined by defining $X$ and $Y$ as follows:

$$x_{ij} = \begin{cases} z_{ij} & \text{for } (i, j) \in L_1 \\ 0 & \text{for } (i, j) \in L_2 \end{cases} \quad \text{and} \quad y_{ij} = \begin{cases} 0 & \text{for } (i, j) \in L_1 \\ z_{ij} & \text{for } (i, j) \in L_2 \end{cases}$$

In other words,

$$X + Y = Z$$

(4.1.1)

Now, for $Z \in S$,

$$S_T(Z) = \max_{(i,j) \in L_1} \{ t_{ij}(z_{ij}) \} + \max_{(i,j) \in L_2} \{ t_{ij}(z_{ij}) \}$$

$$= \max_{X \in S_{L_1}} \{ t_{ij}(x_{ij}) \} + \max_{Y \in S_{L_2}(X)} \{ t_{ij}(y_{ij}) \}$$

$$= T_1(X) + T_2(Y)$$

and

$$C(Z) = \sum_{(i,j) \in I \times J} c_{ij}z_{ij}$$

$$= \sum_{(i,j) \in L_1} \sum_{i,j} c_{ij}z_{ij} + \sum_{(i,j) \in L_2} \sum_{i,j} c_{ij}z_{ij}$$

$$= \sum_{(i,j) \in L_1} \sum_{i,j} c_{ij}x_{ij} + \sum_{(i,j) \in L_2} \sum_{i,j} c_{ij}y_{ij}$$

$$= C_1(X) + C_2(Y)$$

Thus, we see that a feasible solution $Z$ of the problem $P_0$ provides the corresponding feasible solution $(X, Y)$ of the problem BTHTP such that $X + Y = Z$. 


Some definitions

Definition 4.1.1. (Efficient solution). A solution \( \hat{Z} \in S \) of the problem \( P_0 \) is called an efficient solution if \( \nexists \) a feasible solution \( Z \in S \) such that \( (S_T(Z), C(Z)) \leq (S_T(\hat{Z}), C(\hat{Z})) \) where \( \mathbf{a} \leq \mathbf{b} \ (\mathbf{a}, \mathbf{b} \in \mathbb{R}^n) \) implies that \( a_i \leq b_i \), \( \forall \ i \) with strict inequality for at least one \( i \).

If \( \hat{Z} \) is an efficient solution, then the pair \( (S_T(\hat{Z}), C(\hat{Z})) \) is called an efficient pair.

Definition 4.1.2. (Dominated solution). A solution \( \bar{Z} \in S \) of the problem \( P_0 \) is called a dominated solution if \( \exists \) a feasible solution \( Z \in S \) such that \( (S_T(Z), C(Z)) \leq (S_T(\bar{Z}), C(\bar{Z})) \).

If \( \bar{Z} \) is a dominated solution, then the pair \( (S_T(\bar{Z}), C(\bar{Z})) \) is called a dominated pair.

Definition 4.1.3. (Feasible pair). A pair \( (S_T, C) \) is called a feasible pair if \( \exists \) a feasible solution \( Z \in S \) such that \( (S_T, C) = (S_T(Z), C(Z)) \).

To develop an algorithm that records all the efficient pairs of the problem BTHTP, initially a CMTP \( CP_0 \) associated with the problem \( P_0 \) and defined as

\[
\min_{Z \in S} \sum_{(i,j) \in I \times J} c_{ij}z_{ij}
\]

is solved and from its optimal feasible solution (OFS), the optimal cost and the corresponding transportation times of Level-I and Level-II are recorded so that the first pair of sum of transportation times and sum of transportation costs is obtained. Further, this pair is explored by defining various CMTPs which are time restricted versions of the problem \( CP_0 \) with the time restrictions on Level-I and Level-II links imposed in such a way that an OFS of each of these problems, (if exists), gives a pair in which the sum of transportation times is strictly less than that obtained corresponding to the first pair. It is clear that an OFS of each of these problems (if exists), is a feasible solution of the problem \( CP_0 \). Out of these CMTPs, selected number of problems are solved, optimal costs of which justify the efficiency of the first pair. Once the first efficient pair is obtained, the search for the second efficient pair initiates so that sum of transportation times (costs) in the second efficient pair is strictly less (greater) than that obtained in the first efficient pair. This procedure
is continued until a pair with minimum possible sum of transportation times of Level-I and Level-II is obtained.
The detailed explanation of the algorithm is given in the following section.

4.2 Theoretical development

Let an OFS of a time restricted CMTP solved at any step of the algorithm be denoted by $Z$ and the corresponding feasible solutions of Level-I and Level-II problems by $X$ and $Y$ respectively, such that the equation (4.1.1) holds. Then, a feasible pair of sum of transportation times and the optimal transportation cost may be noted as $(S_T, C)$. Clearly, $S_T = S_T(Z) = T_1(X) + T_2(Y)$ and $C = C(Z) = C_1(X) + C_2(Y)$.

To simplify the notations, we assume $T_1(X) = T_1$ and $T_2(Y) = T_2$.

Now, to check the efficiency of the pair $(S_T, C)$, it is explored by defining the following time restricted CMTPs, which are named as ‘sub-problems restricted at $S_T$’.

The time restrictions in these CMTPs are imposed in such a way that corresponding to their OFSs, the sum of transportation times is strictly less than $S_T$, however, the sum of transportation costs may be greater than or equal to $C$.

- Sub-problem $P^1_{[T_1, T_2]}(S_T)$

\[
\min_{Z \in \mathcal{S}} \sum_{i \in I} \sum_{j \in J} c'_{ij} z_{ij}
\]

where

\[
c'_{ij} = \begin{cases} 
  c_{ij} & \text{if for } (i, j) \in L_1, \quad t_{ij} \leq T_1 \\
  M & \text{if for } (i, j) \in L_1, \quad t_{ij} > T_1 \\
  c_{ij} & \text{if for } (i, j) \in L_2, \quad t_{ij} < T_2 \\
  M & \text{if for } (i, j) \in L_2, \quad t_{ij} \geq T_2
\end{cases}
\]

- Sub-problem $P^2_{[T_1, T_2]}(S_T)$

\[
\min_{Z \in \mathcal{S}} \sum_{i \in I} \sum_{j \in J} c'_{ij} z_{ij}
\]
4.2. Theoretical development

where

\[ c'_{ij} = \begin{cases} 
  c_{ij} & \text{if for } (i,j) \in L_1, \quad t_{ij} < T_1 \\
  M & \text{if for } (i,j) \in L_1, \quad t_{ij} \geq T_1 \\
  c_{ij} & \text{if for } (i,j) \in L_2, \quad t_{ij} \leq T_2 \\
  M & \text{if for } (i,j) \in L_2, \quad t_{ij} > T_2
\]

Find \( T_1' = \min \{ S_T, (T_1)_{max} \} \) where \((T_1)_{max}\) denotes the largest time entry in the Level-I links. Also, note down all the time entries in Level-I links that lie between \( T_1 \) and \( T_1' \). Let there be \( r \) such entries i.e.,

\[ T_1 < T_{1,1} < T_{1,2} < \ldots < T_{1,r} < T_1' \]

then, there will be \((r+1)\) sub-problems denoted by

- Sub-problems \( P_{T_1,T_1}^{3,p}(S_T) \), \( p = 1, 2, \ldots, r + 1 \).

\[ \min_{\mathbf{z} \in S} \sum_{i \in I} \sum_{j \in J} c'_{ij} z_{ij} \]

where

\[ c'_{ij} = \begin{cases} 
  c_{ij} & \text{if for } (i,j) \in L_1, \quad t_{ij} \leq T_{1,p} \\
  M & \text{if for } (i,j) \in L_1, \quad t_{ij} > T_{1,p} \\
  c_{ij} & \text{if for } (i,j) \in L_2, \quad t_{ij} < S_T - T_{1,p} \\
  M & \text{if for } (i,j) \in L_2, \quad t_{ij} \geq S_T - T_{1,p}
\]

Clearly, \( T_{1,r+1} = T_1' \).

Now, find \( T_2' = \min \{ S_T, (T_2)_{max} \} \) where \((T_2)_{max}\) denotes the largest time entry in the Level-II links. Also, note down all the time entries in Level-II that lie between \( T_2 \) and \( T_2' \). Let there be \( s \) such entries i.e.,

\[ T_2 < T_{2,1} < T_{2,2} < \ldots < T_{2,s} < T_2' \]
then, there will be \((s + 1)\) sub-problems denoted by

- Sub-problems \(P_{(S_T-T_2,q,T_2)}^{a,q}(S_T)\), \(q = 1, 2, \ldots, s + 1\).

\[
\min_{Z \in \mathcal{S}} \sum_{i \in I} \sum_{j \in J} c'_{ij} z_{ij}
\]

where

\[
c'_{ij} = \begin{cases} 
  c_{ij} & \text{if for } (i, j) \in L_2, \quad t_{ij} \leq T_{2,q} \\
  M & \text{if for } (i, j) \in L_2, \quad t_{ij} > T_{2,q} \\
  c_{ij} & \text{if for } (i, j) \in L_1, \quad t_{ij} < S_T - T_{2,q} \\
  M & \text{if for } (i, j) \in L_1, \quad t_{ij} \geq S_T - T_{2,q}
\end{cases}
\]

Clearly, \(T_{2,s+1} = T'_2\).

**Remark 4.2.1.** The sub-problems restricted at \(S_T\), defined above, can be written either as \(P_{[a_i,b_i]}^{i}(S_T)\), \(P_{[a_i,b_i]}^{i}(S_T)\) or \(P_{[a_i,b_i]}^{i}(S_T)\) for \(i = 1, 2, 3, p, 4, q\) where \(a_i\) and \(b_i\) denote the “upper bounds of the time entries” of various source-destination links in Level-I and Level-II respectively. The links with the time entries less than these upper bounds are associated with their original cost in the respective sub-problems whereas the links with time entries greater than these upper bounds are blocked i.e., not allowed to participate in the transportation process, by assigning them a very high cost ‘M’, however, the parenthesis ‘[’ and ‘(’ decide whether these upper bounds are to be included or excluded while blocking the links. Accordingly, the problem \(CP_0\) can be denoted as \(P_{(\infty,\infty)}\).

**Remark 4.2.2.** Consider a \(3 \times 5\) BTHTP 1 given in Table 4.2.1 to understand the formulation of various sub-problems restricted at a sum \(S_T\), in which the source-destination links with bold entries denote the Level-I links while the others denote Level-II links. Also, at each source-destination link, the number on top left corner denotes the transportation cost \(c_{ij}\), \(\forall i \in I \) and \(j \in J\) whereas the number on bottom right corner denotes the transportation time \(t_{ij}\). The quantities \(a_i\) and \(b_j\) denote the availability at \(i^{th}\) source and demand at \(j^{th}\) destination, respectively.
Definition 4.2.1. (M-feasible solution). A feasible solution $Z = \{z_{ij}\}_{I \times J}$ of a sub-

The distinct time entries in Level-I and Level-II links of the given BTHTP are

Level-I entries: $2 < 3 < 4 < 5 < 6 < 7 < 9$

Level-II entries: $3 < 4 < 6 < 8 < 9 < 12$

One of the feasible pairs of this problem is $(14, 181)$ which corresponds to a feasible solution depicted in Table 4.2.2, i.e., $S_T = 14$ and $C = 181$ with the corresponding transportation times of Level-I and Level-II problems as 5 and 9 respectively. The number written in the circle in Table 4.2.2 denote the corresponding allocation at various source-destination links. Then, the various sub-problems restricted at the sum 14 can be obtained as $P_{[5,9]}^1(14)$, $P_{[5,9]}^2(14)$, $P_{[6,6]}^{3,1}(14)$, $P_{[7,6]}^{3,2}(14)$, $P_{[9,4]}^{3,3}(14)$ and $P_{[2,12]}^{4,1}(14)$. The possible time distributions of the sub-problems $P_{[5,9]}^1(14)$, $P_{[5,9]}^2(14)$, $P_{[6,6]}^{3,1}(14)$, $P_{[7,6]}^{3,2}(14)$ and $P_{[9,4]}^{3,3}(14)$ are depicted in Figure 4.2.1. The time distributions for the sub-problem $P_{[2,12]}^{4,1}(14)$ can also be plotted in the same way.
Figure 4.2.1: Possible time distributions of various sub-problems restricted at the sum 14.
problem is called an M-feasible solution (MFS) if \( z_{ij} = 0, \forall (i, j) \in I \times J \) for which \( c'_{ij} = M \).

A sub-problem is called M-feasible if its M-feasible solution exists.

**Definition 4.2.2.** (Minor/Major sub-problems). Two sub-problems are called minor and major sub-problems respectively, if the links that are open in the former sub-problem are necessarily open in the latter, however, the converse is not true. We denote this relation between two sub-problems by the symbol ‘\( \subset \)’. For example, the sub-problems \( P_i[a_i,b_i](S_T) \) and \( P_i[a_i,b_i](S_T) \) are minor sub-problems for the problem \( P_i[a_i,b_i](S_T) \). Therefore, we can write \( P_i[a_i,b_i](S_T) \subset P_i[a_i,b_i](S_T) \) and \( P_i[a_i,b_i](S_T) \subset P_i[a_i,b_i](S_T) \). From Figure 4.2.1, we see that sub-problems \( P_i[5,9](14), P_i[5,9](14), P_3[7,6](14) \) and \( P_3[9,4](14) \) are major sub-problems and the sub-problem \( P_3[6,6](14) \) is minor for the sub-problem \( P_3[7,6](14) \).

**Remark 4.2.3.** The sub-problems defined above are such that an optimal MFS (OMFS) of each of these problems (if exists), will provide a feasible pair in which the sum of transportation times is strictly less than \( S_T \).

**Remark 4.2.4.** The sub-problem \( P_i[a_i,b_i](S_T) \) (or \( P_i[a_i,b_i](S_T) \)) is equivalent to the sub-problem \( P_i[a_i,b_i](S_T) \) for some values of \( a_i' \) and \( b_i' \), therefore, we may, in general, denote the sub-problems defined at any step of the algorithm by \( P_i[a_i,b_i](S_T) \). For example, the sub-problem \( P_2[5,9](14) \) defined in Remark 4.2.2 is equivalent to the sub-problem \( P_2[4,9](14) \).

**Remark 4.2.5.** The following notations are used for optimal solutions and the optimal costs of various sub-problems.

1. An OFS of the problem \( CP_0 \) is denoted by \( Z^0 \) and an OMFS of the sub-problems \( P_i[a_i,b_i](S_T) \) is denoted \( Z_i \), for all values of \( i \).

2. The optimal cost of \( CP_0 \) is denoted by \( C_0 \) and that of the sub-problems \( P_i[a_i,b_i](S_T) \) is denoted by \( CP_i[a_i,b_i](S_T) \), for all values of \( i \) i.e., \( C_0 = C(Z^0) \) and \( CP_i[a_i,b_i](S_T) = C(Z_i), \forall i \).
Important observations

1. For \( p = r + 1 \), the sub-problem \( P^{3,p}_{(T_1,p,s_T-T_1,p)}(S_T) \) becomes

\[
\min_{Z \in S} \sum_{i \in I} \sum_{j \in J} c'_{ij} z_{ij}
\]

where

\[
c'_{ij} = \begin{cases} 
  c_{ij} & \text{if for } (i, j) \in L_1, & t_{ij} \leq T'_1 \\
  M & \text{if for } (i, j) \in L_1, & t_{ij} > T'_1 \\
  c_{ij} & \text{if for } (i, j) \in L_2, & t_{ij} < S_T - T'_1 \\
  M & \text{if for } (i, j) \in L_2, & t_{ij} \geq S_T - T'_1 
\end{cases}
\]

This problem is of no use for the current study if \( T'_1 = S_T \), because in that case, all the Level-II links would be blocked and an OMFS, even if it exists, will give the Level-II transportation time as zero. Therefore, it follows that the number of relevant sub-problems \( P^{3,p}_{(T_1,p,s_T-T_1,p)}(S_T) \), depends on the value of \( T'_1 \). If \( T'_1 = (T_1)_{max} \), then there are \( r + 1 \) sub-problems otherwise there are \( r \) sub-problems.

Similar argument holds for the subproblems \( P^{4,q}_{(S_T-T_2,q,T_2,q)}(S_T) \) if \( T'_2 = S_T \).

2. At a particular step of the algorithm, out of all the restricted sub-problems, we solve major sub-problems only. The reason for not solving minor sub-problems at this step is that an OMFS of a minor sub-problem is an MFS of the major sub-problem. Therefore, the optimal cost of a minor sub-problem will be greater than or equal to the optimal cost of its major sub-problem. Further, if a minor sub-problem is to provide an efficient pair, then either its OMFS would be OMFS of a major sub-problem at this step only or it would itself occur as a major sub-problem at the subsequent steps (which can be solved there). Initially, the only major sub-problem is CP0.

3. By construction of the sub-problems, it is clear that at any step of the algorithm, if a major sub-problems is not M-feasible, then none of its minor sub-problems would be M-feasible.
4.3 Algorithm

As mentioned earlier, to obtain all the efficient pairs of the problem BTHTP, the proposed algorithm solves various time restricted CMTPs. Initially, a pair consisting of the optimal cost and the corresponding sum of transportation times is obtained, then the sum of transportation times is reduced strictly. This procedure continues until a pair with the minimum possible sum of transportation times of both the levels is obtained. The detailed explanation of the algorithm is as follows.

Initial Step.
Solve the problem \( CP_0 \). From its OFS \( Z^0 \), find the optimal cost \( C_0 \) and note down the corresponding transportation times of Level-I and Level-II problems and hence, find their sum \( S_{T_0} \). For \( k = 0 \), define the set \( N_k = \{(S_{T_k}, C_k)\} \), a collection of non-dominated pairs till now.

General Step (\( k \geq 0 \)).
Explore the pair \( (S_{T_k}, C_k) \) by defining the sub-problems \( P_{[a^k_i,b^k_i]}^i(S_{T_k}) \), \( \forall i \) and solving these sub-problems only for major values of \( i \). An OMFS of the sub-problem
\( P_{[a^k_i, b^k_i]}(S_{T_k}) \) is denoted by \( Z^k_i \), \( \forall \) \( i \) (if exists).

1. (a) If \( P_{[a^k_i, b^k_i]}(S_{T_k}) \) is M-feasible for no major \( i \), then go to the terminal step.

(b) If \( P_{[a^k_i, b^k_i]}(S_{T_k}) \) is M-feasible at least for one major \( i \), then go to step 2.

2. (a) If \( CP_{[a^k_i, b^k_i]}(S_{T_k}) > C_k \), for all those major values of \( i \) for which the sub-
problem \( P_{[a^k_i, b^k_i]}(S_{T_k}) \) is M-feasible, then note \( (S_{T_k}, C_k) \) as an efficient pair and go to step 3, otherwise go to step 2(b).

(b) If \( CP_{[a^k_i, b^k_i]}(S_{T_k}) = C_k \), at least for one major \( i \), then the pair \( (S_{T}(Z^k_i), C(Z^k_i)) \) obtained from an OMFS \( Z^k_i \) of the sub-problem \( P_{[a^k_i, b^k_i]}(S_{T_k}) \) for such major \( i \), would dominate the pair \( (S_{T_k}, C_k) \) as \( S_{T}(Z^k_i) < S_{T_k} \), for all \( i \). Therefore, update the value of \( S_{T_k} \) as

\[
S_{T_k} = \min_{\text{major } i} \{(S_{T}(Z^k_i)|C(Z^k_i) = C_k}\}
\]

Also, update the set of non-dominated pairs from which the original pair \( (S_{T_k}, C_k) \) is selected and repeat the general step for the updated pair \( (S_{T_k}, C_k) \).

3. Construct a set \( N_{k+1} \) which is a collection of non-dominated pairs (NDPs) obtained from the OMFSs of the major sub-problems \( P_{[a^k_i, b^k_i]}(S_{T_k}) \) or obtained at previous steps (if any) that are unexplored till now, i.e.,

\[
N_{k+1} = \left\{ \text{NDPs obtained from the major sub-problems } P_{[a^k_i, b^k_i]}(S_{T_k}) \right\} \cup \left\{ \text{NDPs obtained at previous steps (if any) that are unexplored till now} \right\}
\]

The repeated pairs are ignored while forming the set \( N_{k+1} \). Now, select a pair \( (S_{T_{k+1}}, C_{k+1}) \) from the set \( N_{k+1} \), where \( C_{k+1} = \min_{(S_{T,C}) \in N_{k+1}} C \) and repeat the general step for this pair by setting \( k = k + 1 \).

Terminal Step.
The efficient pairs are \( (S_{T_0}, C_0),(S_{T_1}, C_1), \ldots,(S_{T_k}, C_k) \).

The flow chart of the algorithm is given in Figure 4.3.1.
Some important remarks

1. While solving the problem $CP_0$, if one of the transportation times $T_1^0$ or $T_2^0$ comes out to be zero, i.e., allocation is done only in one of the levels in order to get the optimal cost, then this solution cannot be considered as a feasible solution to the problem BTHTP. In case $T_1^0$ (or $T_2^0$) takes zero value, then we solve the problem $CP_0$ again, with the links blocked in Level-II with time greater than or equal to $T_2^0$ (or $T_1^0$). This is continued until the problem $CP_0$ gives non-zero transportation times $T_1^0$ and $T_2^0$. Further, if such a situation arises for a major sub-problem at a general step, then this problem is discarded and one of its immediate next minor sub-problem is considered as the major sub-problem for that particular step (if any). If we ignore the immediate next minor sub-problem at this step, then there may be a chance that this sub-problem does not occur at subsequent steps and hence, an efficient pair may be missed while applying the algorithm. Immediate next minor sub-problem of
a major sub-problem is the one with time entries immediately lesser than that of the major sub-problem. Grapically, in Figure 4.2.1, the problem $P_{[6,6]}^{3.1}$ is the immediate next minor sub-problem of the sub-problem $P_{[7,6]}^{3.2}$.

2. At a particular step of the proposed algorithm, the sub-problems $P_{[a^l_i,b^l_i]}^i(S_{T_k})$ are constructed in such a way that an OMFS of each of these problems (if exists) will yield a pair in which the sum of transportation times is strictly less than $S_{T_k}$ and as the time entries in both the levels are integers, the sum of transportation times will be reduced by at least one unit.

3. At step 3 of the general step of the algorithm, we select a pair from the set $N_{k+1}$, which is non-dominated till now. However, if condition 2(a) holds for this pair, then it becomes the overall non-dominated or efficient pair of the problem BTHTP.

4. Let $(S_{T_l}, C_l)$, $l \geq 0$ be the last efficient pair recorded by the algorithm, it means none of the major sub-problems $P_{[a^l_i,b^l_i]}^i(S_{T_l})$ is M-feasible. Also, the pair $(S_{T_l}, C_l)$ is selected from the set $N_l$ of non-dominated pairs. Then clearly, the set $N_l$ will be a singleton set. It is obvious for $l = 0$, but for $l \geq 1$, if there would be any pair left in the set $N_l$ after selecting $(S_{T_l}, C_l)$, the cost of that pair should be strictly greater than $C_l$ and being non-dominated, the sum of transportation times in that pair should be less $S_{T_l}$. It means that pair should be obtained from an MFS of one of the major sub-problems $P_{[a^l_i,b^l_i]}^i(S_{T_l})$. It leads to contradiction as none of the major sub-problems $P_{[a^l_i,b^l_i]}^i(S_{T_l})$, is M-feasible.

**Lemma 4.3.1.** Let $(S_T, C)$ be a feasible pair obtained from an OMFS $Z$, of a CMTP solved at any step of the algorithm. If $CP_{[a_i,b_i]}^i(S_T) = C$, at least for one major $i$, then $(S_T, C)$ is not an efficient pair.

Proof. Let $(S_T, C)$ be a feasible pair obtained from an OMFS $Z$, of a CMTP solved at any step of the algorithm i.e., $S_T = S_T(Z)$ and $C = C(Z)$. 

At an OMFS $Z_i$ of $P_{[a_i, b_i]}^i(S_T)$,

$$S_T(Z_i) < S_T(Z), \ \forall \text{ major values of } i \quad (4.3.1)$$

Now, if $CP^i_{[a_i, b_i]}(S_T) = C$, at least for one major $i$, then we can write

$$C(Z_i) = C(Z), \ \text{for that particular } i \quad (4.3.2)$$

By the equations $4.3.1$ and $4.3.2$, it follows that the pair $(S_T(Z_i), C(Z_i))$ dominates the pair $(S_T, C)$ and hence, the pair $(S_T, C)$ is not an efficient pair.

**Theorem 4.3.1.** The feasible pair $(S_T, C)$ obtained from an OMFS of a CMTP solved at any step of the algorithm, is an efficient pair iff $CP^i_{[a_i, b_i]}(S_T) > C, \ \forall \text{ major values of } i$.

Proof. We prove this theorem in two parts.

Part 1. Let $CP^i_{[a_i, b_i]}(S_T) > C, \ \forall \text{ major values of } i$. We need to prove that $(S_T, C)$ is an efficient pair.

Let, if possible, $(S_T, C)$ be not an efficient pair. It means there exists a pair say $(S'_T, C')$, yielded by a feasible solution $(X', Y')$ of the problem BTHTP that dominates the pair $(S_T, C)$, i.e., $(S'_T, C') \leq (S_T, C)$. Now, two cases arise:

Case (i). $S'_T < S_T$ and $C' \leq C$.

Define $Z' = X' + Y'$, then $S'_T = T_1(X') + T_2(Y') = S_T(Z')$ and $C' = C(Z')$. If $S'_T < S_T$, then the solution $Z'$ should be an MFS of a sub-problem $P^i_{[a_i, b_i]}(S_T)$, for some major $i$. Therefore,

$$C' \geq CP^i_{[a_i, b_i]}(S_T), \ \text{ for that major } i \quad (4.3.3)$$

But

$$CP^i_{[a_i, b_i]}(S_T) > C, \ \forall \text{ major values of } i \quad (4.3.4)$$
From the equations [4.3.3] and [4.3.4], we get $C' > C$ which clearly indicates that Case (i) does not hold.

Case (ii). $S_T' \leq S_T$ and $C' < C$

There are two possibilities: either $S_T' < S_T$ and $C' < C$ or $S_T' = S_T$ and $C' < C$.

If $S_T' < S_T$, then as discussed in Case (i), $C'$ cannot be less than $C$. Therefore, the only possibility is $S_T' = S_T$ and $C' < C$.

Let the pair $(S_T, C)$ be obtained by solving a major sub-problem $P_i^a(S_T^*)$ corresponding to the pair $(S_T^*, C^*)$, for some $i = i_1$ (say), where $i_1 \in \{1, 2, 3, p, 4, q\}$ and $(S_T^*, C^*)$ be obtained from an OMFS of a major sub-problem at any other step of the algorithm. Clearly, $S_T' > S_T$. Then, the solution $Z'$ obtained from the solution $(X', Y')$ of the problem BTHTP yielding the pair $(S_T', C')$ with $S_T' = S_T$, will also be an MFS of some of the sub-problems $P_i^a(S_T^*)$.

Now, if $Z'$ is an MFS of the sub-problem $P_i^a(S_T^*)$, then $C' \geq C$, as $C$ is the optimal cost of sub-problem $P_i^a(S_T^*)$. A contradiction to the inequality $C' < C$. But, if $Z'$ is an MFS of $P_i^a(S_T^*)$, for some $i_1 \neq i_1$, then

$$C' \geq CP_{i_1}^{a_1, b_1}(S_T^*)$$  \hspace{1cm} (4.3.5)

Since the pair $(S_T, C)$ is chosen for further exploration from the set of non-dominated pairs, therefore $C$ must be the minimum cost among all the costs obtained by solving $P_i^a(S_T^*)$ i.e.,

$$CP_{i_1}^{a_1, b_1}(S_T^*) \geq C, \quad \forall \ i$$  \hspace{1cm} (4.3.6)

Combining the equations [4.3.5] and [4.3.6], we get $C' \geq C$ which is again a contradiction to the inequality $C' < C$.

Hence, if $CP_i^a(S_T) > C$, $\forall$ major values of $i$, then $(S_T, C)$ is an efficient pair.

Part 2. Let $(S_T, C)$ be an efficient pair obtained from an OMFS $Z$, of a CMTP solved at any step of the algorithm.

We need to show that $CP_i^a(S_T) > C$, $\forall$ major values of $i$.

Let if possible, $CP_i^a(S_T) \leq C$, for some major value of $i$. 
Clearly, at an M-feasible solution $Z_i$ of the sub-problem $P_{[a_i,b_i]}^i(S_T)$,

$$S_T(Z_i) < S_T,$$ for all major values of $i$.

Now, $CP_{[a_i,b_i]}^i(S_T) \leq C$ or $C(Z_i) \leq C$ for some major value of $i$, implies that the pair $(S_T(Z_i), C(Z_i))$ would dominate the pair $(S_T, C)$, for that particular $i$. This is a contradiction as the pair $(S_T, C)$ is an efficient pair.

**Theorem 4.3.2.** Let $(S_T, C)$ be a feasible pair obtained from an OMFS of a CMTP solved at any step of the algorithm. If $CP_{[a_i,b_i]}^i(S_T) = \lambda_i M + \nu_i$, $\forall$ major values of $i$ where $\lambda_i, \nu_i \in \mathbb{R}$ and $\lambda_i > 0$, $\forall$ $i$, then

(i) $\nexists$ an efficient pair $(\hat{S}_T, \hat{C})$ with $\hat{S}_T < S_T$.

(ii) $S_T = \min_{Z \in S} S_T(Z)$.

Proof. Given that $CP_{[a_i,b_i]}^i(S_T) = \lambda_i M + \nu_i$, $\forall$ major values of $i$ and $\lambda_i > 0$, $\forall$ $i$. It means that none of the major sub-problems $P_{[a_i,b_i]}^i(S_T)$ is M-feasible.

(i) If we suppose that there exists a feasible solution $(\hat{X}, \hat{Y})$ of the problem BTHTP that gives an efficient pair $(\hat{S}_T, \hat{C})$ such that $\hat{S}_T < S_T$. Then, the solution $\hat{Z}$ yielded from the solution $(\hat{X}, \hat{Y})$, by using equation (4.1.1), would be an MFS of one of the major sub-problems $P_{[a_i,b_i]}^i(S_T)$, which is a contradiction as none of these major sub-problems is M-feasible. Therefore, $\nexists$ an efficient pair $(\hat{S}_T, \hat{C})$ with $\hat{S}_T < S_T$.

(ii) From (i), we see that $\nexists$ a feasible solution of the problem BTHTP that gives the sum of transportation times strictly less than $S_T$. Hence, $S_T$ is the minimum possible sum of transportation times of Level-I and Level-II.

Moreover, it is worth mentioned here that Theorem 4.3.2 holds only if the pair $(S_T, C)$ is selected from a singleton set of non-dominated pairs.

**Theorem 4.3.3.** Algorithm records all efficient pairs i.e., no efficient pair is missed by the algorithm.

Proof. Let if possible, $(S_T, C)$ be an efficient pair obtained from a feasible solution $(X, Y)$ of the problem BTHTP, not recorded by the algorithm i.e., $\exists$ two consecutive
recorded efficient pairs \((S_{T_p}, C_p)\) and \((S_{T_{p+1}}, C_{p+1})\) say, such that

\[
S_{T_p} > S_T > S_{T_{p+1}}
\]  

(4.3.7)

and

\[
C_p < C < C_{p+1}
\]  

(4.3.8)

To obtain an efficient pair next to \((S_{T_p}, C_p)\), we choose a pair with the cost immediate higher than \(C_p\), from the set \(N_{p+1}\) which is equal to

\[
\{\text{NDPs obtained from the major sub-problems } P^{i}_{[S_{T_p}, S_{T_{p}}]}(S_{T_p})\} \cup \{\text{NDPs obtained at previous steps (if any) that are unexplored till now}\}
\]

Denote this pair as \((S'_{T_p}, C'_p)\) i.e., \(C'_p = \min\limits_{(S_T, C) \in N_{p+1}} C\).

Case (i). Let \((S'_{T_p}, C'_p)\) be a non-dominated pair obtained from an OMFS of one of the sub-problems restricted at \(S_{T_p}\). While applying the algorithm, we solve all the major sub-problems restricted at \(S'_{T_p}\).

Subcase (a). If \(CP^{i}_{[a^{'i}_{p}, b^{'i}_{p}]}(S'_{T_p}) > C'_p\), for all major values of \(i\), then \((S'_{T_p}, C'_p)\) is an efficient pair next to \((S_{T_p}, C_p)\). Therefore, \((S'_{T_p}, C'_p) = (S_{T_{p+1}}, C_{p+1})\).

Since \(S_T < S_{T_p}\) (by (4.3.7)), if we choose \(Z\) satisfying equation (4.1.1) with respect to the solution \((X, Y)\), then \(Z\) will be an MFS of \(P^{i}_{[a^{'i}_{p}, b^{'i}_{p}]}(S_{T_p})\), for some major value of \(i\). Therefore, for that particular major value of \(i\),

\[
C \geq CP^{i}_{[a^{'i}_{p}, b^{'i}_{p}]}(S_{T_p})
\]  

(4.3.9)

Also, \(C'_p\) is the minimum cost among the costs of all the pairs in \(N_{p+1}\), therefore

\[
CP^{i}_{[a^{'i}_{p}, b^{'i}_{p}]}(S_{T_p}) \geq C'_p\), for all major values of \(i\)
\]  

(4.3.10)

From (4.3.9) and (4.3.10), we see that

\[
C \geq C'_p \text{ or } C \geq C_{p+1}
\]  

(4.3.11)
But, $S_T > S_{T+1}$ (by (4.3.7)). Therefore, the pair $(S_T, C)$ is dominated by $(S_{T+1}, C_{p+1})$, which is a contradiction to our assumption.

Subcase (b). If $CP^i_{[a'_i, b'_i]}(S'_T) = C'_p$, at least for one major $i$, then $(S'_T, C'_p)$ can not be an efficient pair. So, we assign $S'_T$, a new updated value given by

$$S'_T = \min_{\text{major } i} \{S_T(Z'_i) | C(Z'_i) = C'_p\}$$

and explore the updated pair $(S'_T, C'_p)$.

If $CP^i_{[a'_i, b'_i]}(S'_T) > C'_p$ for the updated $S'_T$, then record the updated pair $(S'_T, C'_p)$ as $(S_{T+1}, C_{p+1})$, otherwise again update its value.

Continue doing this until the updated pair $(S'_T, C'_p)$ is recorded as $(S_{T+1}, C_{p+1})$.

Clearly, $S_{T+1} < S'_T$ (original) and $C_{p+1} = C'_p$ (original) which means the pair $(S'_T, C'_p)$ (original) is dominated by the pair $(S_{T+1}, C_{p+1})$. Moreover, the pair $(S_T, C)$ is an efficient pair, therefore $(S_T, C)$ and $(S'_T, C'_p)$ (original) are different pairs.

It is clear that (original) $S'_T < S_T$. If possible, assume that

$$S'_T \leq S_T \tag{4.3.12}$$

From the equation (4.3.11)

$$C \geq C'_p$$

This relation is true for original as well as updated $(S'_T, C'_p)$ as after every update $C'_p$ remains the same. Equality can not hold in the equations (4.3.11) and (4.3.12) together as $(S_T, C)$ and $(S'_T, C'_p)$ are different pairs. Hence, the pair $(S_T, C)$ is dominated by the original pair $(S'_T, C'_p)$, which is a contradiction to our assumption.

Now, the only option is that $S'_T > S_T$ or more precisely, $S_T > S'_T > S_T$ or $S_{T+1}$. But then, the pair $(S_T, C)$ is dominated by the pair $(S_{T+1}, C_{p+1})$ as $C \geq C'_p (= C_{p+1})$. Again, a contradiction to our assumption.

Case (ii). Let $(S'_T, C'_p)$ be a non-dominated pair obtained from an OMFS of one of the sub-problems solved at the previous steps, that is unexplored till now.
Since the pair \((S_T^p, C_p)\) is explored before the pair \((S_T^{p'}, C_p')\), it means that \(C_p < C_p'\).

Also, by the choice criteria,

\[ CP^i_{[a^i_p, b^i_p]}(S_T^p) > C_p', \quad \text{for all major values of } i \quad (4.3.13) \]

This is because if \(CP^i_{[a^i_p, b^i_p]}(S_T^p) = C_p'\), for some major \(i\), then the sum in the corresponding pair can neither be less than \(S_T^{p'}\) (as \((S_T^{p'}, C_p')\) is non-dominated) nor greater than \(S_T^{p'}\) (as \(N_{p+1}\) contains only non-dominated pairs). However, if the sum is equal to \(S_T^{p'}\), it would become Case (i) which has been discussed earlier.

Now, clearly, \(S_T^{p'} < S_T^p\). Therefore, \(\forall \alpha, \beta \in \{1, 2, 3.p, 4.q\},\)

\[ P^\alpha_{[a^\alpha_p, b^\alpha_p]}(S_T^{p'}) \subset P^\beta_{[a^\beta_p, b^\beta_p]}(S_T^p) \]

and consequently, \(CP^\alpha_{[a^\alpha_p, b^\alpha_p]}(S_T^{p'}) \geq CP^\beta_{[a^\beta_p, b^\beta_p]}(S_T^p)\) (It should be noted that \(\alpha\) may or may not be equal to \(\beta\)). Using the equation \((4.3.13)\), we get

\[ CP^\alpha_{[a^\alpha_p, b^\alpha_p]}(S_T^{p'}) > C_p', \quad \forall \alpha \in \{1, 2, 3.p, 4.q\} \]

Therefore,

\[ CP^i_{[a^i_p, b^i_p]}(S_T^{p'}) > C_p' \quad \forall \text{ major values of } i. \]

Thus, \((S_T^{p'}, C_p')\) is the next efficient pair, i.e., \((S_T^{p'}, C_p') = (S_{T_{p+1}}^p, C_{p+1})\).

Further, \(S_T < S_T^p\), therefore, \(C \geq CP^i_{[a^i_p, b^i_p]}(S_T^p)\). Using \((4.3.13)\), we see that \(C > C_p' = C_{p+1}\).

But \(S_T > S_{T_{p+1}}\), which means that the pair \((S_T, C)\) is dominated by the pair \((S_{T_{p+1}}, C_{p+1})\), again a contradiction to our assumption.

Therefore, we conclude that our assumption is wrong. Hence, the algorithm records all the efficient pairs. \(\square\)
4.4 Numerical illustration

Consider a $3 \times 5$ BTHTP given in Table 4.4.1 where the source-destination links with bold entries denote the Level-I links while the others denote Level-II links. Here, the index sets of sources and destinations are $I = \{1, 2, 3\}$ and $J = \{1, 2, 3, 4, 5\}$ respectively. Also, at each source-destination link, the number on top left corner denotes the transportation cost $c_{ij}$, for $i \in I$, $j \in J$ whereas the number on bottom right corner denotes the transportation time $t_{ij}$. The quantities $a_i$ and $b_j$ denote the availability at $i$th source and demand at $j$th destination, respectively.

The time entries in Level-I and Level-II links can be arranged in ascending order as follows:

$$t_1^1(=2) < t_2^1(=3) < t_3^1(=4) < t_4^1(=5) < t_5^1(=6)$$

and

$$t_1^2(=2) < t_2^2(=3) < t_3^2(=4) < t_4^2(=5) < t_5^2(=8)$$

Initial Step. Solve the problem $CP_0$ whose OFS $Z^0$, is obtained as $z_0 = 6$, $z_1 = 1$, $z_2 = 2$, $z_3 = 3$, $z_4 = 5$, $z_5 = 3$.

From the solution $Z^0$, the optimal cost $C_0$ is obtained as 211 and the corresponding feasible solutions $X^0$ and $Y^0$, defined as in the equation (4.1.1), of Level-I and Level-II transportation problems respectively, give $T_1(X^0) = 5$ and $T_2(Y^0) = 8$.

Therefore, the pair $(S_{T_0}, C_0)$ is noted as $(13, 211)$ where $S_{T_0} = T_1(X^0) + T_2(Y^0) = 5 + 8 = 13$. Set $N_0 = \{(S_{T_0}, C_0)\} = \{(13, 211)\}$.

Step 1. Explore the pair $(13, 211)$ by defining the sub-problems $P_i^{a_i, b_i}(13), \forall i,$
Figure 4.4.1: Possible time distributions of the sub-problems restricted at the sum 13.
given below. The possible time distributions of these sub-problems are depicted in Figure 4.4.1

1. $P^1_{[5,5]}(13)$

2. $P^2_{[4,8]}(13)$

3. Here $T'_1 = \min\{S_{T_0}, (T_1)_{\text{max}}\} = \min\{13, 6\} = 6$ and there is no time entry in Level-I that lie between $T_1(X^0) = 5$ and $T'_1 = 6$, i.e., $r = 0$. Therefore, $T_{1,1}$ and $T'_1$ coincide. So, there is only one sub-problem viz., $P^3_{[6,5]}(13)$.

4. Also, $T'_2 = \min\{S_{T_0}, (T_2)_{\text{max}}\} = \min\{13, 8\} = 8$ and there is no time entry in Level-II that lie between $T_2(Y^0) = 8$ and $T'_2 = 8$, i.e., $s = 0$. Therefore, $T_{2,1}$ and $T'_2$ coincide. So there is only one sub-problem viz., $P^4_{[4,8]}(13)$ which is exactly same as the problem $P^2_{[4,8]}(13)$.

Out of these sub-problems, only $P^2_{[4,8]}(13)$ and $P^3_{[6,5]}(13)$ are major sub-problems whose optimal solutions are obtained as follows:

$P^2_{[4,8]}(13)$: An OMFS $Z^0_2$ of this problem is obtained as

$$(z_{14})^0_2 = 6, \ (z_{15})^0_2 = 1, \ (z_{21})^0_2 = 0, \ (z_{22})^0_2 = 5, \ (z_{25})^0_2 = 3, \ (z_{31})^0_2 = 3, \ (z_{33})^0_2 = 6$$

which gives $T_1(X^0_2) = 4, \ T_2(Y^0_2) = 8$ and $C(Z^0_2) = 214$. In other words, we say $CP^2_{[4,8]}(13) = 214$.

Therefore, $(S_T(Z^0_2), C(Z^0_2)) = (4 + 8, 214) = (12, 214)$.

$P^3_{[6,5]}(13)$: An OMFS $Z^0_{3,1}$ of this problem is obtained as

$$(z_{14})^0_{3,1} = 5, \ (z_{15})^0_{3,1} = 2, \ (z_{23})^0_{3,1} = 6, \ (z_{25})^0_{3,1} = 2, \ (z_{31})^0_{3,1} = 3, \ (z_{32})^0_{3,1} = 3, \ (z_{34})^0_{3,1} = 1$$

which gives $T_1(X^0_{3,1}) = 5, \ T_2(Y^0_{3,1}) = 4$ and $C(Z^0_{3,1}) = 217$. In other words, we say $CP^3_{[6,5]}(13) = 217$.

Therefore, $(S_T(Z^0_{3,1}), C(Z^0_{3,1})) = (5 + 4, 217) = (9, 217)$.

We see that $CP^i_{[a_i,b_i]}(13) > 211, \ \forall$ major values of $i$. Therefore, the pair $(13, 211)$ is the first efficient pair.
Now, construct the set

\[ N_1 = \{ \text{NDPs obtained by solving major } P_{[a_i^0, b_i^0]}(S_{T_0}) \} \cup \{ \text{NDPs obtained in previous steps that are unexplored till now} \} \]

\[ = \{(12, 214), (9, 217)\} \cup \emptyset \]

Out of \(N_1\), choose the pair (12, 214) as it has minimum cost. Call it \((S_{T_1}, C_1)\) and explore. Note that the pair \((S_{T_1}, C_1)\) is obtained from the solution \(Z_0^0\) of CMTP solved at the Step 1.

Step 2. Explore the pair (12, 214) by defining the sub-problems \(P_{[a_i^1, b_i^1]}(12), \forall i\), given below. The possible time distributions of these sub-problems are depicted in Figure 4.4.2.

1. \(P_{[4,5]}^1(12)\)
2. \(P_{[3,8]}^2(12)\)
3. \(P_{[5,5]}^{3,1}(12)\) and \(P_{[6,5]}^{3,2}(12)\)
4. \(P_{[3,8]}^{4,1}(12)\)

Out of these sub-problems, \(P_{[3,8]}^2(12)\) and \(P_{[6,5]}^{3,2}(12)\) are major sub-problems whose optimal solutions are obtained as follows:

\(P_{[3,8]}^2(12)\): An OMFS \(Z_2^1\) of this problem is
\[(z_{14})^1_2 = 3, \ (z_{15})^1_2 = 4, \ (z_{21})^1_2 = 3, \ (z_{22})^1_2 = 5, \ (z_{24})^1_2 = 0, \ (z_{33})^1_2 = 6, \ (z_{34})^1_2 = 3\]
which gives \(T_1(X_2^1) = 3, \ T_2(Y_2^1) = 8\) and \(C(Z_2^1) = 220 = CP_{[3,8]}^2(12)\).

Therefore, \((S_T(Z_2^1), C(Z_2^1)) = (3 + 8, 220) = (11, 220)\). This pair is dominated by the pair (9, 217).

\(P_{[6,5]}^{3,2}(12)\): An OMFS \(Z_{3,2}^1\) of this problem is
\[(z_{14})^1_{3,2} = 5, \ (z_{15})^1_{3,2} = 2, \ (z_{23})^1_{3,2} = 6, \ (z_{25})^1_{3,2} = 2, \ (z_{31})^1_{3,2} = 3, \ (z_{32})^1_{3,2} = 5, \ (z_{34})^1_{3,2} = 1\]
which gives \(T_1(X_{3,2}^1) = 5, \ T_2(Y_{3,2}^1) = 4\) and \(C(Z_{3,2}^1) = 217 = CP_{[6,5]}^{3,2}(12)\).

Therefore, \((S_T(Z_{3,2}^1), C(Z_{3,2}^1)) = (5 + 4, 217) = (9, 217)\) (Repeated).
Figure 4.4.2: Possible time distributions of the sub-problems restricted at the sum 12.
We see that $CP_{i}^{j}(12) > C_{1}$, $\forall$ major values of $i$. Therefore $(12, 214)$ is the second efficient pair.

Now, the set $N_2$ of non-dominated pairs is noted as $N_2 = \emptyset \cup \{(9, 217)\}$.

Set $(S_{T_2}, C_2) = (9, 217)$ and explore.

Step 3. Explore the pair $(9, 217)$ by defining the sub-problems $P_{i}^{j}[a_{i}, b_{j}]^{2}(9)$, $\forall i$, given below. The possible time distributions of these sub-problems are depicted in Figure 4.4.3.

1. $P_{[5,3]}^{1}(9)$
2. $P_{[4,4]}^{2}(9)$
3. $P_{[6,2]}^{3,1}(9)$
4. $P_{[3,5]}^{4,1}(9)$ and $P_{[1,8]}^{4,2}(9)$ (Not possible as there is no time entry in Level-I which is strictly less than 1)

Out of these sub-problems, $P_{[5,3]}^{1}(9)$, $P_{[4,4]}^{2}(9)$, $P_{[6,2]}^{3,1}(9)$ and $P_{[3,5]}^{4,1}(9)$ are major sub-problems whose optimal solutions are obtained as follows:

$P_{[5,3]}^{1}(9)$: An OMFS $Z_{2}^{1}$ of this problem is

$(z_{13})_{2}^{2} = 1$, $(z_{14})_{2}^{2} = 2$, $(z_{15})_{2}^{2} = 4$, $(z_{21})_{2}^{2} = 3$, $(z_{23})_{2}^{2} = 5$, $(z_{32})_{2}^{2} = 5$, $(z_{34})_{2}^{2} = 4$

which gives $T_{1}(X_{2}^{2}) = 5$, $T_{2}(Y_{2}^{2}) = 3$ and $C(Z_{2}^{1}) = 225 = CP_{[a,b]}^{1}(9)$.

Therefore, $(S_{T}(Z_{2}^{1}), C(Z_{2}^{1})) = (5 + 3, 225) = (8, 225)$.

$P_{[4,4]}^{2}(9)$: An OMFS $Z_{2}^{2}$ of this problem is

$(z_{13})_{2}^{2} = 6$, $(z_{14})_{2}^{2} = 0$, $(z_{15})_{2}^{2} = 1$, $(z_{22})_{2}^{2} = 5$, $(z_{25})_{2}^{2} = 3$, $(z_{31})_{2}^{2} = 3$, $(z_{34})_{2}^{2} = 6$

which gives $T_{1}(X_{2}^{2}) = 4$, $T_{2}(Y_{2}^{2}) = 4$ and $C(Z_{2}^{2}) = 244 = CP_{[a,b]}^{2}(9)$.

Therefore, $(S_{T}(Z_{2}^{2}), C(Z_{2}^{2})) = (4 + 4, 244) = (8, 244)$. This pair is dominated by the pair (8, 225).

$P_{[6,2]}^{3,1}(9)$: An OMFS $(Z_{3,1}^{2})$ of this problem is

$(z_{13})_{3,1}^{2} = 5$, $(z_{14})_{3,1}^{2} = 2$, $(z_{21})_{3,1}^{2} = 3$, $(z_{23})_{3,1}^{2} = 1$, $(z_{25})_{3,1}^{2} = 4$, $(z_{32})_{3,1}^{2} = 5$, $(z_{34})_{3,1}^{2} = 4$ which gives $T_{1}(X_{3,1}^{2}) = 5$, $T_{2}(Y_{3,1}^{2}) = 2$ and $C(Z_{3,1}^{2}) = 233 = CP_{[a,b]}^{3,1}(9)$.

Therefore, $(S_{T}(Z_{3,1}^{2}), C(Z_{3,1}^{2})) = (5 + 2, 233) = (7, 233)$.

$P_{[3,5]}^{4,1}(9)$: There is no MFS to this problem.
We see that $CP^i_{[a_i^1,b_i^1]}(9) > 217$, $\forall$ major values of $i$. Therefore, the pair $(9, 217)$ is the third efficient pair.

Now, the set $N_3$ of non-dominated pairs is noted as $N_3 = \{(8, 225)(7, 233)\} \cup \emptyset$.

Set $(S_{T_2}, C_2) = (8, 225)$ and explore.

**Step 4.** Explore the pair $(8, 225)$ by defining the sub-problems $P^i_{[a_i^1,b_i^1]}(8)$, $\forall i$, given below. The possible time distributions of these sub-problems are depicted in Figure 4.4.4.

1. $P^1_{[5,2]}(8)$

2. $P^2_{[4,3]}(8)$
3. \( P^{3.1}_{[6,2]}(8) \) (Not possible as there is no time entry in Level-II which is strictly less than 2).

4. \( P^{4.1}_{[3,4]}(8) \) and \( P^{4.2}_{[2,5]}(8) \)

Out of these sub-problems, \( P^{1}_{[5,2]}(8) \), \( P^{2}_{[4,3]}(8) \), \( P^{4.1}_{[3,4]}(8) \) and \( P^{4.2}_{[2,5]}(8) \) are major sub-problems whose optimal solutions are obtained as follows:

\( P^{1}_{[5,2]}(8) \): An OMFS \( Z_1^3 \) of this problem is
\( (z_{13})^3_1 = 5, (z_{14})^3_1 = 2, (z_{21})^3_1 = 3, (z_{23})^3_1 = 1, (z_{25})^3_1 = 4, (z_{32})^3_1 = 5, (z_{34})^3_1 = 4 \)
which gives \( T_1(X_1^3) = 5, T_2(Y_1^3) = 2 \) and \( C(Z_1^3) = 233 = CP^{1}_{[5,2]}(8) \).
Therefore, \( (S_r(Z_1^3), C(Z_1^3)) = (5 + 2, 233) = (7, 233) \) (Repeated).

\( P^{2}_{[4,3]}(8) \): There is no feasible solution to this problem.
Numerical illustration

The sub-problems $P_{[3,4]}^4(8)$ and $P_{[2,5]}^4(8)$ are contained in $P_{[3,5]}^4(9)$ which has no M-feasible solution. Therefore, these problems also, do not have an M-feasible solution. Note that $CP^i_{[a^i_3, b^i_3]}(8) > 225, \forall$ major values of $i$ for which the solution exists. Therefore, the pair $(8, 225)$ is the fourth efficient pair. Now, the set $N_4$ of non-dominated pairs becomes $N_4 = \emptyset \cup \{(7, 233)\}$. Out of $N_4$, only choice is the pair $(7, 233)$. Call it $(S_{T_4}, C_4)$ and explore. Note that the pair $(S_{T_4}, C_4)$ is initially obtained from the solution $Z^2_{3,1}$ of CMTP solved at the step 3.

Step 5. Explore the pair $(7, 233)$ by defining the sub-problems $P^i_{[a^i_4, b^i_4]}(7), \forall i$, given below. The possible time distributions of these sub-problems are depicted in Figure 4.4.5.

1. $P^1_{[5,2]}(7)$ (Not possible as there is no time entry in Level-II which is strictly less than 2)

2. $P^2_{[5,2]}(7)$

3. $P^3_{[6,1]}(7)$ (Not Possible as there is no time entry in Level-II which is strictly less than 1)

4. $P^4_{[3,3]}(7), P^4_{[2,4]}(7)$ and $P^4_{[2,3]}(7)$ (Not Possible as there is no time entry in Level-I which is strictly less than 2)

Out of these sub-problems, $P^2_{[5,2]}(7), P^4_{[3,3]}(7)$ and $P^4_{[2,4]}(7)$ are major sub-problems. None of these sub-problems is M-feasible. Therefore, (i) $\not\exists$ an efficient pair with sum of transportation times strictly less than 7. (ii) $7 = \min_{Z \in S} S_T(Z)$.

and the algorithm terminates as an efficient pair with the minimum sum of times of Level-I and Level-II is obtained. Hence, the set of efficient pairs is

$$\{(13, 211), (12, 214), (9, 217), (8, 225), (7, 233)\}$$

All of these efficient pairs are equally good as no additional subjective preference is considered.
Figure 4.4.5: Possible time distributions of the sub-problems restricted at the sum 7.
4.5 Computational experiments

The algorithm given in Section 4.3 have been coded in MATLAB and experiments were conducted using Intel Processor i5 with 2.5 GHz, 4 GB RAM on 64-bit windows operating system. The algorithm runs successfully for randomly generated BTHTP instances of different sizes. The integral values of $t_{ij}$ and $c_{ij}$ in these problems have been drawn from a uniform distribution of the intervals $[1, 40]$ and $[5, 60]$ respectively. Table 4.5.1 narrates the computational behavior of these problems with different percentages of Level-I links. In a problem of size $m \times n$, we report the average running time (in seconds) (taken over 50 instances) of the algorithm with different percentage of Level-I links where the number of Level-I links is taken as greatest integral value of the number corresponding to a particular percentage of $mn$. The running time of the algorithm depends upon the variation in time entries of Level-I and Level-II sets and the number of sub-problems solved for a particular instance of the problem BTHTP, which further depends upon the sum of transportation times corresponding to the minimum cost and the minimum possible sum of transportation times. This is due to the reason that the algorithm is constructed in such a way that sum of transportation times is reduced at each step, strictly by at least one unit. Therefore, greater the difference between sum of transportation times corresponding to the minimum cost and the minimum possible sum of transportation times, larger number of sub-problems are expected to be constructed which results in higher running time of the algorithm.

To analyze the computational behavior of the proposed algorithm more precisely, we have conducted the computational experiments, in particular, for the $10 \times 10$ instance, by changing the interval from which the integral values of $t_{ij}$ and $c_{ij}$ have been drawn, as follows:

1. $t_{ij} \in [10, 20], c_{ij} \in [5, 60]$
2. $t_{ij} \in [1, 40], c_{ij} \in [5, 20]$
3. $t_{ij} \in [10, 20], c_{ij} \in [5, 20]$
Table 4.5.2 narrates the computational behavior of the algorithm for the $10 \times 10$ instance with the aforementioned variations from which we observe that if the variation in integral values of $t_{ij}$ and $c_{ij}$ is less, the running time of the algorithm is reduced significantly in comparison to that listed in Table 4.5.1. For instance, a $10 \times 10$ BTHTP with 40% Level-I links is solvable in 0.696354 secs with $t_{ij} \in [10, 20], c_{ij} \in [5, 60]$, which is almost 18% of the running time of the algorithm for the same sized problem with $t_{ij} \in [1, 40], c_{ij} \in [5, 60]$ (given in Table 4.5.1). Thus, for the problem BTHTP instances with lesser variation in integral values of $t_{ij}$ and $c_{ij}$, our algorithm performs far better than it does for the instances with larger variations.

Worst case complexity of the proposed algorithm

The problem BTHTP has been studied through various CMTPs which are defined according to the possible time distributions in Level-I and Level-II sets, at a particular step of the algorithm. These possible time distributions are obtained by exploring all the distinct time entries in Level-I and Level-II sets such that the sum of transportation times of both the levels corresponding to the optimal solutions of these problems is strictly less than that obtained at the previous step. Therefore, the maximum number of CMTPs solved by the proposed algorithm to yield all the Pareto optimal solutions of the problem BTHTP is equal to the maximum possible combinations of Level-I and Level-II time entries. If the number of distinct time entries in Level-I and Level-II are $s_1$ and $s_2$ say, where $s_1 + s_2 \leq mn$, then their maximum possible combinations are $s_1 s_2$. Equivalently, the algorithm needs to solve at the most $s_1 s_2$ CMTPs. Furthermore, the value of $s_1 s_2$ for the given values of $m$ and $n$ is maximum when $s_1 = \left\lfloor \frac{mn}{2} \right\rfloor$ and $s_2 = \left\lceil \frac{mn}{2} \right\rceil$ or vice-versa where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ represent the floor and ceiling functions, respectively. Since the best time complexity to solve a CMTP of size $m \times n$, obtained by Orlin [Orl93] is $O(n \log n (m + n \log n))$ where $m$ and $n$ are the number of sources and destinations respectively, therefore the time complexity of the proposed algorithm is $O\left(\left\lfloor \frac{mn}{2} \right\rfloor \left\lceil \frac{mn}{2} \right\rceil n \log n (m + n \log n)\right)$. 
Table 4.5.1: Running time of proposed algorithm

<table>
<thead>
<tr>
<th>Size((m \times n))</th>
<th>Percentage of Level-I links</th>
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*Continued on next page*
Table 4.5.1 – Continued from previous page

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<th>Size ((m \times n))</th>
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### Table 4.5.1 – Continued from previous page

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<th>Size ($m \times n$)</th>
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<th>60</th>
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Table 4.5.2: Running time for $10 \times 10$ BTHTP for various time and cost distributions

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4.6 Concluding remarks

In this chapter, a bi-criteria two-level hierarchical transportation problem with the objective of minimizing ‘sum of transportation times’ and ‘sum of transportation costs’ of both the levels is explored. An algorithm based on solving sub-problems with restricted combinations of Level-I and Level-II transportation times is proposed which generates all the efficient pairs. These time restrictions are imposed in such a way that, at each step, the sum of transportation times decreases strictly whereas
Concluding remarks

the sum of transportation costs increases. The problem BTHTP with one of the objectives as concave function and the other as linear function has been studied for the first time in literature, therefore a comparative study cannot be carried out. This problem can be explored further as a multi-level and multi-criteria transportation problem for which more efficient techniques can be developed.