Chapter 3

Some Arithmetic Operations on Some Special Types of Hexagonal Fuzzy Matrices

Abstract

In this chapter, the definition of HFM has been used to propose ACHFM. The definition of a trace of ACHFM is established and its properties are verified. The special types of notions called Determinant, Permanent and Inverse with the aid of Hexagonal Fuzzy Matrices are highlighted. Some of the relevant properties are derived from the same notions.

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(4) Fuzzy Information and Engineering, (Communicated).
3.1 Introduction

The $\alpha$-cut of fuzzy number has represented by the membership function which is in the different forms in the capability to decompose fuzzy sets into a collection of crisp sets. This decomposition along with the extension principle (EP) form a methodology for extending mathematical concepts from crisp sets to fuzzy sets. Yager[117] generalized the $\alpha$-cuts to interval-valued fuzzy sets and provided a representation and a way of applying the EP to these sets. Zeng.et.al[124] discussed different mathematical formulations of the EP for the interval valued fuzzy set, they also defined an $\alpha$-cut decomposition for such set. Recently Kalaichelvi and Janofer[50, 51] discussed the $\alpha$-cut with the theory of matrices in the triangular fuzzy number using some new operations. Dinagar and Latha[96] established the trace of type-2 triangular fuzzy number matrices.

Fuzzy matrices play an important in science and technology. It plays an important role fuzzy set theory. It is well known that the matrix formulation of a mathematical formula gives an extra advantage to study the problem. When some problems are not solved classical matrices, then the concept of fuzzy matrices are used. The properties of the determinant of a square fuzzy matrix are analogous to the properties of the determinant of the determinant of matrices in general. It is important to mention here that the properties are studied taking into consideration of the complementation of fuzzy matrices in our way because we are not in a position
to accept the existing definition of complementation of fuzzy sets due to lack of logical backgrounds. If it is so the same should be followed in case of fuzzy matrices also because it has already been mentioned that matrix has been proposed to represent fuzzy set theory. The new definition determinant of the square fuzzy matrix by [19, 20]. Dinagar and Latha [101] discussed the determinant of type-2 triangular fuzzy number matrices. Jaisankar and Mani [47] presented the determinant of trapezoidal fuzzy number matrices. Permanent matrix theory is very similar to determinant only the sign is associated.

The system of linear equations is important for solving a large proposition of the problems in many topics in applied mathematics. The system of simultaneous linear equations plays a major role in various area such as operation research, physics, statistics, engineering, social science and theory of matrices. Usually, in many applications, some of the parameters in the problems are represented by fuzzy number rather than the crisp number and hence it is important to develop mathematical models and numerical procedures that would appropriate treat general fuzzy linear systems and solve them. When the inverse of a singular and non-singular matrix whose entries are real numbers is concerned, one of the computing methods is that one writes it down in the form of linear equation system composed of the product of $n \times n$ matrices where the former denotes the coefficients and latter represents the unknown. To find the solution, the
original $n \times n$ fuzzy linear system is replaced with the $2n \times 2n$ crisp linear system. They are proposed to find the solution of $2n \times 2n$ crisp linear systems. The exact solution of the fuzzy linear system can be found by solving this interval system\cite{8} proposed fuzzy linear systems. Whose all parameters are the triangular fuzzy number. Abbasbandy.et.al\cite{1} proposed LU decomposition method for solving a fuzzy system of linear equation. Murganandam and Abdul Raza\cite{73} proposed matrix inversion method for solving fully fuzzy linear systems with triangular fuzzy numbers. Basaran\cite{12} proposed calculating fuzzy inverse matrix using fuzzy linear equation system. Dinagar and Latha\cite{98} proposed invertible on Type-2 triangular fuzzy matrices. Dehghan.et.al\cite{22} presented inverse of a fuzzy matrix of fuzzy numbers. Khan and Anita Pal\cite{54}, proposed the generalized inverse of intuitionistic fuzzy matrices, In this chapter, we discuss various special types of matrices using the Hexagonal fuzzy number. Also, some of the properties are verified.

3.2 Hexagonal Fuzzy Matrices (HFM)\textbf{S}\textbf{S}

\textbf{Definition 3.2.1:} A hexagonal fuzzy matrix of order $m \times n$ is defined as $\hat{A} = (\hat{a}_{hij})_{m \times n}$ where $(\hat{a}_{hij}) = (\hat{a}_{ij1}, \hat{a}_{ij2}, \hat{a}_{ij3}, \hat{a}_{ij4}, \hat{a}_{ij5}, \hat{a}_{ij6})$ is $i \, j^{th}$ element of $\hat{A}$. 
3.2.1 Operations on Hexagonal Fuzzy Matrices (HFM)

Let \( \hat{A} = (\tilde{a}_{hij})_{m \times n} \) and \( \hat{B} = (\tilde{b}_{hij})_{m \times n} \) be two HFMs of same order. Then we have the following:

1. \( \hat{A} + \hat{B} = (\tilde{a}_{hij})_{m \times n} = \left( (\tilde{a}_{hij}) + (\tilde{b}_{hij}) \right) \)

2. \( \hat{A} - \hat{B} = (\tilde{a}_{hij})_{m \times n} = \left( (\tilde{a}_{hij}) - (\tilde{b}_{hij}) \right) \)

3. For \( \hat{A} = (\tilde{a}_{hij})_{m \times n} \) and \( \hat{B} = (\tilde{b}_{hij})_{n \times k} \) then \( \hat{A}\hat{B} = (\tilde{c}_{hij})_{m \times k} \) where

   \[
   (\tilde{c}_{hij})_{m \times k} = \sum_{p=1}^{n} (\tilde{a}_{hiq})(\tilde{b}_{qjp}), \text{ i=1,2,...,m and j=1,2,...,k}
   \]

4. \( \hat{A}^T \) or \( \hat{A}' = (\tilde{a}_{hji}) \)

5. \( k\hat{A} = (k(\tilde{a}_{hij})), \text{ where k is scalar.} \)

**Definition 3.2.2: Square HFM**

An HFM \( \hat{A} = (\tilde{a}_{hij}) \) of order \( m \times n \) is said to be a square HFM if the number of rows is equal to the number of columns, (i.e.), \( m=n \). Otherwise, it is called non-square HFM.

**Definition 3.2.3: Symmetric HFM**

A square HFM \( \hat{A} = (\tilde{a}_{hij}) \) is said to be symmetric HFM about the principal diagonal if \( \tilde{a}_{hij} = \tilde{a}_{hji} \) for all \( i, j = 1, 2, \ldots n \). (i.e) \( \hat{A}^T = \hat{A} \).

**Definition 3.2.4: Diagonal HFM**

A square HFM \( \hat{A} = (\tilde{a}_{hij}) \) is said to be a diagonal HFM if all the elements outside the principle diagonal are 0.
Definition 3.2.5: Diagonal-Equivalent HFM

A square HFM $\hat{A} = (\tilde{a}_{hij})$ is said to be a diagonal-equivalent HFM if all the elements outside the principle diagonal are $\tilde{0}$.

Definition 3.2.6: Scalar HFM

A diagonal HFM $\hat{A} = (\tilde{a}_{hij})$ is said to be a scalar HFM if every entry $\tilde{a}_{hii}$ in the principal diagonal is same.

Definition 3.2.7: Scalar-equivalent HFM

A diagonal HFM $\hat{A} = (\tilde{a}_{hij})$ is said to be a scalar-equivalent HFM if the value of $\tilde{R}(\tilde{a}_{hii})$ is the same for every entry $\tilde{a}_{hii}$ in the principal diagonal.

Definition 3.2.8: Unit or Identity HFM

A scalar HFM $\hat{A} = (\tilde{a}_{hij})$ is said to be a Unit or Identity HFM if $\tilde{a}_{hij} = 1$ for every entry $\tilde{a}_{hii}$ in the principal diagonal. It is denoted by $I$.

Definition 3.2.9: Unit-equivalent or Identity-equivalent HFM

A scalar-equivalent HFM $\hat{A} = (\tilde{a}_{hij})$ is said to be an Unit-equivalent or Identity-equivalent HFM if $\tilde{a}_{hij} = \tilde{1}$ for every entry $\tilde{a}_{hii}$ in the principal diagonal. It is denoted by $\hat{I}$.

Definition 3.2.10: Zero or Null HFM

Let $m \times n$ HFM with which each entry 0 is called the zero or Null HFM. It is denoted by $O$. 
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Definition 3.2.11: Zero-equivalent or Null-equivalent HFM

Let m\times n HFM with which each entry 0 is called the zero-equivalent or Null-equivalent HFM. It is denoted by \( \hat{\mathbf{O}} \).

3.3 Alpha cut Of Hexagonal Fuzzy Number Matrices (ACHFMs)

In this section we introduce the notion of Hexagonal Fuzzy Number Matrices using \( \alpha \)–cut method and its arithmetic operations.

Definition 3.3.1: \( \alpha \)–cut of Hexagonal Fuzzy Number Matrix

An \( \alpha \)–cut of hexagonal fuzzy matrix of order m\times n is defined as \( \hat{A}_\alpha = \left[ a_{ij}^L, a_{ij}^U \right]_{m \times n} \) where,
\[
a_{ij}^L = 2\alpha (a_{ij2} - a_{ij1}) + a_{ij1}, -2\alpha (a_{ij6} - a_{ij5}) + a_{ij6} \quad \text{for} \quad \alpha \in [0, 0.5)
\]
\[
a_{ij}^U = 2\alpha (a_{ij3} - a_{ij2}) - a_{ij3} + 2a_{ij2}, -2\alpha (a_{ij5} - a_{ij4}) + 2a_{ij5} - a_{ij4} \quad \text{for} \quad \alpha \in [0.5, 1]
\]
is \( ij^{th} \) element of \( \hat{A}_\alpha \).

3.3.1 Operations on \( \alpha \)–Cut of Hexagonal Fuzzy Matrices

1. \( \hat{A}_\alpha + \hat{B}_\alpha = \left[ a_{ij}^L + b_{ij}^L, a_{ij}^U + b_{ij}^U \right] \)
2. \( \hat{A}_\alpha - \hat{B}_\alpha = \left[ a_{ij}^L - b_{ij}^L, a_{ij}^U - b_{ij}^U \right] \)
3. For \( \hat{A}_\alpha = \left[ a_{ij}^L, a_{ij}^U \right]_{m \times n} \) and \( \hat{B}_\alpha = \left[ b_{ij}^L, b_{ij}^U \right]_{n \times k} \) then \( \hat{A}_\alpha \hat{B}_\alpha = \hat{C}_\alpha = \left[ c_{ij}^L, c_{ij}^U \right]_{m \times k} \), where \( c_{ij}^L = \sum_{p=1}^{n} \left[ a_{ip}^L, a_{ip}^U \right] \left[ b_{pj}^L, b_{pj}^U \right], i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, k \).
4. $\hat{A}_\alpha \lor \hat{B}_\alpha = \left[ a^L_{ij}, a^U_{ij} \right] \lor \left[ b^L_{ij}, b^U_{ij} \right] = \left[ \max(a^L_{ij}, b^L_{ij}), \max(a^U_{ij}, b^U_{ij}) \right]$. 

5. $\hat{A}_\alpha \land \hat{B}_\alpha = \left[ a^L_{ij}, a^U_{ij} \right] \land \left[ b^L_{ij}, b^U_{ij} \right] = \left[ \min(a^L_{ij}, b^L_{ij}), \min(a^U_{ij}, b^U_{ij}) \right]$. 

6. $\hat{A}^T_\alpha$ (or) $\hat{A}'_\alpha = \left[ a^L_{ij}, a^U_{ij} \right]$. 

7. $k\hat{A}_\alpha = \left[ ka^L_{ij}, ka^U_{ij} \right]$. 

### 3.3.2 Numerical Illustration

**Example 3.3.2:** Let us examine the addition of $\alpha$–cut of two hexagonal fuzzy matrices

\[
\hat{A} = \begin{bmatrix}
(1, 2, 3, 5, 6, 7) & (-1, 0, 1, 3, 4, 5) \\
(1, 3, 5, 7, 9, 11) & (-1, 0, 1, 5, 6, 7)
\end{bmatrix}
\]

and

\[
\hat{B} = \begin{bmatrix}
(2, 4, 6, 8, 10, 12) & (-1, 0, 1, 5, 6, 7) \\
(-1, 0, 1, 3, 4, 5) & (1, 3, 5, 7, 9, 11)
\end{bmatrix}
\]

Now to find the alpha cut of HFMs are

\[
\hat{A}_\alpha = \begin{bmatrix}
a^L_{11}, a^U_{11} \\
a^L_{12}, a^U_{12} \\
a^L_{21}, a^U_{21} \\
a^L_{22}, a^U_{22}
\end{bmatrix}
\]

Where

\[
\begin{cases}
2\alpha (a_{ij2} - a_{ij1}) + a_{ij1}, -2\alpha (a_{ij6} - a_{ij5}) + a_{ij6} & \text{for } \alpha \in [0, 0.5] \\
2\alpha (a_{ij3} - a_{ij2}) - a_{ij3} + 2a_{ij2}, -2\alpha (a_{ij5} - a_{ij4}) + 2a_{ij5} - a_{ij4} & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]

\[
\begin{bmatrix}
a^L_{ij}, a^U_{ij} \\
a^L_{i1}, a^U_{i1}
\end{bmatrix} = \begin{bmatrix}
[2\alpha + 1, -2\alpha + 7] & \text{for } \alpha \in [0, 0.5] \\
[2\alpha + 1, -2\alpha + 7] & \text{for } \alpha \in [0.5, 1]
\end{bmatrix}
\]
\[
\begin{align*}
\begin{bmatrix} a_{12}^L, a_{12}^U \end{bmatrix} &= \begin{bmatrix} 2\alpha - 1, -2\alpha + 5 \end{bmatrix} \text{ for } \alpha \in [0, 0.5) \\
&= \begin{bmatrix} 2\alpha - 1, -2\alpha + 5 \end{bmatrix} \text{ for } \alpha \in [0.5, 1] \\
\begin{bmatrix} a_{21}^L, a_{21}^U \end{bmatrix} &= \begin{bmatrix} 4\alpha + 1, -4\alpha + 11 \end{bmatrix} \text{ for } \alpha \in [0, 0.5) \\
&= \begin{bmatrix} 4\alpha + 1, -4\alpha + 11 \end{bmatrix} \text{ for } \alpha \in [0.5, 1] \\
\begin{bmatrix} a_{22}^L, a_{22}^U \end{bmatrix} &= \begin{bmatrix} 2\alpha - 1, -2\alpha + 7 \end{bmatrix} \text{ for } \alpha \in [0, 0.5) \\
&= \begin{bmatrix} 2\alpha - 1, -2\alpha + 7 \end{bmatrix} \text{ for } \alpha \in [0.5, 1]
\end{align*}
\]

Since the alpha cut of coefficient matrix is equal interval then,

\[\hat{A}_\alpha = \begin{bmatrix} 2\alpha + 1, -2\alpha + 7 \\ 4\alpha + 1, -4\alpha + 11 \end{bmatrix} \text{ and } \hat{B}_\alpha = \begin{bmatrix} 2\alpha - 1, -2\alpha + 7 \\ 4\alpha + 1, -4\alpha + 11 \end{bmatrix}\]

similarly we have,

\[\hat{A}_\alpha = \begin{bmatrix} 4\alpha + 2, -4\alpha + 12 \\ 2\alpha - 1, -2\alpha + 7 \end{bmatrix} \text{ and } \hat{B}_\alpha = \begin{bmatrix} 4\alpha + 2, -4\alpha + 12 \\ 2\alpha - 1, -2\alpha + 7 \end{bmatrix}\]

Now the addition of \(\alpha\)-cut of two hexagonal fuzzy matrix is

\[
\hat{A}_\alpha + \hat{B}_\alpha = \begin{bmatrix} 6\alpha + 3, -6\alpha + 19 \\ 6\alpha, -6\alpha + 16 \end{bmatrix} \\
\begin{bmatrix} 4\alpha - 2, -4\alpha + 12 \\ 6\alpha, -6\alpha + 18 \end{bmatrix}
\]

Put \(\alpha = 0 \Rightarrow \hat{A}_0 + \hat{B}_0 = \begin{bmatrix} 3, 19 \\ -2, 12 \end{bmatrix} \begin{bmatrix} 0, 16 \\ 0, 18 \end{bmatrix}\]

\(\alpha = 0.5 \Rightarrow \hat{A}_{0.5} + \hat{B}_{0.5} = \begin{bmatrix} 6, 16 \\ 0, 10 \end{bmatrix} \begin{bmatrix} 3, 13 \\ 3, 15 \end{bmatrix}\]

\(\alpha = 1 \Rightarrow \hat{A}_1 + \hat{B}_1 = \begin{bmatrix} 9, 13 \\ 2, 8 \end{bmatrix} \begin{bmatrix} 6, 10 \\ 6, 12 \end{bmatrix}\]
Hence the alphacut of addition of two hexagonal fuzzy matrix is

\[
\hat{A}_\alpha + \hat{B}_\alpha = \begin{bmatrix}
(3, 6, 9, 13, 16, 19) & (-2, 0, 2, 8, 10, 12) \\
(0, 3, 6, 10, 13, 16) & (0, 3, 6, 12, 15, 18)
\end{bmatrix}
\]

Example 3.3.3: Similarly we have to prove the difference operation of matrix in above example 3.3.2.

Example 3.3.4: Let us examine the multiplication of \(\alpha\)-cut of two hexagonal fuzzy matrices

\[
\hat{A} = \begin{bmatrix}
(1, 2, 3, 5, 6, 7) & (-1, 0, 1, 3, 4, 5) \\
(1, 3, 5, 7, 9, 11) & (-1, 0, 1, 5, 6, 7)
\end{bmatrix}
\text{ and }
\hat{B} = \begin{bmatrix}
(2, 4, 6, 8, 10, 12) & (-1, 0, 1, 5, 6, 7) \\
(-1, 0, 1, 3, 4, 5) & (1, 3, 5, 7, 9, 11)
\end{bmatrix}
\]

Now to find the alpha cut of HFMs are

\[
\hat{A}_\alpha = \begin{bmatrix}
[a^L_{11}, a^U_{11}] & [a^L_{12}, a^U_{12}] \\
[a^L_{21}, a^U_{21}] & [a^L_{22}, a^U_{22}]
\end{bmatrix}
\]

Where

\[
[a^L_{ij}, a^U_{ij}] = \begin{cases}
2\alpha(a_{ij2} - a_{ij1}) + a_{ij1}, -2\alpha(a_{ij6} - a_{ij5}) + a_{ij6} & \text{for } \alpha \in [0, 0.5) \\
2\alpha(a_{ij3} - a_{ij2}) - a_{ij3} + 2a_{ij2}, -2\alpha(a_{ij5} - a_{ij4}) + 2a_{ij5} - a_{ij4} & \text{for } \alpha \in [0.5, 1] \\
[2\alpha + 1, -2\alpha + 7] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha + 1, -2\alpha + 7] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]
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\[
\begin{bmatrix}
 a_{12}^L, a_{12}^U \\
 a_{21}^L, a_{21}^U \\
 a_{22}^L, a_{22}^U
\end{bmatrix} =
\begin{bmatrix}
 [2\alpha - 1, -2\alpha + 5] & \text{for } \alpha \in [0, 0.5) \\
 [2\alpha - 1, -2\alpha + 5] & \text{for } \alpha \in [0.5, 1]
\end{bmatrix}
\]

\[
\begin{bmatrix}
 a_{12}^L, a_{12}^U \\
 a_{21}^L, a_{21}^U \\
 a_{22}^L, a_{22}^U
\end{bmatrix} =
\begin{bmatrix}
 [4\alpha + 1, -4\alpha + 11] & \text{for } \alpha \in [0, 0.5) \\
 [4\alpha + 1, -4\alpha + 11] & \text{for } \alpha \in [0.5, 1]
\end{bmatrix}
\]

\[
\begin{bmatrix}
 a_{12}^L, a_{12}^U \\
 a_{21}^L, a_{21}^U \\
 a_{22}^L, a_{22}^U
\end{bmatrix} =
\begin{bmatrix}
 [2\alpha - 1, -2\alpha + 7] & \text{for } \alpha \in [0, 0.5) \\
 [2\alpha - 1, -2\alpha + 7] & \text{for } \alpha \in [0.5, 1]
\end{bmatrix}
\]

Since the alpha cut of coefficient matrix is equal interval then,

therefore

\[
\hat{A}_\alpha =
\begin{bmatrix}
 [2\alpha + 1, -2\alpha + 7] & [2\alpha - 1, -2\alpha + 5] \\
 [4\alpha + 1, -4\alpha + 11] & [2\alpha - 1, -2\alpha + 7]
\end{bmatrix}
\]

similarly we have,

\[
\hat{B}_\alpha =
\begin{bmatrix}
 [4\alpha + 2, -4\alpha + 12] & [2\alpha - 1, -2\alpha + 7] \\
 [2\alpha - 1, -2\alpha + 5] & [4\alpha + 1, -4\alpha + 11]
\end{bmatrix}
\]

Now the multiplication of \( \alpha \)-cut of two hexagonal fuzzy matrix is

\[
\hat{A}_\alpha \times \hat{B}_\alpha =
\begin{bmatrix}
 [18\alpha + 5, -18\alpha + 59] & [18\alpha - 3, -18\alpha + 51] \\
 [32\alpha + 5, -32\alpha + 91] & [24\alpha - 3, -24\alpha + 75]
\end{bmatrix}
\]

Put \( \alpha = 0 \Rightarrow \hat{A}_0 \times \hat{B}_0 =
\begin{bmatrix}
 [5, 91] & [-3, 75]
\end{bmatrix}
\]

\( \alpha = 0.5 \Rightarrow \hat{A}_{0.5} \times \hat{B}_{0.5} =
\begin{bmatrix}
 [14, 50] & [6, 42] \\
 [21, 75] & [9, 63]
\end{bmatrix}
\]

\( \alpha = 1 \Rightarrow \hat{A}_1 \times \hat{B}_1 =
\begin{bmatrix}
 [23, 41] & [15, 33] \\
 [37, 59] & [21, 51]
\end{bmatrix}
\]
Hence the alphacut of multiplication of two hexagonal fuzzy matrix is

\[\hat{A}_\alpha \times \hat{B}_\alpha = \begin{bmatrix}
(5, 14, 23, 41, 50, 59) & (-3, 6, 15, 33, 42, 51) \\
(5, 21, 37, 59, 75, 91) & (-3, 9, 21, 51, 63, 75)
\end{bmatrix}\]

**Example 3.3.5:** Let us examine the join operation of \(\alpha\)-cut of two hexagonal fuzzy matrices

\(\hat{A} = \begin{bmatrix}
(1, 2, 3, 5, 6, 7) & (-1, 0, 1, 3, 4, 5) \\
(1, 3, 5, 7, 9, 11) & (-1, 0, 1, 5, 6, 7)
\end{bmatrix}\) and

\(\hat{B} = \begin{bmatrix}
(2, 4, 6, 8, 10, 12) & (-1, 0, 1, 5, 6, 7) \\
(-1, 0, 1, 3, 4, 5) & (1, 3, 5, 7, 9, 11)
\end{bmatrix}\)

Now to find the alpha cut of HFMs are

\[\hat{A}_\alpha = \begin{bmatrix}
[a_{11}^L, a_{11}^U] & [a_{12}^L, a_{12}^U] \\
[a_{21}^L, a_{21}^U] & [a_{22}^L, a_{22}^U]
\end{bmatrix}\]

Where

\[\begin{bmatrix}
a_{ij}^L, a_{ij}^U
\end{bmatrix} = \begin{bmatrix}
2\alpha(a_{ij2} - a_{ij1}) + a_{ij1}, -2\alpha(a_{ij6} - a_{ij5}) + a_{ij6} \\
2\alpha(a_{ij3} - a_{ij2}) - a_{ij3} + 2a_{ij2}, -2\alpha(a_{ij5} - a_{ij4}) + 2a_{ij5} - a_{ij4}
\end{bmatrix}\]

for \(\alpha \in [0, 0.5]\)

\[\begin{bmatrix}
a_{11}^L, a_{11}^U
\end{bmatrix} = \begin{bmatrix}
[2\alpha + 1, -2\alpha + 7] & for \alpha \in [0, 0.5] \\
[2\alpha + 1, -2\alpha + 7] & for \alpha \in [0.5, 1]
\end{bmatrix}\]

\[\begin{bmatrix}
a_{12}^L, a_{12}^U
\end{bmatrix} = \begin{bmatrix}
[2\alpha - 1, -2\alpha + 5] & for \alpha \in [0, 0.5] \\
[2\alpha - 1, -2\alpha + 5] & for \alpha \in [0.5, 1]
\end{bmatrix}\]
\[
\begin{bmatrix}
\alpha_L^{21}, \alpha_U^{21}
\end{bmatrix} = \begin{bmatrix}
4\alpha + 1, -4\alpha + 11 \\
4\alpha + 1, -4\alpha + 11
\end{bmatrix} \text{ for } \alpha \in [0, 0.5)
\]

\[
\begin{bmatrix}
\alpha_L^{21}, \alpha_U^{21}
\end{bmatrix} = \begin{bmatrix}
2\alpha - 1, -2\alpha + 7 \\
2\alpha - 1, -2\alpha + 7
\end{bmatrix} \text{ for } \alpha \in [0, 0.5)
\]

Since the alpha cut of coefficient matrix is equal interval then, therefore

\[
\hat{A}_\alpha = \begin{bmatrix}
2\alpha + 1, -2\alpha + 7 \\
2\alpha + 1, -2\alpha + 7
\end{bmatrix}
\]

\[
B_\alpha = \begin{bmatrix}
4\alpha + 2, -4\alpha + 12 \\
4\alpha + 2, -4\alpha + 12
\end{bmatrix}
\]

Now the join operation of \(\alpha\)-cut of two hexagonal fuzzy matrix is

\[
\hat{A}_\alpha \lor \hat{B}_\alpha = \begin{bmatrix}
\max(2\alpha + 1, 4\alpha + 2), \max(-2\alpha + 7, -4\alpha + 12) \\
\max(2\alpha - 1, 2\alpha - 1), \max(-2\alpha + 5, -2\alpha + 7) \\
\max(4\alpha + 1, 2\alpha - 1), \max(-4\alpha + 11, -2\alpha + 5) \\
\max(2\alpha - 1, 4\alpha + 1), \max(-2\alpha + 7, -4\alpha + 11)
\end{bmatrix}
\]

Put \(\alpha = 0 \Rightarrow \hat{A}_0 \lor \hat{B}_0 = \begin{bmatrix}
2, 12 \\
1, 11
\end{bmatrix}
\]

\(\alpha = 0.5 \Rightarrow \hat{A}_{0.5} \lor \hat{B}_{0.5} = \begin{bmatrix}
4, 10 \\
3, 9
\end{bmatrix}
\]

\(\alpha = 1 \Rightarrow \hat{A}_1 \lor \hat{B}_1 = \begin{bmatrix}
6, 8 \\
5, 7
\end{bmatrix}
\]

Hence the alphacut of addition of two hexagonal fuzzy matrix is

\[
\hat{A}_\alpha \lor \hat{B}_\alpha = \begin{bmatrix}
2, 4, 6, 8, 10, 12 \\
1, 3, 5, 7, 9, 11
\end{bmatrix}
\]

**Example 3.3.6:** Similarly we have to prove the meet operation of matrix
3.4 α—Cut of Trace Of Hexagonal Fuzzy Matrix

Definition 3.4.1: α—cut of trace of HFM

The α—cut of trace of a square HFM \( \hat{A}_\alpha = [a^L_{ij}, a^U_{ij}] \) of order \( n \) is the sum of the principle diagonal elements and is denoted by \( \text{tr}(\hat{A}_\alpha) \). (i.e),
\[
\text{tr}(\hat{A}_\alpha) = \sum_{i=1}^{n}[a^L_{ii}, a^U_{ii}].
\]

3.5 Properties of α—Cut of Trace of HFM

Property 3.5.1. If \( \hat{A}_\alpha = [a^L_{ij}, a^U_{ij}] \) and \( \hat{B}_\alpha = [b^L_{ij}, b^U_{ij}] \) are any two square ACHFMs of order \( n \) then \( \text{tr}(\hat{A}_\alpha) + \text{tr}(\hat{B}_\alpha) = \text{tr}(\hat{A}_\alpha + \hat{B}_\alpha) \).

Proof. Let \( \hat{A}_\alpha = [a^L_{ij}, a^U_{ij}] \) and \( \hat{B}_\alpha = [b^L_{ij}, b^U_{ij}] \) are any two square ACHFMs of order \( n \) where,
\[
[a^L_{ij}, a^U_{ij}] = \begin{cases} 
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_6 - a_5) + a_6] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(a_3 - a_2) - a_3 + 2a_2, -2\alpha(a_5 - a_4) + 2a_5 - a_4] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]
and
\[
[b^L_{ij}, b^U_{ij}] = \begin{cases} 
[2\alpha(b_2 - b_1) + b_1, -2\alpha(b_6 - b_5) + b_6] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(b_3 - b_2) - b_3 + 2b_2, -2\alpha(b_5 - b_4) + 2b_5 - b_4] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]
Let \( \hat{A}_\alpha + \hat{B}_\alpha = \hat{C}_\alpha = [c^L_{ij}, c^U_{ij}] = [a^L_{ij}, a^U_{ij}] + [b^L_{ij}, b^U_{ij}] = [a^L_{ij} + b^L_{ij}, a^U_{ij} + b^U_{ij}] \)
\[
[c^L_{ij}, c^U_{ij}] =
\]
\[
\left\{
\begin{array}{l}
[2\alpha(c_2 - c_1) + c_1, -2\alpha(c_6 - c_5) + c_6] \quad \text{for } \alpha \in [0, 0.5) \\
[2\alpha(c_3 - c_2) - c_3 + c_2, -2\alpha(c_5 - c_4) + 2c_5 - c_4] \quad \text{for } \alpha \in [0.5, 1].
\end{array}
\right.
\]

\[
[c_{ij}^L, c_{ij}^U] = \begin{cases}
2\alpha((a_{ij2} + b_{ij2}) - (a_{ij1} + b_{ij1})) + (a_{ij1} + b_{ij1}), \\
-2\alpha((a_{ij6} + b_{ij6}) - (a_{ij5} + b_{ij5})) + (a_{ij6} + b_{ij6})
\end{cases}
\]

for \(\alpha \in [0, 0.5)\)

\[
\begin{cases}
2\alpha((a_{ij3} + b_{ij3}) - (a_{ij2} + b_{ij2})) - (a_{ij3} + b_{ij3}) + 2(a_{ij2} + b_{ij2}), \\
-2\alpha((a_{ij5} + b_{ij5}) - (a_{ij4} + b_{ij4})) + 2(a_{ij5} + b_{ij5}) - (a_{ij4} + b_{ij4})
\end{cases}
\]

for \(\alpha \in [0.5, 1].\)

Now \(\text{tr}(\hat{A}_\alpha) = \sum_{i=1}^{n}[a_{ii}^L, a_{ii}^U]\)

\[
\begin{cases}
2\alpha((a_{ii2} + b_{ii2}) - (a_{ii1} + b_{ii1})) + (a_{ii1} + b_{ii1}), \\
-2\alpha((a_{ii6} + b_{ii6}) - (a_{ii5} + b_{ii5})) + (a_{ii6} + b_{ii6})
\end{cases}
\]

for \(\alpha \in [0, 0.5)\)

\[
\begin{cases}
2\alpha((a_{ii3} + b_{ii3}) - (a_{ii2} + b_{ii2})) - (a_{ii3} + b_{ii3}) + 2(a_{ii2} + b_{ii2}), \\
-2\alpha((a_{ii5} + b_{ii5}) - (a_{ii4} + b_{ii4})) + 2(a_{ii5} + b_{ii5}) - (a_{ii4} + b_{ii4})
\end{cases}
\]

for \(\alpha \in [0.5, 1].\)

\[
\begin{cases}
2\alpha(a_{ii2} + b_{ii2} - a_{ii1} - b_{ii1}) + a_{ii1} + b_{ii1}, \\
-2\alpha(a_{ii6} + b_{ii6} - a_{ii5} - b_{ii5}) + a_{ii6} + b_{ii6}
\end{cases}
\]

for \(\alpha \in [0, 0.5)\)

\[
\begin{cases}
2\alpha(a_{ii3} + b_{ii3} - a_{ii2} - b_{ii2}) - a_{ii3} - b_{ii3} + 2a_{ii2} + 2b_{ii2}, \\
-2\alpha(a_{ii5} + b_{ii5} - a_{ii4} - b_{ii4}) + 2a_{ii5} + 2b_{ii5} - a_{ii4} - b_{ii4}
\end{cases}
\]

for \(\alpha \in [0.5, 1].\)
Hence \( \text{tr}(\hat{A}) = \sum_{i=1}^{n} [2\alpha(a_{i2} - a_{i1}) + a_{i1}, -2\alpha(a_{i6} - a_{i5}) + a_{i6}] + [2\alpha(b_{i2} - b_{i1}) + b_{i1}, -2\alpha(b_{i6} - b_{i5}) + b_{i6}] \)

for \( \alpha \in [0, 0.5) \)

\[ \sum_{i=1}^{n} [2\alpha(a_{i3} - a_{i2}) - a_{i3} + 2a_{i2}, -2\alpha(a_{i5} - a_{i4}) + 2a_{i5} - a_{i4}] + [2\alpha(b_{i3} - b_{i2}) - b_{i3} + 2b_{i2}, -2\alpha(b_{i5} - b_{i4}) + 2b_{i5} - b_{i4}] \]

for \( \alpha \in [0.5, 1] \)

\( \sum_{i=1}^{n} [2\alpha(a_{i2} - a_{i1}) + a_{i1}, -2\alpha(a_{i6} - a_{i5}) + a_{i6}] + [2\alpha(b_{i2} - b_{i1}) + b_{i1}, -2\alpha(b_{i6} - b_{i5}) + b_{i6}] \)

for \( \alpha \in [0, 0.5) \)

\[ \sum_{i=1}^{n} [2\alpha(a_{i3} - a_{i2}) - a_{i3} + 2a_{i2}, -2\alpha(a_{i5} - a_{i4}) + 2a_{i5} - a_{i4}] + [2\alpha(b_{i3} - b_{i2}) - b_{i3} + 2b_{i2}, -2\alpha(b_{i5} - b_{i4}) + 2b_{i5} - b_{i4}] \]

for \( \alpha \in [0.5, 1] \)

\( = \sum_{i=1}^{n} [a_{ii}^{L}, a_{ij}^{U}] + \sum_{j=1}^{n} [b_{ji}^{L}, b_{ij}^{U}] \)

\( = \text{tr}(\hat{A}_\alpha) + \text{tr}(\hat{B}_\alpha) \)

Hence \( \text{tr}(\hat{A}_\alpha) + \text{tr}(\hat{B}_\alpha) = \text{tr}(\hat{A}_\alpha + \hat{B}_\alpha). \)

Property 3.5.2. If \( \hat{A}_\alpha = [a_{ij}^{L}, a_{ij}^{U}] \) is a \( \alpha \)-cut of HFM of order \( n \) then \( \text{tr}(\hat{A}_\alpha) = \text{tr}(\hat{A}_\alpha^T) \).

Proof. Let \( \hat{A}_\alpha = [a_{ij}^{L}, a_{ij}^{U}] \) be a square HFM of order \( n \) and let \( \hat{A}_\alpha^T = \hat{B}_\alpha = [b_{ij}^{L}, b_{ij}^{U}] \).

Then \( [b_{ij}^{L}, b_{ij}^{U}] = [a_{ij}^{L}, a_{ij}^{U}] \). When \( j = i \), we have \( [b_{ii}^{L}, b_{ii}^{U}] = [a_{ii}^{L}, a_{ii}^{U}] \).

Now, \( \text{tr}(\hat{A}_\alpha^T) = \text{tr}(\hat{B}_\alpha) \)

\[ = \sum_{i=1}^{n} [b_{ii}^{L}, b_{ii}^{U}] = \sum_{i=1}^{n} [a_{ii}^{L}, a_{ii}^{U}] = \text{tr}(\hat{A}_\alpha) \]
Hence \( \text{tr}(\hat{A}_\alpha) = \text{tr}(\hat{A}_\alpha^T) \).

Property 3.5.3. If \( \hat{A}_\alpha = [a^L_{ij}, a^U_{ij}] \) and \( \hat{B}_\alpha = [b^L_{ij}, b^U_{ij}] \) are any two square ACHFMs of order \( n \) then \( \text{tr}(\hat{A}_\alpha \hat{B}_\alpha) = \text{tr}(\hat{B}_\alpha \hat{A}_\alpha) \).

Proof. Let \( \hat{A}_\alpha = [a^L_{ij}, a^U_{ij}] \) and \( \hat{B}_\alpha = [b^L_{ij}, b^U_{ij}] \) are any two square ACHFMs of order \( n \) where,

\[
[a^L_{ij}, a^U_{ij}] = \begin{cases} 
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_6 - a_5) + a_6] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(a_3 - a_2) - a_3 + 2a_2, -2\alpha(a_5 - a_4) + 2a_5 - a_4] & \text{for } \alpha \in [0.5, 1].
\end{cases}
\]

\[
b^L_{ij}, b^U_{ij} = \begin{cases} 
[2\alpha(b_2 - b_1) + b_1, -2\alpha(b_6 - b_5) + b_6] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(b_3 - b_2) - b_3 + b_2, -2\alpha(b_5 - b_4) + 2b_5 - b_4] & \text{for } \alpha \in [0.5, 1].
\end{cases}
\]

Let \( \hat{A}_\alpha \hat{B}_\alpha = \hat{C}_\alpha = [c^L_{ij}, c^U_{ij}] \) then \( [c^L_{ij}, c^U_{ij}] = \sum_{p=1}^{n} [a^L_{ip}, a^U_{ip}][b^L_{pj}, b^U_{pj}] \).

Also let \( \hat{B}_\alpha \hat{A}_\alpha = \hat{D}_\alpha = [d^L_{ij}, d^U_{ij}] \) then \( [d^L_{ij}, d^U_{ij}] = \sum_{p=1}^{n} [b^L_{ip}, b^U_{ip}][a^L_{pj}, a^U_{pj}] \).

Now,

\[
\text{tr}(\hat{A}_\alpha \hat{B}_\alpha) = \text{tr}(\hat{C}_\alpha)
= \sum_{i=1}^{n} [c^L_{ii}, c^U_{ii}]
= \sum_{i=1}^{n} \sum_{p=1}^{n} [a^L_{ip}, a^U_{ip}][b^L_{pi}, b^U_{pi}] \quad \text{(interchangeing the indices } i \text{ and } p)
= \sum_{i=1}^{n} \sum_{p=1}^{n} [b^L_{pi}, b^U_{pi}][a^L_{ip}, a^U_{ip}]
= \sum_{i=1}^{n} [d^L_{ii}, d^U_{ii}] = \text{tr}(\hat{D}_\alpha) = \text{tr}(\hat{B}_\alpha \hat{A}_\alpha)
\]

Hence \( \text{tr}(\hat{A}_\alpha \hat{B}_\alpha) = \text{tr}(\hat{B}_\alpha \hat{A}_\alpha) \).

\[\blacksquare\]
Ch.3: Some Arithmetic Operations on Some Special Types of HFM

3.6 Determinant of HFM

Definition 3.6.1: Determinant of HFM

The Determinant of $n \times n$ HFM $\hat{A} = (\tilde{a}_{hij})$ is denoted by $|\hat{A}|$ or $\text{det}(\hat{A})$ and is defined as follows

$$|\hat{A}| = \sum_{q \in s_n} \prod_{i=1}^{n} \tilde{a}_{hiq(i)} = \sum_{q \in s_n} \text{sgn}q \tilde{a}_{h1q(1)}, \tilde{a}_{h2q(2)}, \ldots, \tilde{a}_{hnq(n)}$$

where $\tilde{a}_{hiq(i)} = (a_{iq(i)1}, a_{iq(i)2}, a_{iq(i)3}, a_{iq(i)4}, a_{iq(i)5}, a_{iq(i)6})$ are hexagonal fuzzy number (HFN) and $s_n$ denotes the symmetric group of all permutations of the indices $\{1, 2, \ldots, n\}$ and $\text{sgn} q = 1$ or $-1$ according as the permutations $q = \begin{pmatrix} 1 & 2 & \ldots & n \\ q(1) & q(2) & \ldots & q(n) \end{pmatrix}$ is even or odd respectively.

Definition 3.6.2: Minor of HFM

Let $\hat{A} = (\tilde{a}_{hij})$ be a square HFM of order $n$. The minor of an element $\tilde{a}_{hij}$ in $\hat{A}$ is a determinant of order $(n - 1) \times (n - 1)$ which is obtained by deleting the $i^{th}$ row and $j^{th}$ column from $\hat{A}$ and is denoted by $\tilde{M}_{hij}$.

Definition 3.6.3: Co-factor of HFM

Let $\hat{A} = (\tilde{a}_{hij})$ be a square HFM of order $n$. The cofactor of an element $\tilde{a}_{hij}$ in $\hat{A}$ is denoted by $\tilde{A}_{hij}$ and is defined as $\tilde{A}_{hij} = (-1)^{i+j} \tilde{M}_{hij}$.

Definition 3.6.4: Alternative Definition of Determinant of HFM

Alternately, the determinant of square HFM $\hat{A} = (\tilde{a}_{hij})$ of order $n$ may be
expanded in the form

$$|\hat{A}| = \sum_{j=1}^{n} \tilde{a}_{hi,j} \tilde{A}_{hi,j} \quad i \in \{1, 2, \ldots, n\}$$

where $\tilde{A}_{hi,j}$ is the cofactor of $\tilde{a}_{hi,j}$. Thus the determinant is the sum of the products of the elements of any row (or column) and the cofactors of the corresponding elements of the same row (or column).

**Definition 3.6.5: Adjoint of HFM**

let $\hat{A} = (\tilde{a}_{hi,j})$ be a square HFM of order $n$. Find the cofactor $\tilde{A}_{hi,j}$ for every element $\tilde{a}_{hi,j}$ in $\hat{A}$ and replace every $\tilde{a}_{hi,j}$ by its cofactor $\tilde{A}_{hi,j}$ in $\hat{A}$ and let $\hat{B}$, i.e., $\hat{A} = (\tilde{A}_{hi,j})$. Then the transpose of $\hat{B}$ is called the adjoint or adjugate of $\hat{A}$ and is denoted by $\text{adj} \hat{A}$, i.e., $\hat{B}' = \tilde{A}_{hi,j} = \text{adj} \hat{A}$.

### 3.6.1 Properties of Determinant of HFMs

**Property 3.6.6.** Let $\hat{A} = (\tilde{a}_{hi,j})$ be a square HFM of order $n$. If all the elements of a row (or column) of $\hat{A}$ are 0 then $|\hat{A}|$ is also 0.

**Proof.** Let $\hat{A} = (\tilde{a}_{hi,j})$ be a square HFM of order $n$ and let all the elements of the $r^{th}$ row, where $1 \leq r \leq n$ be 0. i.e, $\tilde{a}_{hi,j} = (0, 0, 0, 0, 0, 0)$, $j = 1, 2, \ldots, n$. Then $|\hat{A}| = \sum_{j=1}^{n} \tilde{a}_{hi,j} \tilde{A}_{hi,j} \quad i \in \{1, 2, \ldots, n\}$ where $\tilde{A}_{hi,j}$ is the cofactor of $\tilde{a}_{hi,j}$.

Particularly if we expand through the $r^{th}$ row then we have $|\hat{A}| = \sum_{j=1}^{n} \tilde{a}_{hr,j} \tilde{A}_{hr,j}$, which is the sum of the products of the elements of the $r^{th}$ row and the cofactor of the corresponding elements of the $r^{th}$ row.
Since all $\tilde{a}_{hrj}$ are 0, each term in this summation is 0 and hence $|\hat{A}|$ is also 0.

**Property 3.6.7.** Let $\hat{A} = (\tilde{a}_{hij})$ be a square HFM of order n. If all the elements of a row(or column) of $\hat{A}$ are $\tilde{0}$ then $|\hat{A}|$ is either 0 or $\tilde{0}$.

**Proof.** Let $\hat{A} = (\tilde{a}_{hij})$ be a square HFM of order n and let all the elements of $r^{th}$ row be $\tilde{0}$. Since for any hexagonal fuzzy number $\tilde{a}_h \neq 0, \tilde{a}_h * \tilde{0} = 0$ and $\tilde{0} * \tilde{a}_h = \tilde{0}$, if we expand through $r^{th}$ row then in this case $|\hat{A}| = \tilde{0}$ and we expand through other than we have $|\hat{A}| = 0$.

**Property 3.6.8.** Let $\hat{A} = (\tilde{a}_{hij})$ be a square HFM of order n, where $\tilde{a}_{hij} = (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}, a_{ij5}, a_{ij6})$. If a row be multiplied by scalar $k$, then $|\hat{A}|$ is multiplied by $k$.

**Proof.** case(i): $k = 0$. If $k = 0$, then the result is obvious since $|\hat{A}| = 0$ when $\hat{A}$ has a 0 row.

case(ii):$k \neq 0$. Let $\hat{B} = (\tilde{b}_{hij})_{n \times n}$ where $\tilde{b}_{hij} = (b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4}, b_{ij5}, b_{ij6})$ obtained from $\hat{A} = (\tilde{a}_{hij})_{n \times n}$ by multiplying $r^{th}$ row by a scalar $k \neq 0$.

Obviously

Let $(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4}, b_{ij5}, b_{ij6}) = (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}, a_{ij5}, a_{ij6})$ for all $i \neq r$ and

when $k > 0$

$(b_{rj1}, b_{rj2}, b_{rj3}, b_{rj4}, b_{rj5}, b_{rj6}) = (ka_{rj1}, ka_{rj2}, ka_{rj3}, ka_{rj4}, ka_{rj5}, ka_{rj6})$

when $k < 0$
Then by the definition

\[
| \mathbf{B} | = \sum_{q \in S_n} sgnq \prod_{i=1}^{n} \tilde{b}_{hiq(i)} \\
= \sum_{q \in S_n} sgnq[(b_{1q(1)1}, b_{1q(1)2}, b_{1q(1)3}, b_{1q(1)4}, b_{1q(1)5}, b_{1q(1)6}) \\
(b_{rq(r)1}, b_{rq(r)2}, b_{rq(r)3}, b_{rq(r)4}, b_{rq(r)5}, b_{rq(r)6}) \\
(b_{nq(n)1}, b_{nq(n)2}, b_{nq(n)3}, b_{nq(n)4}, b_{nq(n)5}, b_{nq(n)6})]
\]

when \( k > 0 \)

\[
| \hat{B} | = \sum_{q \in S_n} sgnq[(a_{1q(1)1}, a_{1q(1)2}, a_{1q(1)3}, a_{1q(1)4}, a_{1q(1)5}, a_{1q(1)6}) \\
(ka_{rq(r)1}, ka_{rq(r)2}, ka_{rq(r)3}, ka_{rq(r)4}, ka_{rq(r)5}, ka_{rq(r)6}) \\
(a_{nq(n)1}, a_{nq(n)2}, a_{nq(n)3}, a_{nq(n)4}, a_{nq(n)5}, a_{nq(n)6})]
\]

\[
k \sum_{q \in S_n} sgnq[(a_{1q(1)1}, a_{1q(1)2}, a_{1q(1)3}, a_{1q(1)4}, a_{1q(1)5}, a_{1q(1)6}) \\
(a_{rq(r)1}, a_{rq(r)2}, a_{rq(r)3}, a_{rq(r)4}, a_{rq(r)5}, a_{rq(r)6}) \\
(a_{nq(n)1}, a_{nq(n)2}, a_{nq(n)3}, a_{nq(n)4}, a_{nq(n)5}, a_{nq(n)6})]
\]

\[
k \sum_{q \in S_n} sgnq \prod_{i=1}^{n} \tilde{a}_{hiq(i)}
\]

\[
| \hat{B} | = k | \hat{A} | \quad \text{when } k < 0
\]

\[
| \hat{B} | = \sum_{q \in S_n} sgnq[(a_{1q(1)1}, a_{1q(1)2}, a_{1q(1)3}, a_{1q(1)4}, a_{1q(1)5}, a_{1q(1)6}) \\
(ka_{rq(r)6}, ka_{rq(r)5}, ka_{rq(r)4}, ka_{rq(r)3}, ka_{rq(r)2}, ka_{rq(r)1}) \\
(a_{nq(n)1}, a_{nq(n)2}, a_{nq(n)3}, a_{nq(n)4}, a_{nq(n)5}, a_{nq(n)6})]
\]

\[
k \sum_{q \in S_n} sgnq[(a_{1q(1)1}, a_{1q(1)2}, a_{1q(1)3}, a_{1q(1)4}, a_{1q(1)5}, a_{1q(1)6}) \\
(a_{rq(r)1}, a_{rq(r)2}, a_{rq(r)3}, a_{rq(r)4}, a_{rq(r)5}, a_{rq(r)6})]
\]
(a_{nq(1)}, a_{nq(2)}, a_{nq(3)}, a_{nq(4)}, a_{nq(5)}, a_{nq(6)})

= k \sum_{q \in s_n} sgnq \prod_{i=1}^{n} \tilde{a}_{hiq(i)}

| \hat{B} | = k | \hat{A} | \text{ Hence the result.}

**Property 3.6.9.** The determinant of triangular HFM is given by the product of diagonal elements.

**Proof.** Let \( \hat{A} = (\tilde{a}_{hij}) \) be a square triangular HFM without loss of generality, let us assume that \( \hat{A} \) is a lower triangular HFM. i.e \( \tilde{a}_{hij} = 0 \) for \( i < j \).

Take a term \( t \) of \( | \hat{A} | \cdot t = \tilde{a}_{h1q(1)} \cdot \tilde{a}_{h2q(2)} \cdot \ldots \cdot \tilde{a}_{hnq(n)} \).

Let \( q(1) \neq 1 \). So that \( 1 < q(1) \) and so \( \tilde{a}_{h1q(1)} = 0 \) and thus \( t = 0 \). This means that each term is 0 if \( q(1) \neq 1 \).

Now let \( q(1) = 1 \) but \( q(2) \neq 1 \). Then \( 2 < q(2) \) and so \( \tilde{a}_{h2q(2)} = 0 \) and thus \( t = 0 \). This means that each term is 0 if \( q(2) \neq 1 \).

However in a similar manner we can see that each term must be 0 if \( q(1) \neq 1 \), or \( q(2) \neq 2 \) or...\( q(n) \neq n \).

Consequently,

\[
| \hat{A} | = \tilde{a}_{h1q(1)} \cdot \tilde{a}_{h2q(2)} \cdot \ldots \cdot \tilde{a}_{hnq(n)}
= \prod_{i=1}^{n} \tilde{a}_{hii}
= \text{Product of its diagonal elements}

\]

Similarly when \( \hat{A} \) is an upper triangular HFM, the result follows. ■
3.7 A Comparison Between Permanent and Determinant of Matrices

we investigate the relation between permanent and determinant of matrices using crisp matrix and find out the new results are verified due to the counterexamples.

**Definition 3.7.1: Determinant** Let $A = (a_{ij})_{n \times n}$ is a crisp matrix of order $n \times n$, then the determinant of A is denoted by det($A$) or $|A|$ and defined as

$$|A| = \sum_{\sigma \in s_n} sgn\sigma \prod_{i=1}^{n} a_i\sigma(i)$$

where $s_n$ denotes the symmetric group of all permutations of the indices $\{1, 2, \ldots, n\}$ and $sgn\sigma$ is +1 for even permutations and -1 for odd permutations.

**Definition 3.7.2: Permanent** If $A = (a_{ij})_{n \times n}$ is a crisp matrix of order $n \times n$, then the permanent of A is denoted by per($A$) and defined as

$$per(A) = \sum_{\sigma \in s_n} \prod_{i=1}^{n} a_i\sigma(i)$$

where $s_n$ denotes the symmetric group of order $n$.

The definition of permanent is similar to the definition of determinant except the sign of each term in summation. The number of terms over summation are both cases but the sign associated in each term are all positive in case of permanent. The permanent cannot compete with determinant,
in terms of the depth of theory and breadth of applications, but it is safe
to say that the permanent also exhibits both these characteristic in ample
measure, a fact that has not receive enough attention.

Here three ways to calculate per(A) for general $3 \times 3$ matrix for
\[
A = \begin{pmatrix}
a & d & g \\
b & e & h \\
c & f & i
\end{pmatrix}
\]
The classical formula using all the permutation in $S_3$ is
\[
\text{Per}(A) = ae i + bfg + cdh + af h + bdi + ceg
\]
Ryser’s Method gives
\[
\text{Per}(A) = (a + b + c)(d + e + f)(g + h + i) - (a + b)(d + e)(g + h) - (a + c)(d + f)(g + i) - (b + c)(e + f)(h + i) + adg + bef + cfi.
\]
Glynn Method gives
\[
2^2 \text{Per}(A) = (a + b + c)(d + e + f)(g + h + i) - (a - b + c)(d - e + f)(g - h + i) - (a + b - c)(d + e - f)(g + h - i) + (a - b - c)(d - e - f)(g - h - i).
\]

Let us consider an example to illustrate both the determinant and per-
manent of crisp matrix.

**Example 3.7.3:** Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 2 & 1 & 6 \end{pmatrix}$

Then $|A| = -13$ and $\text{per}(A) = 153$. 
3.8 Special Properties of permanent

Properties of permanent are also presented below.

1. For a crisp matrix \( A = (a_{ij})_{n \times n} \) is the positive square matrix of order \( n \) then, \( \text{per}(A) \geq \text{det}(A) \).

2. For a crisp matrix \( A = -(a_{ij})_{n \times n} \) then,
   - If \( A = -(a_{ij})_{n \times n} \) is even square matrix of order \( n \) then, \( \text{per}(A) \geq \text{det}(A) \).
   - If \( B = -(b_{ij})_{n \times n} \) is odd square matrix of order \( n \) then, \( \text{per}(B) \leq \text{det}(B) \).

3. If any one of the row (or) column of a crisp matrix \( A = (a_{ij})_{n \times n} \) of order \( n \) is negative the permanent of a matrix is negative.

4. If any one of the row (or) column of a crisp matrix \( A = (a_{ij})_{n \times n} \) of order \( n \) is negative then, \( \text{per}(A) \leq \text{det}(A) \).

**Verifications:**

The above said special properties of permanent have been verified by the counter examples.

**Example 3.8.1:** Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \)

\[ \text{per}(A) = 10 \quad (3.1) \]
\[ \text{det}(A) = -2 \quad (3.2) \]

from (3.1) and (3.2)

\[ \text{per}(A) > \text{det}(A) \]

In particular, the zero matrix and triangular matrix then, the \( \text{per}(A) = \text{det}(A) \).

Therefore, \( \text{per}(A) \geq \text{det}(A) \).
Example 3.8.2: (i) Let $A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}_{2\times2}$

$\text{per}(A) = 10$ \hfill (3.3)

$\text{det}(A) = -2$ \hfill (3.4)

from (3.3) and (3.4)

$\text{per}(A) > \text{det}(A)$

In particular, matrix will become zero matrix and Triangular matrix then, the $\text{per}(A)=\text{det}(A)$.

Therefore, $\text{per}(A) \geq \text{det}(A)$.

(ii) Let $B = \begin{bmatrix} -1 & -2 & -1 \\ 0 & -3 & -2 \\ -5 & -3 & -2 \end{bmatrix}_{3\times3}$

$\text{per}(A) = -47$ \hfill (3.5)

$\text{det}(A) = -5$ \hfill (3.6)

from (3.5) and (3.6)

$\text{per}(A) < \text{det}(A)$

In particular, matrix will become zero matrix and Triangular matrix then, the $\text{per}(B)=\text{det}(B)$.

Therefore, $\text{per}(B) \leq \text{det}(B)$.

Example 3.8.3: Let $A = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 3 & 2 \\ 5 & 3 & 2 \end{bmatrix}$
per(A) = -1.3.2 + 1.3.2 + -2.0.2 + -2.5.2 + -1.0.3 + -1.5.3.
per(A) = -6-6+0-20+0-15.
per(A) = -47.

Therefore, the permanent of a matrix is negative.

**Example 3.8.4:** Let \( A = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 3 & 2 \\ 5 & 3 & 2 \end{bmatrix} \)

\( \text{per}(A) = -47 \) \hspace{1cm} (3.7)

\( \text{det}(A) = -5 \) \hspace{1cm} (3.8)

form (3.7) and (3.8)

\( \text{per}(A) < \text{det}(A) \).

In particular, matrix will become zero matrix and Triangular matrix then, \( \text{per}(A) = \text{det}(A) \).

Therefore, \( \text{per}(A) \leq \text{det}(A) \).

### 3.9 Matrix Inversion Method For Fuzzy Linear System (FLS) Using Hexagonal Fuzzy Number

In this section, we define the concept of fuzzy linear system is justify in matrix inversion method with the aid of hexagonal fuzzy numbers and the relevant definitions are recalled in nature.

Consider the system of n fuzzy linear non- homogeneous HFN equations in n unknown HFN vectors \( x_{h1}, x_{h2}, \ldots, x_{hn} \).
where $\tilde{a}_{hi}, x_{hi}, b_{hi}$ hexagonal fuzzy numbers.

\[
\begin{align*}
\tilde{a}_{h11}x_{h1} \oplus \tilde{a}_{h12}x_{h2} \oplus \ldots \ldots \ldots \oplus \tilde{a}_{h1n}x_{hn} &= b_{h1} \\
\tilde{a}_{h21}x_{h1} \oplus \tilde{a}_{h22}x_{h2} \oplus \ldots \ldots \ldots \oplus \tilde{a}_{h2n}x_{hn} &= b_{h2} \\
&\vdots \\
\tilde{a}_{hn1}x_{h1} \oplus \tilde{a}_{hn2}x_{h2} \oplus \ldots \ldots \ldots \oplus \tilde{a}_{hnn}x_{hn} &= b_{hn}
\end{align*}
\]

The above linear system is represented in the form is given by

\[\hat{A}x_h = b_h \quad (3.9)\]

Where $\hat{A} = (\tilde{a}_{hi})$, $1 \leq i, j \leq n$ is ntimesn hexagonal fuzzy matrix and $\tilde{a}_{hi} \in F(R)$ and $(x_{hi}, b_{hi}) \in F(R)$, for all $i=1,2,\ldots,n$ and $j=1,2,\ldots,n$. This system is called fuzzy linear system. If the coefficient Hexagonal fuzzy matrix $\hat{A}$ is non singular, then $\hat{A}^{-1}$ we get

$x_h = \hat{A}^{-1}b_h$ This is the solution of FLS will be represented by $x_h = \hat{A}^{-1}b_h$

**Definition 3.9.1: (Singular HFM)**

Let $\hat{A} = (\tilde{a}_{hi})$ be a square HFM of order n, then it is said to be singular HFM if $|\hat{A}| = \tilde{0}$.

**Definition 3.9.2: (Non-Singular HFM)**

Let $\hat{A} = (\tilde{a}_{hi})$ be a square HFM of order n, then it is said to be non-singular HFM if $|\hat{A}| \neq \tilde{0}$.

**Definition 3.9.3: (Inverse of HFM)**

A non-singular HFM $\hat{A} = (\tilde{a}_{hi})$ of order n is said to be invertible if there
exist a HFM \( \hat{B} \) of order \( n \) such that \( \hat{A}\hat{B} = \hat{I} = \hat{B}\hat{A} \).

Then \( \hat{B} \) is called the inverse of \( \hat{A} \) and is denoted by \( \hat{A}^{-1} \). Thus \( \hat{A}\hat{A}^{-1} = \hat{I} = \hat{A}^{-1}\hat{A} \). Also \( \hat{A}^{-1} = \frac{1}{\hat{R}(\hat{A})} \text{adj} \hat{A} \).

### 3.10 Numerical Example

In this section, an examples is given in order to illustrate the proposed method.

**Example 3.10.1:** Consider the following fuzzy linear system and solve by matrix inversion method.

\[
\begin{align*}
(−1, 1, 3, 5, 7, 9)_{x_1} &⊕ (−1, 0, 1, 2, 4, 6)_{x_2} &⊕ (−2 − 1, 0, 1, 2, 6)_{x_3} = (0, 2, 4, 6, 8, 10) \\
(−1, 0, 1, 3, 6, 9)_{x_1} &⊕ (−1, 0, 1, 2, 4, 6)_{x_2} &⊕ (−2 − 1, 0, 1, 2, 6)_{x_3} = (−1, 0, 1, 3, 6, 9) \\
(−2 − 1, 0, 1, 2, 6)_{x_1} &⊕ (−2 − 1, 0, 1, 2, 6)_{x_2} &⊕ (−2 − 1, 0, 1, 2, 6)_{x_3} = (−1, 1, 3, 5, 7, 9).
\end{align*}
\]

**Solution:**

The given linear system may be written as

\[
\begin{pmatrix}
(−1, 1, 3, 5, 7, 9) & (−1, 0, 1, 2, 4, 6) & (−2 − 1, 0, 1, 2, 6)
\end{pmatrix}
\begin{pmatrix}
x_1
\end{pmatrix}
=
\begin{pmatrix}
(0, 2, 4, 6, 8, 10)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(−1, 0, 1, 3, 6, 9) & (−1, 0, 1, 2, 4, 6) & (−2 − 1, 0, 1, 2, 6)
\end{pmatrix}
\begin{pmatrix}
x_2
\end{pmatrix}
=
\begin{pmatrix}
(−1, 0, 1, 3, 6, 9)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(−2 − 1, 0, 1, 2, 6) & (−2 − 1, 0, 1, 2, 6) & (−2 − 1, 0, 1, 2, 6)
\end{pmatrix}
\begin{pmatrix}
x_3
\end{pmatrix}
=
\begin{pmatrix}
(−1, 1, 3, 5, 7, 9)
\end{pmatrix}
\]

\[x_h = \hat{A}^{-1}b_h\]

Now, \( |\hat{A}| = (−15, −8, −1, 4, 9, 17) \).

Then, \( \hat{R}(|\hat{A}|) = 1 ≠ 0 \).

Since \( \hat{A} \) is non-singular, then \( \hat{A}^{-1} \) exists,

\[
\hat{A}^{-1} = \frac{1}{\hat{R}(|\hat{A}|)} \text{adj} \hat{A}.
\]
$\hat{A} = \begin{bmatrix}
(-7, -2, 0, 2, 5, 8) & (-8, -5, -2, 0, 2, 7) & (-7, -4, -1, 1, 4, 7) \\
(-6, -4, -2, -1, 0, 1) & (-7, -1, 2, 5, 8, 11) & (-10, -7, -4, 0, 5, 10) \\
(-13, -4, -1, 3, 8, 13) & (-13, -9, -5, -1, 3, 13) & (-20, -10, 0, 8, 14, 20)
\end{bmatrix}$

$\hat{A}^{-1} = \begin{bmatrix}
(-7, -2, 0, 2, 5, 8) & (-8, -5, -2, 0, 2, 7) & (-7, -4, -1, 1, 4, 7) \\
(-6, -4, -2, -1, 0, 1) & (-7, -1, 2, 5, 8, 11) & (-10, -7, -4, 0, 5, 10) \\
(-13, -4, -1, 3, 8, 13) & (-13, -9, -5, -1, 3, 13) & (-20, -10, 0, 8, 14, 20)
\end{bmatrix}$

The solution is, $x_h = \hat{A}^{-1} b_h$

\[
\begin{bmatrix}
x_{h1} \\
x_{h2} \\
x_{h3}
\end{bmatrix} = \begin{bmatrix}
(-87, -41, -10, 14, 47, 89) \\
(-91, -51, -20, 10, 44, 78) \\
(-184, -87, -20, 44, 56, 184)
\end{bmatrix}
\]

The solution is,

$x_{h1} = (-87, -41, -10, 14, 47, 89)$;

$x_{h2} = (-91, -51, -20, 10, 44, 78)$;

$x_{h3} = (-184, -87, -20, 44, 56, 184)$. 