CHAPTER – I
INTRODUCTION

1.1. INTRODUCTION

A complex sequence, whose $k^{th}$ term is $x_k$, is denoted by {$x_k$} or simply $x$. Let $\phi$ be the set of all finite sequences. A sequence $x=\{x_i\}$ is said to be analytic if $\sup_k(1^{\frac{1}{k}} |x_k|) < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$. A sequence $x$ is called chi sequence if $\lim_{k \to \infty} k! x_k = 0$. The vector space of all chi sequences will be denoted by $\chi$. Let $\sigma$ be a one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1, 2, 3, ...$

A continuous linear functional $\phi$ on $\Lambda$ is said to be an invariant mean or a, $\sigma$–mean if and only if

1. $\phi(x) \geq 0$ when the sequence $x = \{x_n\}$ has $x_n \geq 0$ for all $n$

2. $\phi(e) = 1$ where $e = (1, 1, 1, ...)$ and

3. $\phi(\{x_{\sigma}(n)\}) = f(\{x_{\sigma}(n)\}) = \phi(\{x_n\})$ for all $x \in \Lambda$.

For certain kinds of mappings $\sigma$, every invariant mean $\phi$ extends the limit functional on the space $C$ of all real convergent sequences in the sense that $\phi(x) = \lim x$ for all $x \in C$. Consequently $C \subset V_\sigma$, where $V_\sigma$ is the set of analytic sequences all of those $\sigma$–means are equal.
If \( x = (x_n) \), set \( T_x = (Tx)^\frac{1}{n} = (x_\alpha(n)) \). It can be shown that

\[
V_\sigma = \left\{ x = (x_n) : \lim_{m \to \infty} t_{mn}(x_n)^\frac{1}{n} = L \text{ uniformly in } n, \quad L = \sigma - \lim_{n \to \infty} (x_n)^\frac{1}{n} \right\}
\]

where \( t_{mn}(x) = t_{mn}(x) = \frac{(x_n + Tx_n + \ldots + T^mx_n)^\frac{1}{n}}{m + 1} \)

Given a sequence \( x = \{x_k\} \), its \( n^{\text{th}} \) section is the sequence \( x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} \), \( \delta^{(n)} = (0, 0, \ldots, 1, 0, 0, \ldots) \), 1 in the \( n^{\text{th}} \) place and zeros elsewhere. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals \( p_k(x) = x_k \ (k = 1, 2, \ldots) \) are continuous.

Throughout \( \omega, \chi \) and \( \Lambda \) denote the classes of all, gai and analytic scalar valued single sequences respectively.

We write \( \omega^2 \) for the set of all complex sequences \((x_{mn})\), where \( m, n \in \mathbb{N} \), the set of positive integers. Then, \( \omega^2 \) is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [6]. Later on, they were investigated by Hardy [14], Moricz [21], Moricz and Rhoades [22], Basarir and Solankan [3], Tripathy [35], Colak and Turkmenoglu [9], Turkmenoglu [42] and many others.
Let us define the following sets of double sequence $s$:

$$
\mathcal{M}_u(t) := \left\{ (x_{mn}) \in \omega^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}
$$

$$
\mathcal{C}_p(t) := \left\{ (x_{mn}) \in \omega^2 : p - \lim_{m,n \to \infty} |x_{mn} - 1|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\},
$$

$$
\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in \omega^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 1 \right\},
$$

$$
\mathcal{L}_u(t) := \left\{ (x_{mn}) \in \omega^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},
$$

$$
\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);
$$

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim’ sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$, and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [4, 5] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-$, $\beta-$, $\gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her Ph.D thesis, Zeltser [46] has essentially studied both the sequences. Mursaleen and Edely [26] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesáro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [27] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core
for double sequences and determined those four dimensional matrices transforming every bounded double sequences \( x = (x_{jk}) \) into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces \( BS, BS(t), ES_p, ES_{bp}, ES_r \) and \( BV \) and BV of double sequences consisting of all double series whose sequence of partial sums are in the spaces \( \mathcal{M}_u, \mathcal{M}_a(t), \mathcal{E}_p, \mathcal{E}_{bp}, \mathcal{E}_r \) and \( \mathcal{L}_u \), respectively, and also examined some properties of those sequence spaces and determined the \( \alpha \)-duals of the spaces \( BS, BV, ES_{bp} \) and the \( \beta(v) \) – duals of the spaces \( ES_{bp} \) and \( ES_r \) of double series. Quite recently Başar and Sever [2] have introduced the Banach space \( \mathcal{L}_q \) of double sequences corresponding to the well-known space \( \ell_q \) of single sequences and examined some properties of the space \( \mathcal{L}_q \). Quite sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For \( a, b, \geq 0 \) and \( 0 < p < 1 \), we have \( (a + b)^p \leq a^p + b^p \)

The double series \( \sum_{m,n=1}^{\infty} x_{mn} \) is called convergent if and only if the double sequence \( (s_{mn}) \) is convergent, where \( s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \) \( (m, n \in \mathbb{N}) \) [1].

A sequence \( x = (x_{mn}) \) is said to be double analytic if \( \sup_{m,n} \left| x_{mn} \right| \frac{1}{m+n} < \infty \). The vector space of all double analytic sequences will be denoted by \( \Lambda^2 \). A sequence \( x = (x_{mn}) \) is called double entire sequence if \( \left| x_{mn} \right| \frac{1}{m+n} \to 0 \) as \( m, n \to \infty \). The double entire sequences will be denoted by \( \Gamma^2 \). A sequence
\( x = (x_{mn}) \) is called double chi sequence if \( \left( (m + n)! x_{mn} \frac{1}{m+n} \right) \to 0 \) as \( m, n \to \infty \).

The double chi sequences will be denoted by \( \chi^2 \). Let \( \phi = \{ \text{all finite sequences} \} \).

Consider a double sequence \( x = (x_{ij}) \). The \((m, n)\)th section \( x^{[m, n]} \) of the sequence is defined by \( x^{[m, n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathbb{I} \) for all \( m, n \in \mathbb{N} \); where \( \mathbb{I} \) denotes the double sequence whose only non zero term is a \( \frac{1}{(i+j)!} \) in the \((i, j)\)th place for each \( i, j \in \mathbb{N} \).

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings \( x = (x_k) \to (x_{mn}) \) \( (m, n \in \mathbb{N}) \) are also continuous.

Orlicz [30] used the idea of Orlicz function to construct the space \((L^M)\). Lindenstrauss and Tzafriri [19] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p \) \((1 \leq p < \infty)\). Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [8], Mursaleen et al. [24], Bektas and Altin [9], Tripathy et al. [36], Rao and Subramanian [7] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [18].

Recalling [30] and [18], an orlciz function is a function \( M : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing, and convex with \( M(0) = 0, M(x) > 0, \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function, \( M \) is
replaced by sub additive of M, then this function is called modulus function, defined by Nakano [29] and further discussed by Ruckle [33] and Maddox [20], and many others.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$ if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)(u \geq 0)$. The $\Delta_2$-condition is equivalent to $M(\ell_u) \leq K\ell M(u)$, for all values of $u$ and for $\ell < 1$.

Lindenstrauss and Tzafriri [19] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space $\ell_M$ with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p \leq \infty)$, the spaces $\ell_M$ coincide with the classical sequence space $\ell_p$. If $X$ is a sequence space, we give the following definitions:

(i) $X' = \text{the continuous dual of } X$;

(ii) $X^\alpha = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\}$;

(iii) $X^\beta = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X \right\}$;
(iv) \(X^\gamma = \left\{ a = (a_{mn}) : \sup_{m,n \geq 1} \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, \text{ for each } x \in X \right\}\)

(v) let \(X\) be an FK-space \(\supset \phi\); then \(X^\ell = \left\{ f \left( \mathcal{J}_{mn} \right) : f \in X' \right\}\)

(vi) \(X^\delta = \left\{ a = (a_{mn}) : \sup_{mn} \left| a_{mn} x_{mn} \right|^\frac{1}{m+n} < \infty, \text{ for each } x \in X' \right\}\)

\(X^\alpha, X^\beta, X^\gamma\) are called \(\alpha\)-(or Köthe – Toeplitz) dual of \(X\), \(\beta\)-(or generalized-Köthe - Toeplitz) dual of \(X\), \(\gamma\)-dual of \(X\) respectively. \(X^\alpha\) is defined by Gupta and Kamptan [15]. It is clear that \(x^\alpha \subset X^\beta\) and \(X^\alpha \subset X^\gamma\), but \(X^\alpha \subset X^\ell\) does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [16] as follows

\[ Z(\Delta) = \left\{ x = (x_k) \in \omega : (\Delta x_k) \in Z \right\} \]

for \(Z = c, c_0\) and \(\ell_\infty\), where \(\Delta x_k = x_k - x_{k+1}\) for all \(k \in \mathbb{N}\). Here \(\omega, c, c_0\) and \(\ell_\infty\) denote the classes \(\omega\) of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

\[ \|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \]

Later on the notion was further investigated by many others. Hence, the following difference double sequence spaces defined by

\[ Z(\Delta) = \left\{ x = (x_{mn}) \in \omega^2 : (\Delta x_{mn}) \in Z \right\} \]

where \(Z = \Lambda^2, \Gamma^2\) and \(\chi^2\) respectively.
\[ \Delta x_{mn} = (x_{mn} - x_{m+1n}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1} \]

for all \( m, n \in \mathbb{N} \)

Let \( r \in \mathbb{N} \) be fixed, then

\[ Z(D^r) = \{ (x_{mn}) : (\Delta^r x_{mn}) \in Z \} \text{ for } Z = \chi^2, \Gamma^2 \text{ and } \Lambda^2 \]

where \( \Delta^r x_{mn} = \Delta^{r-1} x_{mn} - \Delta^{r-1} x_{m,n+1} - \Delta^{r-1} x_{m+1,n} + \Delta^{r-1} x_{m+1,n+1} \).

Therefore, introduced a generalized difference double operator as follows:

Let \( r, \gamma \in \mathbb{N} \) be fixed, then

\[ Z(\Delta^r) = \{ (x_{mn}) : (\Delta^r x_{mn}) \in Z \} \text{ for } Z = \chi^2, \Gamma^2 \text{ and } \Lambda^2 \]

where \( \Delta^r x_{mn} = \Delta^r x_{mn} - \Delta^{r-1} x_{m,n+1} - \Delta^{r-1} x_{m+1,n} + \Delta^{r-1} x_{m+1,n+1} \) and \( \Delta^0 x_{mn} = x_{mn} \)

for all \( m, n \in \mathbb{N} \).

The notion of a modulus function was introduced by Nakano [19]. We recall that a modulus \( f \) is a function from \( [0, \infty) \to [0, \infty) \), such that

(i) \( f(x) = 0 \) if and only if \( x = 0 \)

(ii) \( f(x + y) \leq f(x) + f(y) \), for all \( x \geq 0, y \geq 0 \),

(iii) \( f \) is increasing

(iv) \( f \) is continuous from the right at 0.

since, \( |f(x) - f(y)| \leq f(|x - y|) \), it follows from condition (iv) that \( f \) is continuous on \( [0, \infty) \).

It is immediate from (ii) and (iv) that \( f \) is continuous on \( [0, \infty) \). Also from condition (ii), we have \( f(nx) \leq nf(x) \) for all \( n \in \mathbb{N} \) and \( n^{-1}f(x) \leq f(xn^{-1}) \),

for all \( n \in \mathbb{N} \).
The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [45], fuzzy logic has become an important area of research in various branches of Mathematics such as metric and topological spaces, theory of functions, approximation theory etc. Subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. The concept of fuzziness has been applied in various fields such as Statistics, Cybernetics, Artificial intelligence, Operation research, Decision making, Agriculture, Weather forecasting, Quantum physics. Similarity relations of fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming etc.

The space $\Lambda^2$ is a metric space with the metric

$$d(x, y) = \sup_{mn} \left\{ x_{mn} - y_{mn} \right\}_{m,n \in \mathbb{N}}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\Lambda^2$.

The space $\chi^2$ is a metric space with the metric

$$d(x, y) = \sup_{mn} \left\{ (m + n)! |x_{mn} - y_{mn}| \right\}_{m,n \in \mathbb{N}}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\chi^2$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x_{[m,n]}$ of the sequence is defined by $x_{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m, n \in \mathbb{N}$,
with 1 in the \((m,n)\)th position and zero otherwise. An FK-space (or a metric space) \(X\) is said to have AK property if \((\delta_{mn})\) is a Schauder basis for \(X\) or equivalently \(x^{[m,n]} \to x\).

First Chapter discuss about the concepts of single sequence space and double sequence spaces.

Second chapter analyses the rate of \(\chi\)–space defined by a modulus.

Third chapter deals the Semi Normed Space defined by \(\chi\) Sequences.

Fourth chapter investigates the Analytic Fuzzy \(I\)–Convergent of \(\chi_{(\Lambda,p)}^{2I(F)}\) Space defined by Modulus.

Fifth chapter focuses the Generalization of \(\chi^2\) spaces by Cesáro method of order one and Modulus Function.

Sixth chapter consists of the modular sequence space of \(\chi^2\).

Seventh chapter contains the modular sequence space of \(\int \chi_{\alpha}^2\).

Eighth chapter examines the \(\chi^{2I}\) Convergent sequence spaces defined by a moduli.

Ninth chapter summarizes the Random of Lacunary Statistical on \(\chi^2\) over P–Metric Spaces defined by Musielak.