CHAPTER – V
ONE MODULO THREE MEAN LABELING OF SOME SPECIAL GRAPHS

5.1. INTRODUCTION

In chapters III and IV, we have discussed necessary conditions, properties and labeling schemes of one modulo three mean labeling of graphs.

In this chapter, we establish one modulo three meanness of the following graphs

- $M(P_n)$
- $spl(P_n)$
- $K_{1, n} \bowtie K_{1, n}$
- $P_m \times P_n$
- $L_m \odot K_{1, n}$
- $P_{m, n}$
- $C_n \circ P_m$
- $C_n \odot P_m$
- $VD(P_n)$
- $ED(P_n)$

Further, we have also introduced one modulo three mean number for a non-one modulo three mean graph. We have found this number for the graphs $C_3 \circ K_{1, n}$, $C_4 \circ K_{1, n}$, $K_{1, n}$, $S(B_{m,n})$, for $m = n$ and $S(B_{m,n})$, for $m \geq n + 2$. 
5.2. ONE MODULO THREE MEAN LABELING OF SPECIAL GRAPHS

Definition 5.2.1

Let $G$ be a graph. Let $G'$ be a copy of $G$. The mirror graph $M(G)$ of $G$ is defined as the disjoint union of $G$ and $G'$ with additional edges joining each vertex of $G$ to its corresponding vertex in $G'$.

Theorem 5.2.2

The mirror graph $M(P_n)$ is a one modulo three mean graph if and only if $n$ is odd.

Proof

Assume $n$ is odd. Let $\{v_i, v'_i, 1 \leq i \leq n\}$ be the vertices and $\{e_i, 1 \leq i \leq 3n - 2\}$ be the edges which are denoted as in Figure 5.1.

![Figure 5.1: Ordinary labeling of $M(P_n)$](image)

First we label the vertices as follows:

Define $f : V \rightarrow \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\}$ by

$$f(v_{2i-1}) = 6(i - 1) \quad 1 \leq i \leq \frac{n + 1}{2}$$

$$f(v_{2i}) = 6(i - 1) + 1 \quad 1 \leq i \leq \frac{n - 1}{2}$$
\[ f(v_{2i-1}) = 6(n - 1) + 6(i - 1) + 1 \quad 1 \leq i \leq \frac{n+1}{2} \]

\[ f(v_{2i}) = 6n + 6(i - 1) \quad 1 \leq i \leq \frac{n-1}{2} \]

Then the induced edge labels are:

\[ f^*(e_i) = 3i - 2 \quad 1 \leq i \leq 3n - 2 \]

The above defined function \( f \) provides one modulo three mean labeling of the graph \( M(P_n) \). Hence the graph \( M(P_n) \) is a one modulo three mean graph when \( n \) is odd.

Conversely, let us assume that \( M(P_n) \) is a one modulo three mean graph. Suppose \( n \) is even, then the number of edges \( q = 3n - 2 \) is even. Now, by Corollary 3.3.5, \( M(P_n) \) is not a one modulo three mean graph, a contradiction to our assumption.

Hence \( n \) is odd.

One modulo three mean labeling of the graph \( M(P_5) \) is shown in Figure 5.2.

\[ \text{Figure 5.2: OMTML of } M(P_5) \]
One modulo three mean labeling of the graph $M(P_g)$ is shown in Figure 5.3.

![Fig 5.3: OMTML of $M(P_g)$](image)

**Definition 5.2.3**

For a graph $G$, the split graph is obtained by adding to each vertex $v$, a new vertex $v'$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$. The resultant graph is denoted as $spl(P_n)$.

**Theorem 5.2.4**

The splitting graph $spl(P_n)$ is a one modulo three mean graph if and only if $n$ is even.

**Proof**

Assume $n$ is even. Let $\{u_i, v_i, 1 \leq i \leq n\}$ be the vertices and $\{a_i, b_i, c_i, 1 \leq i \leq n-1\}$ be the edges of $spl(P_n)$ which are denoted as in Figure 5.4.

![Fig 5.4: Ordinary labeling of $spl(P_n)$](image)
First we label the vertices as follows:

Define \( f : V \rightarrow \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\} \) by

For \( 1 \leq i \leq n \),

\[
\begin{align*}
  f(v_i) &= \begin{cases} 
  9(i-1) & \text{if } i \text{ odd} \\
  9i - 11 & \text{if } i \text{ even}
  \end{cases} \\
  f(u_i) &= \begin{cases} 
  9i - 3 & \text{if } i \text{ odd} \\
  9i - 17 & \text{if } i \text{ even}
  \end{cases}
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
  f^*(a_i) &= 9i - 5 & 1 \leq i \leq n - 1 \\
  f^*(b_i) &= 9i - 8 \\
  f^*(c_i) &= 9i - 2
\end{align*}
\]

The above defined function \( f \) provides one modulo three mean labeling of the graph \( spl(P_n) \). Hence the graph \( spl(P_n) \) is a one modulo three mean graph when \( n \) is even.

Conversely, let us assume that \( spl(P_n) \) is a one modulo three mean graph. Suppose \( n \) is odd, then the numbers of edges \( q = 3(n-1) \) is even. Now, by Corollary 3.3.5, \( spl(P_n) \) is not a one modulo three mean graph, a contradiction to our assumption.

Hence \( n \) is even.
One modulo three mean labeling of the graph $spl(P_6)$ is shown in Figure 5.5.

![Figure 5.5: OMTML of $spl(P_6)$](image)

One modulo three mean labeling of the graph $spl(P_{10})$ is shown in Figure 5.6.

![Figure 5.6: OMTML of $spl(P_{10})$](image)

**Definition 5.2.5**

Consider two copies of Graph $G$ (wheel, star, fan and friendship) namely $G_1$ and $G_2$. Then the graph $G' = \langle G_1 \triangle G_2 \rangle$ is the graph obtained by joining the apex vertices of $G_1$ and $G_2$ by an edge as well as to a new vertex $v'$.

**Theorem 5.2.6**

The graph $\langle K_{1,n} \triangle K_{1,n} \rangle$ is a one modulo three mean graph for all $n$.

**Proof**

Let $\{u,v,w,u_i,v_i,1 \leq i \leq n\}$ be the vertices and $\{a_i,b_i,1 \leq i \leq n,c,d,e\}$ be the edges which are denoted as in Figure 5.7.
First we label the vertices as follows:

Define \( f : V \rightarrow \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\} \) by

\[
\begin{align*}
  f(u) &= 1 \quad ; \quad f(v) = 6(n + 1) \\
  f(w) &= 6n + 1 \quad ; \quad f(v_n) = 6(n + 1) + 1 \\
  f(u_i) &= 6(i - 1) \quad 1 \leq i \leq n \\
  f(v_i) &= 6i + 1 \quad 1 \leq i \leq n - 1
\end{align*}
\]

Then the induced edge labels are:

\[
\begin{align*}
  f^*(c) &= 3n + 1 \quad ; \quad f^*(e) = 3(n + 1) + 1 \\
  f^*(d) &= 6(n + 1) - 2 \quad ; \quad f^*(b_n) = 6(n + 1) + 1 \\
  f^*(a_i) &= 3i - 2 \quad 1 \leq i \leq n \\
  f^*(b_i) &= 3i + 3(n + 1) + 1 \quad 1 \leq i \leq n - 1
\end{align*}
\]
The above defined function $f$ provides one modulo three mean labeling of $\langle K_{1,n} \triangle K_{1,n} \rangle$. Hence the graph $\langle K_{1,n} \triangle K_{1,n} \rangle$ is one modulo three mean graph.

One modulo three mean labeling of the graph $\langle K_{1,6} \triangle K_{1,6} \rangle$ is shown in Figure 5.8.

![Figure 5.8: Ordinary labeling of $\langle K_{1,6} \triangle K_{1,6} \rangle$](image)

One modulo three mean labeling of the graph $\langle K_{1,9} \triangle K_{1,9} \rangle$ is shown in Figure 5.9.

![Figure 5.9: Ordinary labeling of $\langle K_{1,9} \triangle K_{1,9} \rangle$](image)
Theorem 5.2.7

The graph \( \langle K_{1,m} \uplus K_{1,n} \rangle \) is not a one modulo three mean graph if \( m = n + 1 \).

Proof

Suppose \( G = \langle K_{1,m} \uplus K_{1,n} \rangle \) is a one modulo three mean graph. The number of edges \( q = m + n + 3 \) is even, a contradiction to Corollary 3.3.5. Hence the theorem.

Theorem 5.2.8

The graph \( \langle K_{1,m} \uplus K_{1,n} \rangle \), \( m \geq n + 2 \) is not a one modulo three mean graph.

Proof

Suppose \( \langle K_{1,m} \uplus K_{1,n} \rangle \), \( m \geq n + 2 \) is a one modulo three mean graph.

Case 1: \( m \) even, \( n \) odd (or) \( m \) odd, \( n \) even

In this case, the number of edges in \( q = m + n + 3 \) is even, a contradiction to Corollary 3.3.5.

Case 2: \( m \) even, \( n \) even (or) \( m \) odd, \( n \) odd

In this case, the number of edges in \( q = m + n + 3 \) is odd. Therefore by Theorem 3.3.9.

\[
\Delta(G) \leq \frac{q+1}{2}
\]

Here, \( G = \langle K_{1,m} \uplus K_{1,n} \rangle \), \( \Delta = m + 2 \), \( q = m + n + 3 \)
\[ m + 2 \leq \frac{m + n + 3 + 1}{2} \]

\[ 2m + 4 \leq m + n + 4 \]

\[ m \leq n \]

a contradiction to \( m \geq n + 2 \).

**Theorem 5.2.9**

The graph \( \langle K_{1,m} \boxtimes K_{1,n} \rangle \) is a one modulo three mean graph if and only if \( m = n \).

**Proof**

Follows from Theorems 5.2.6, 5.2.7 and 5.2.8.

**Definition 5.2.10**

The Cartesian Product of two paths \( P_m \) and \( P_n \) is known as a grid graph. It is denoted by \( P_m \times P_n \).

**Theorem 5.2.11**

The planar grid \( P_m \times P_n \) is a one modulo three mean graph if \( m \) even and \( n \) odd (or) \( m \) odd and \( n \) even.

**Proof**

Let \( \{a_{ij} : 1 \leq i \leq m, 1 \leq j \leq n \} \) be the vertices and \( \{e_i, 1 \leq i \leq 2mn - (m + n)\} \) be the edges which are denoted as in the Figure 5.10.
First we label the vertices as follows:

Define $f : V \rightarrow \{0, 1, 3, 4, ... , 3q - 3, 3q - 2\}$ by

For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f(a_{ij}) = \begin{cases} 
3(j - 1) + 3(i - 1)(2n - 1) & \text{i odd, j odd} \\
3(j - 1) + 3(i - 2)(2n - 1) + 6(n - 1) + 1 & \text{i even, j odd} \\
3(j - 2) + 3(i - 1)(2n - 1) + 1 & \text{i odd, j even} \\
3(j - 2) + 3(i - 2)(2n - 1) + 6n & \text{i even, j even}
\end{cases}$$

Then the induced edge labels are:

$$f^*(e_i) = 3i - 2 \quad 1 \leq i \leq 2mn - (m + n)$$

The above defined function $f$ provides one modulo three mean labeling of the graph $P_m \times P_n$. Hence $P_m \times P_n$ is a one modulo three mean graph.

One modulo three mean labeling of the graph $P_6 \times P_5$ is shown in Figure 5.11.
One modulo three mean labeling of the planar grid $P_9 \times P_6$ is shown in Figure 5.12.
Theorem 5.2.12

The planar grid $P_m \times P_n$ is not a one modulo three mean graph if both $m$ and $n$ are odd (or) even.

Proof

Suppose the graph $P_m \times P_n$ is a one modulo three mean graph. The number of edges $q = 2mn - (m + n)$ is even, a contradiction to Corollary 3.3.5.

Hence the theorem.

Theorem 5.2.13

The planar grid $P_m \times P_n$ is a one modulo three mean graph if and only if $m$ even and $n$ odd (or) $m$ odd and $n$ even.

Proof

Follows from Theorems 5.2.11 and 5.2.12.

Theorem 5.2.14

The graph $L_m \odot K_{1,n}$ is a one modulo three mean graph if and only if $m$ is odd for all $n$.

Proof

Assume $m$ is odd. Let $\{u_i, v_i, 1 \leq i \leq m, u_j, v_j, 1 \leq i \leq m, 1 \leq j \leq n\}$ be the set of vertices and $\{a_i, 1 \leq i \leq m, b_j, c_j, 1 \leq i \leq m - 1, e_j, g_j, 1 \leq i \leq m, 1 \leq j \leq n\}$ be the edges which are denoted as in the Figure 5.13.
Figure 5.13: Ordinary labeling of $\left( L_m \circ K_{1,n} \right)$

First we label the vertices as follows:

Define $f : V \rightarrow \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\}$ by

For $1 \leq i \leq m$, $1 \leq j \leq n$,

\[
f(u_{ij}) = \begin{cases} 
(6n + 9)(i - 1) + 6(j - 1) & \text{i odd} \\
(6n + 9)(i - 2) + 6(j - 1) + 6(n + 3) + 1 & \text{i even} 
\end{cases}
\]

\[
f(v_{ij}) = \begin{cases} 
(6n + 9)(i - 1) + 6j + 1 & \text{i odd} \\
(6n + 9)(i - 2) + 6(j - 1) + 6(n + 1) & \text{i even} 
\end{cases}
\]
For $1 \leq i \leq m$,

$$f(u_i) = \begin{cases} 
(6n+9)(i-1)+1 & \text{i odd} \\
(6n+9)(i-2)+6(n+1)+6n & \text{i even}
\end{cases}$$

$$f(v_i) = \begin{cases} 
(6n+9)(i-1)+6n & \text{i odd} \\
(6n+9)(i-2)+6(n+2)+1 & \text{i even}
\end{cases}$$

Then the induced edge labels are:

For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f^*(e_{ij}) = \begin{cases} 
(6n+9)(i-1)+3(j-1)+1 & \text{i odd} \\
(6n+9)(i-2)+3(i-1)+9n+13 & \text{i even}
\end{cases}$$

$$f^*(g_{ij}) = \begin{cases} 
(6n+9)(i-1)+3(j-1)+3(n+1)+1 & \text{i odd} \\
(6n+9)(i-2)+3(j-1)+6(n+1)+4 & \text{i even}
\end{cases}$$

$$f^*(a_i) = (6n+9)(i-1)+3n+1 \quad 1 \leq i \leq m$$

$$f^*(b_i) = (6n+9)(i-1)+6(n+1)-2 \quad 1 \leq i \leq m-1$$

$$f^*(c_i) = (6n+9)(i-1)+6(n+1)+1 \quad 1 \leq i \leq m-1$$

The above defined function $f$ provides one modulo three mean labeling of the graph $L_m \odot K_{1,n}$. Hence the graph $L_m \odot K_{1,n}$ is a one modulo three mean graph when $m$ is odd.

Conversely, let us assume that $L_m \odot K_{1,n}$ is a one modulo three mean graph. Suppose $m$ is even, then the number of edges $q = 2mn-3n-2$ is even. Now, by Corollary 3.3.5, $L_m \odot K_{1,n}$ is not a one modulo three mean graph, a contradiction to our assumption.

Hence $m$ is odd.
One modulo three mean labeling of the graph $L_7 \odot K_{1,4}$ is shown in Figure 5.14.

Figure 5.14: Ordinary labeling of $L_7 \odot K_{1,4}$
One modulo three mean labeling of graph $L_5 \odot K_{1,7}$ is shown in Figure 5.15.

Figure 5.15: OMTML of $L_5 \odot K_{1,7}$
Theorem 5.2.15

The graph $P_{m,n}$ is a one modulo three mean graph if and only if $m$ and $n$ are both odd.

Proof

Assume $m$ and $n$ are both odd. Let $\{u, v, v_i, 1 \leq i \leq n, 1 \leq j \leq m-1\}$ be the vertices and $\{e_j, 1 \leq i \leq n, 1 \leq j \leq m\}$ be the edges which are denoted as in Figure 5.16.

First we label the vertices as follows:

Define $f: V \to \{0, 1, 3, \ldots, 3q-3, 3q-2\}$ by

$$f(u) = 0; \quad f(v) = 3mn - 2$$

$$f(v_j) = \begin{cases} 
3nj + 6i - 3n - 5 & 1 \leq i \leq n, 1 \leq j \leq m - 1, j \text{ odd} \\
3nj + 6i - 6 & 1 \leq i \leq \frac{n+1}{2}, 1 \leq j \leq m - 1, j \text{ even} \\
3nj - 6n + 6i - 6 & \frac{n+3}{2} \leq i \leq n, 1 \leq j \leq m - 1, j \text{ even}
\end{cases}$$

Figure 5.16: Ordinary labeling of $P_{m,n}$
Then the induced edge labels are:

\[ f^*(e_{ij}) = \begin{cases} 
3i - 2 & 1 \leq i \leq n, j = 1 \\
3nj + 6i - 3n - 5 & 1 \leq i \leq \frac{n+1}{2}, 2 \leq j \leq m - 1 \\
3nj + 6i - 6n - 5 & \frac{n+3}{2} \leq i \leq n, 2 \leq j \leq m - 1 \\
3i - 3\left(\frac{n+1}{2}\right) + 3mn - 2 & 1 \leq i \leq \frac{n+1}{2}, j = m \\
3(i-n) + 3mn - 3\left(\frac{n+1}{2}\right) - 2 & \frac{n+3}{2} \leq i \leq n, j = m 
\end{cases} \]

The above defined function \( f \) provides one modulo three mean labeling of the graph \( P_{m,n} \).

Conversely, let us assume that \( P_{m,n} \) is a one modulo three mean graph.

Suppose \( m \) even and \( n \) odd (or) \( m \) odd and \( n \) even (or) \( m \) and \( n \) are both even, then the number of edges \( q = mn \) is even. Now, by Corollary 3.3.5, \( P_{m,n} \) is not a one modulo three mean graph, a contradiction to our assumption.

Hence \( m \) and \( n \) are both odd.

One modulo three mean labeling of the graph \( P_{5,5} \) is shown in Figure 5.17.
One modulo three mean labeling of the graph $P_{7,9}$ is shown Figure 5.18.

Definition 5.2.16

A dragon is a graph formed by joining an end vertex of a path $P_m$ to a vertex of the cycle $C_n$. It is denoted as $C_n \oplus P_m$.

Theorem 5.2.17

The graph dragon $C_n \oplus P_m$ is a one modulo three mean graph for $n \equiv 0 \pmod{4}$ and $m$ is even.

Proof

Let $\{v_i, 1 \leq i \leq m + n - 1\}$ be the vertices and $\{e_i, 1 \leq i \leq m + n - 1\}$ be the edges which are denoted as in Figure 5.19.
First we label the vertices as follows:

Define $f: V \rightarrow \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\}$ by

$$f(v_{2i-1}) = 6(i - 1) \quad 1 \leq i \leq \frac{m+n}{2}$$

$$f(v_{2i}) = \begin{cases} 
6(i -1) + 1 & 1 \leq i \leq \frac{n}{4} \\
6i + 1 & \frac{n}{4} + 1 \leq i \leq \frac{m+n}{2} -1 
\end{cases}$$

Then the induced edge labels are:

$$f^*(e_i) = \begin{cases} 
3i - 2 & 1 \leq i \leq \frac{n}{2} \\
3i + 1 & \frac{n}{2} + 1 \leq i \leq n-1 \\
3i - 2 & n+1 \leq i \leq m+n-1 
\end{cases}$$

$$f^*(e_n) = \frac{3n+2}{2}$$

The above defined function $f$ provides one modulo three mean labeling of the graph $C_n \circ P_m$. Hence the graph $C_n \circ P_m$ is a one modulo three mean graph when $n \equiv 0 \pmod{4}$ and $n$ is even.
One modulo three mean labeling of the graph $C_8 @ P_{10}$ is shown in Figure 5.20.

![Figure 5.20: -OMTML of $C_8 @ P_{10}$](image)

One modulo three mean labeling of the graph $C_{12} @ P_8$ is shown in Figure 5.21.

![Figure 5.21: -OMTML of $C_{12} @ P_8$](image)

**Theorem 5.2.18**

The graph dragon $C_n @ P_m$ is not a one modulo three mean graph for $n \equiv 0 \pmod{4}$ and $m$ is odd.

**Proof**

Suppose the graph $C_n @ P_m$ is a one modulo three mean graph. Then the number of edges $q = m + n - 1$ is even, a contradiction to Corollary 3.3.5.
Theorem 5.2.19

The graph dragon $C_n @ P_m$ is a one modulo three mean graph for $n \equiv 1 \pmod{4}$ and $m$ is odd.

Proof

Let $\{v_i, 1 \leq i \leq m + n - 1\}$ be the vertices and $\{e_i, 1 \leq i \leq m + n - 1\}$ be the edges which are denoted as in Figure 5.19.

First we label the vertices as follows:

Define $f: V \to \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\}$ by

$$f(v_{2i-1}) = \begin{cases} 6(i-1) & 1 \leq i \leq \frac{n-1}{4} \\ 6(i-1)+1 & \frac{n-1}{4} + 1 \leq i \leq \frac{m+n}{2} \end{cases}$$

$$f(v_{2i}) = \begin{cases} 6(i-1)+1 & 1 \leq i \leq \frac{n-1}{4} \\ 6i & \frac{n-1}{4} + 1 \leq i \leq \frac{m+n-1}{2} \end{cases}$$

Then the induced edge labels are:

$$f^*(e_i) = \begin{cases} 3i - 2 & 1 \leq i \leq \frac{n-1}{2} \\ 3i + 1 & \frac{n-1}{2} + 1 \leq i \leq n-1 \\ 3i - 2 & n+1 \leq i \leq m + n - 1 \end{cases}$$

$$f^*(e_n) = \frac{3n-1}{2}$$

The above defined function $f$ provides one modulo three mean labeling of the graph $C_n @ P_m$. Hence the graph $C_n @ P_m$ is a one modulo three mean graph when $n \equiv 1 \pmod{4}$ and $m$ is odd.
One modulo three mean labeling of the graph of $C_5@P_7$ is shown in Figure 5.22.

![Figure 5.22: OMTML of $C_5@P_7$](image)

One modulo three mean labeling of the graph of $C_9@P_{11}$ is shown in Figure 5.23.

![Figure 5.23: OMTML of $C_9@P_{11}$](image)

**Theorem 5.2.20**

The graph dragon $C_n@P_m$ is not a one modulo three mean graph for $n \equiv 1 \pmod{4}$ and $m$ is even.

**Proof**

Suppose the graph $C_n@P_m$ is a one modulo three mean graph. Then the number of edges $q = m + n - 1$ is even, a contradiction to Corollary 3.3.5.
Theorem 5.2.21

The graph $C_n \circ P_m$ is a one modulo three mean graph when $n \equiv 1 \pmod{4}$ and $m$ is odd.

Proof

Let \( \{u_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} \) be the vertices and \( \{e_{ij}, 1 \leq i \leq m - 1, 1 \leq j \leq n, e_j, 1 \leq j \leq n\} \) be the edges which are denoted as in Figure 5.24.

![Graph](image_url)

**Figure 5.24: Ordinary Labeling of $C_n \circ P_m$**

First we label the vertices as follows:

Define $f: V \to \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\}$ by
Case 1: both $i$ and $j$ are even (or) odd

$$f(v_{ij}) = \begin{cases} 
3m(j-1) + 3i - 3 & 1 \leq i \leq m, 1 \leq j \leq \frac{n-1}{2} \\
3m(j-1) + 3i - 3 & 1 \leq i \leq m-1, j = \frac{n+1}{2} \\
3m(j-1) + 3i - 2 & 1 \leq i \leq m, \frac{n+3}{2} \leq j \leq n
\end{cases}$$

Case 2: $i$ even, $j$ odd (or) $i$ odd, $j$ even

$$f(v_{ij}) = \begin{cases} 
3m(j-1) + 3i - 5 & 1 \leq i \leq m, 1 \leq j \leq \frac{n-1}{2} \\
3m(j-1) + 3i - 5 & 1 \leq i \leq m-1, j = \frac{n+1}{2} \\
3m(j-1) + 3i & 1 \leq i \leq m, \frac{n+3}{2} \leq j \leq n
\end{cases}$$

$$v_{m(n+1)} = \frac{3m(n+1) - 4}{2}$$

Then the induced edge labels are:

$$f^*(e_i) = \begin{cases} 
3m(j-1) + 3i - 2 & 1 \leq i \leq m-1, 1 \leq j \leq \frac{n+1}{2} \\
3m(j-1) + 3i + 1 & 1 \leq i \leq m, \frac{n+3}{2} \leq j \leq n
\end{cases}$$

$$f^*(e_j) = \begin{cases} 
3mi - 2 & 1 \leq j \leq \frac{n-1}{2} \\
3mi + 1 & \frac{n+1}{2} \leq j \leq n-1
\end{cases}$$

$$f^*(e_n) = \frac{3m(n+1) - 4}{2}$$

The above defined function $f$ provides one modulo three mean labeling the graph $C_n \odot P_m$ when $n \equiv 1 \pmod{4}$ and $m$ is odd.
One modulo three mean labeling of the graph $C_{13} \bigcirc P_3$ is shown in Figure 5.25.

![Figure 5.25: OMTML of $C_{13} \bigcirc P_3$](image)

One modulo three mean labeling of the graph $C_9 \bigcirc P_5$ is shown in Figure 5.26.

![Figure 5.26: OMTML of $C_9 \bigcirc P_5$](image)
Theorem 5.2.22

The graph $C_n \odot P_m$ is not a one modulo three mean graph when $n \equiv 1 \pmod{4}$ and $m$ is even.

Proof

Suppose the graph $C_n \odot P_m$ is a one modulo three mean graph when $n \equiv 1 \pmod{4}$ and $m$ is even. Then the number of edges $q = mn$ is even, a contradiction to Corollary 3.3.5.

Theorem 5.2.23

The graph $C_n \odot P_m$ is not a one modulo three mean graph when $n \equiv 0, 2 \pmod{4}$ and for all $m$.

Proof

Suppose the graph $C_n \odot P_m$ is a one modulo three mean graph when $n \equiv 0, 2 \pmod{4}$ and for all $m$. Then the number of edges $q = mn$ is even, a contradiction to Corollary 3.3.5.

Definition 5.2.24

Let $G$ be a graph and $v$ be any vertex of $G$. A new vertex $v'$ is said to be duplication of $v$ if all the vertices which are adjacent to $v$ are adjacent to $v'$. The graph obtained by duplication $v$ is denoted by $VD(G)$. 
Theorem 5.2.25

The graph $VD(P_n) (n \geq 4)$ is a one modulo mean graph if and only if $n$ is even.

Proof

Let $P_n$ denoted a path on $n$ vertices $\{v_i, 1 \leq i \leq n, v_2\}$ be the vertices and $\{e_i, 1 \leq i \leq n - 1, e_1, e_2\}$ be the edges of $VD(P_n)$ obtained by the duplication of the vertex $v_2$ which are denoted as in Figure 5.27.

First we label the vertices as follows:

\[
    f(v_1) = 0; \quad f(v_2) = 1; \quad f(v'_2) = 7
\]

\[
    f(v_i) = \begin{cases} 
        3i + 3 & 3 \leq i \leq n, \ i \ \text{odd} \\
        3i + 1 & 4 \leq i \leq n, \ i \ \text{even} 
    \end{cases}
\]

Then the induced edge labels are:

\[
    f^*(e_1) = 1; \quad f^*(e_2) = 7
\]

\[
    f^*(e'_1) = 4; \quad f^*(e'_2) = 10
\]

\[
    f^*(e_i) = 3i + 4 \quad 3 \leq i \leq n - 1
\]

The above defined function $f$ provides one modulo three mean labeling of the graph $VD(P_n)$. Hence the graph $VD(P_n)$ is a one modulo three mean graph when $n$ is even.
Conversely, let us assume that $VD(P_n)$ is a one modulo three mean graph. Suppose $n$ is odd, then the number of edges $q = n + 1$ is even, a contradiction to Corollary 3.3.5. Hence $n$ is even.

One modulo three mean labeling of the graph $VD(P_6)$ is shown in Figure 5.28.

![Figure 5.28: OMTML of $VD(P_6)$](image)

One modulo three mean labeling of the graph $VD(P_{10})$ is shown in Figure 5.29.

![Figure 5.29: OMTML of $VD(P_{10})$](image)

**Definition 5.2.26**

Let $G$ be a graph and $e$ be any edge. A new edge $e'$ is said to be duplication of an edge $e$ if all the edges which are incident to $e$ in $G$ are incident to $e'$. The graph obtained by duplicating $e$ is denoted by $ED(G)$.

**Theorem 5.2.27**

The graph $ED(P_n)$ ($n \geq 4$) is a one modulo three mean graph for $n \equiv 1$(mod 4).

**Proof**

Let $P_n$ denote a path on $n$ vertices. Let \{\(v_i, 1 \leq i \leq n, \ v'_1, \ v'_2\} \) be the vertices and \{\(e_i, 1 \leq i \leq n - 1, \ e'_1, \ e'_2, \ e'_3\} \) be the edges of $ED(P_n)$ which are denoted as in Figure 5.30.
First we label the vertices as follows:

Define \( f: V \rightarrow \{0, 1, 3, 4, ..., 3q - 3, 3q - 2\} \) by

\[
\begin{align*}
  f(v_1) &= 6; & f(v_2) &= 7 \\
  f(v_{4i+1}) &= 12i & 1 \leq i \leq \frac{n-1}{4} \\
  f(v_{4i-1}) &= 12i + 6 & 1 \leq i \leq \frac{n-1}{4} \\
  f(v_{4i+1}) &= 12i + 13 & 1 \leq i \leq \frac{n-1}{4} - 1 \\
  f(v_{4i+2}) &= 12i + 19 & 1 \leq i \leq \frac{n-1}{4} - 1
\end{align*}
\]

For \( i = 3 \), \( f(v_{i+1}) = 6i + 1; \quad f(v_i^{'}) = 1; \quad f(v_{i}^{'}) = 0 \)

Then the induced edge labels are:

\[
\begin{align*}
  f^*(e_i) &= \begin{cases} 
    6i+1 & 1 \leq i \leq 3 \\
    3i+7 & 5 \leq i \leq n-1, i \text{ odd}
  \end{cases} \\
  f^*(e_{4i+1}) &= 12i + 13 & 1 \leq i \leq \frac{n-1}{4} - 1 \\
  f^*(e_{4i+2}) &= 12i + 19 & 1 \leq i \leq \frac{n-1}{4} - 1
\end{align*}
\]
The above defined function $f$ provides one modulo three mean labeling of graph $ED(P_n)$. Hence the graph $ED(P_n)$ is a one modulo three mean graph when $n \equiv 1 \pmod{4}$.

One modulo three mean labeling of the graph $ED(P_5)$ is shown in Figure 5.31.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.31.png}
\caption{OMTML of $ED(P_5)$}
\end{figure}

One modulo three mean labeling of the graph $ED(P_9)$ is shown in Figure 5.32.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.32.png}
\caption{OMTML of $ED(P_9)$}
\end{figure}

**Theorem 5.2.28**

The graph $ED(P_n)$ ($n \geq 4$) is not a one modulo three mean graph for $n \equiv 0, 2 \pmod{4}$.

**Proof**

Suppose the graph $ED(P_n)$ is a one modulo three mean graph. Then the number of edges $q = m + n - 1$ is even, a contradiction to Corollary 3.3.5.
5.3. ONE MODULO THREE MEAN NUMBER

Definition 5.3.1

Let \( f : V(G) \to \{0, 1, 3, 4, ..., n\} \) be a function such that the label of any edge \( uv \) induced by 
\[
\left\lceil \frac{f(u) + f(v)}{2} \right\rceil
\]
and \( \{ f^*(e), e = uv \in E(G) \} \subseteq \{0, 1, 3, 4, ..., n\} \).

If \( n \) is the smallest positive integer satisfying the above condition and the label of \( E(G) \) are distinct then \( n \) is called the one modulo three mean number and it is denoted by OMTM(G) where \( n \equiv 0 \) (or) 1 (mod 3).

Theorem 5.3.2

\[ \text{OMTM} (C_3 \hat{\odot} K_{1,n}) = 6(n + 1), \text{ for } n \geq 1. \]

Proof

From Theorem 3.5.11, we know that \( C_3 \hat{\odot} K_{1,n}, n \geq 1 \) is not a one modulo three mean graph. We now find its one modulo three mean number.

Let \( \{ u, u_i, 1 \leq i \leq n, v_1, v_2 \} \) be the vertices and \( \{ a, b, c, e_i, 1 \leq i \leq n \} \) be the edges which are denoted as in Figure 5.33.

\[ \text{Figure 5.33: Ordinary labeling of } C_3 \hat{\odot} K_{1,n} \]

First we label the vertices as follows:
\[ f(u) = 1; \quad f(u_i) = 0 \]
\[ f(u_i) = 6i \quad 2 \leq i \leq n \]

\[ f(v_i) = 7; \quad f(v_2) = 6(n + 1) \]

Then the induced edge labels are:

\[ f^*(e_i) = 3i + 1 \quad 2 \leq i \leq n \]

\[ f^*(e_1) = 1; \quad f^*(a) = 3n + 7 \]

\[ f^*(b) = 4; \quad f^*(c) = 3(n + 1) + 1 \]

In this case, the maximum vertex label used is \(6(n + 1)\). Therefore, one modulo three mean number of \(C_3 \circ K_{1,n} = 6(n + 1)\), for \(n \geq 1\).

One modulo three mean number of the graph \(C_3 \circ K_{1,4}\) is shown in Figure 5.34.

![Figure 5.34: OMTM(\(C_3 \circ K_{1,4}\))](image)

**Theorem 5.3.3**

\[
\text{OMTM}\ (C_4 \circ K_{1,n}) = 6(n + 1), \text{ for } n \geq 2. 
\]

**Proof**

From Theorem 3.5.12, we know that \(C_4 \circ K_{1,n}, \ n \geq 2\) is not a one modulo three mean graph. We now find its mean number.
Let \{u, u_i, 1 \leq i \leq n, v_i, 1 \leq i \leq 3\} be the vertices and \{a, b, c, d, e_i, 1 \leq i \leq n\} be the edges which are denoted as in Figure 5.35.

![Figure 5.35: Ordinary labeling of $C_4 \circ K_{1,n}$](image)

First we label the vertices as follows:

\[
f(u) = 13; \quad f(u_i) = 6i + 6 \quad 1 \leq i \leq n
\]

\[
f(v_i) = 0; \quad f(v_i + 1) = 5i - 4 \quad i = 1, 2
\]

Then the induced edge labels are:

\[
f^*(e_i) = 3i + 10 \quad 1 \leq i \leq n
\]

\[
f^*(a) = 1; \quad f^*(b) = 4
\]

\[
f^*(c) = 7; \quad f^*(d) = 10
\]

In this case, the maximum vertex label used is $6(n + 1)$. Therefore, one modulo three mean number of $C_4 \circ K_{1,n} = 6(n + 1)$, for $n \geq 2$.

One modulo three mean number of the graph $C_4 \circ K_{1,8}$ is shown in Figure 5.36.
Theorem 5.3.4

\[ \text{OMTM}(K_{1,n}) = 6n - 5, \text{ for } n \geq 2. \]

Proof

From Theorem 3.4.16, we know that \( K_{1,n}, n \geq 2 \) is not a one modulo three mean graph. We now find its mean number.

Let \( \{u, u_i, 1 \leq i \leq n\} \) be the vertices and \( \{e_i, 1 \leq i \leq n\} \) be the edges which are denoted as in Figure 5.37.

First we label the vertices as follows:

\[ f(u) = 6n - 5 \]

\[ f(u_i) = 6(i-1) \quad 1 \leq i \leq n \]
Then the induced edge labels are:

\[ f^*(e_i) = 3i + 3n - 5 \quad 1 \leq i \leq n \]

In this case, the maximum vertex label used is \( 6n - 5 \). Therefore, one modulo three mean number of \( K_{1,n} = 6n - 5 \), for \( n \geq 2 \).

One modulo three mean number of the graph \( K_{1,6} \) is shown in Figure 5.38.

**Theorem 5.3.5**

\[ \text{OMTM}(S(B_{n,n})) = 6n + 7. \]

**Proof**

From Theorem 4.2.8, we know that \( S(B_{n,n}) \) is not a one modulo three mean graph. We now find its mean number.

Let \( \{u, v, w, u_i, 1 \leq i \leq n, v_i, 1 \leq i \leq n\} \) be the vertices and \( \{a, b, e_i, g_i, 1 \leq i \leq n\} \) be the edges which are denoted as in Figure 5.39.

---

**Figure 5.38: OMTM\((K_{1,6})\)**

**Figure 5.39: Ordinary labeling of \( S(B_{n,n}) \)**
First we label the vertices as follows:

For $1 \leq i \leq n$,

$$f(v_i) = 6i + 1; \quad f(u) = 1; \quad f(u_i) = 6(i - 1)$$

$$f(v) = 6(n + 1) + 1; \quad f(w) = 6(n + 1)$$

Then the induced edge labels are:

$$f^*(a) = 3(n + 1) + 1; \quad f^*(b) = 6(n + 1) + 1$$

For $1 \leq i \leq n$,

$$f^*(e_i) = 3i - 2$$

$$f^*(g_i) = 3(i - 1) + 3(n + 2) + 1$$

In this case, the maximum vertex label used is $6n - 5$. Therefore, one modulo three mean number of $S(B_{n,n}) = 6n + 7$.

One modulo three mean number of the graph $S(B_{7,7})$ is shown in Figure 5.40.

---

**Figure 5.40: OMTM(S(B_{7,7}))**
Theorem 5.3.6

$$\text{OMTM} \left( S(B_{m,n}) \right) = 6m + 1, \text{ for } m \geq n + 2.$$ 

Proof

From Theorem 4.2.9, we know that $S(B_{m,n})$, $m \geq n + 2$ is not a one modulo three mean graph. We now find its one modulo three mean number.

Let $\{u, v, w, u_i, 1 \leq i \leq m, v_i, 1 \leq i \leq n\}$ be the vertices and $\{a, b, e_i, 1 \leq i \leq m, g_i, 1 \leq i \leq n\}$ be the edges which are denoted as in Figure 5.41.

First we label the vertices as follows:

$$f(u) = 1; \quad f(v) = 6m + 1; \quad f(w) = 6m$$

$$f(u_i) = 6(i - 1) \quad 1 \leq i \leq m$$

$$f(v_i) = 6i + 1 \quad 1 \leq i \leq n$$

Then the induced edge labels are:

$$f^* (a) = 3m + 1; \quad f^* (b) = 6m + 1$$

$$f^* (e_i) = 3i - 2 \quad 1 \leq i \leq m$$

$$f^* (g_i) = 3(i - 1) + 3(n + 1) + 1 \quad 1 \leq i \leq n$$
In this case, the maximum vertex label used is $6m + 1$. Therefore, one modulo three mean number of $S(B_{m,n}) = 6m + 1$, for $m \geq n + 2$.

One modulo three mean number of the graph $S(B_{8,5})$ is shown in Figure 5.42.