CHAPTER - III
THE SEMI NORMED SPACE DEFINED
BY $\chi$ SEQUENCES

This Chapter starts with the definition of semi normed space $(\chi, q)$, followed by the analysis of some of the properties of sequence space.

3.1. PRELIMINARIES AND DEFINITIONS

Definition 3.1.1

The space consisting of all those sequences $x$ in $w$ such that $(k!|x_k|)^{\frac{1}{k}} \to 0$ as $k \to \infty$ is denoted by $\chi$. In other words $(k!|x_k|)^{\frac{1}{k}}$ is a null sequence $\chi$ is called the space of chi sequences. The space $\chi$ is a metric space with the metric $d(x, y) = \left\{ \sup_{k} (k!|x_k - y_k|)^{\frac{1}{k}}, k = 1, 2, 3, \ldots \right\}$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in $\chi$.

Definition 3.1.2

The space consisting of all those sequence $x$ in $w$ such that $\left( \sup_{k} (|x_k|)^{\frac{1}{k}} \right) < \infty$ is denoted by $\Lambda$. In other words $\left( \sup_{k} (|x_k|)^{\frac{1}{k}} \right)$ is a bounded sequence.
Definition 3.1.3

Let \( p, q \) be semi norms on a vector space \( X \). then \( p \) is said to be stronger than \( q \) if whenever \((x_n)\) is a sequence such that \( p(x_n) \to 0 \), then also \( q(x_n) \to 0 \).

If each is stronger than the other, then \( p \) and \( q \) are said to be equivalent.

Lemma 3.1.4

Let \( p \) and \( q \) be semi norms on a linear space \( X \). Then \( p \) is stronger than \( q \) if and only if there exists a constant \( M \) such that \( q(x) \leq Mp(x) \) for all \( x \in X \).

Definition 3.1.5

A sequence space \( E \) is said to be solid or normal if \((\alpha_k x_k) \in E \) whenever \((x_k) \in E \) an for all sequences of scalars \((\alpha_k)\) with \( |\alpha_k| \leq 1 \), for all \( k \in \mathbb{N} \).

Definition 3.1.6

A sequence space \( E \) is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark

From the above two definitions, it is clear that a sequence space \( E \) is solid implies that \( E \) is monotone.

Definition 3.1.7

A sequence \( E \) is said to be convergence free if \((y_k) \in E \) whenever \((x_k) \in E \) and \( x_k = 0 \) implies that \( y_k = 0 \).
Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < \sup p_k = G$.

Let $D = \max (1, 2^{G-1})$.

Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$ we have

$$|a_k + b_k|^{\frac{1}{p_k}} \leq D \left\{ |a_k|^{\frac{1}{p_k}} + |b_k|^{\frac{1}{p_k}} \right\} \quad (1)$$

Let $(X, q)$ be a semi normed space over the field $\mathbb{C}$ of complex numbers with the semi norm $q$. The symbol $\Lambda(X)$ denotes the space of all analytic sequences defined over $X$. We define the following sequence spaces:

$$\Lambda(p, \sigma, q, s) = \left\{ x \in \Lambda(X) : \sup_{n,k} k^{-s} \left[ q \left( \left| x_{\sigma^{(n)}(k)} \right|^{\frac{1}{p_k}} \right) \right]^{p_k} < \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \right\}$$

$$\chi(p, \sigma, q, s) = \left\{ x \in \chi(X) : k^{-s} \left[ q \left( \left| x_{\sigma^{(n)}(k)} \right|^{\frac{1}{p_k}} \right) \right] \rightarrow 0, \text{ as } k \rightarrow \infty, \text{ uniformly in } n \geq 0, s \geq 0 \right\}$$

### 3.2. MAIN RESULTS

**Theorem 3.2.1**

$\chi(p, \sigma, q, s)$ is a linear space over the set of complex numbers.

**Proof**

It is routine verification. Therefore the proof is omitted.

**Theorem 3.2.2**

$\chi(p, \sigma, q, s)$ is a paranormed space with
\[ g^*(x) = \sup_{k \geq 1} \left\{ q^k(n) \left| x^{\sigma^+(n)} \right|^{1/k}, \text{ uniformly in } n > 0 \right\} \]

where \( H = \max \left( 1, \sup_k p_k \right) \).

**Proof**

Clearly \( g(x) = g(-x) \) and \( g(\theta) = 0 \), where \( \theta \) is the zero sequence. It can be easily verified that \( g(x + y) \leq g(x) + g(y) \). Next \( x \to \theta, \lambda \) fixed implies \( g(\lambda x) \to 0 \). Also \( x \to \theta \) and \( \lambda \to 0 \) imply \( g(\lambda x) \to 0 \). The case \( \lambda \to 0 \) and \( x \) fixed implies that \( g(\lambda x) \to 0 \) follows from the following expressions.

\[ g(\lambda x) = \left\{ \sup_{k \geq 1} q^k(n) \left| x^{\sigma^+(n)} \right|^{1/k}, \text{ uniformly in } n, m \in \mathbb{N} \right\} \]

where \( r = \frac{1}{|\lambda|^1} \). Hence \( \chi(p, \sigma, q, s) \) is a paranormed space. This completes the proof.

**Theorem 3.2.3**

\[ \chi(p, \sigma, q, s) \cap \Lambda(p, \sigma, q, s) \subseteq \chi(p, \sigma, q, s). \]

**Proof**

If is routine verification. Therefore the proof is omitted.

**Remark**

(i) Let \( q_1 \) and \( q_2 \) be two semi norms on \( X \), we have

\[ \chi(p, \sigma, q_1, s) \cap \chi(p, \sigma, q_2, s) \subseteq \chi(p, \sigma, q_1 + q_2, s); \]
(ii) If $q_1$ is stronger than $q_2$, then $\chi(p, \sigma, q_1, s) \subseteq \chi(p, \sigma, q_2, s)$;

(iii) If $q_1$ is equivalent to $q_2$, then $\chi(p, \sigma, q_1, s) = \chi(p, \sigma, q_2, s)$;

**Theorem 3.2.4(i)**

Let $0 \leq p_k \leq r_k$ and \( \left\{ \frac{r_k}{p_k} \right\} \) be bounded.

Then (i) $\chi(r, \sigma, q, s) \subset \chi(p, \sigma, q, s)$;

(ii) $s_1 \leq s_2$ implies $\chi(p, \sigma, q, s_1) \subset \chi(p, \sigma, q, s_2)$;

**Proof (Proof of (i))**,

Let $x \in \chi(r, \sigma, q, s)$

$$k^{-s} \left[ q \left( \sigma^k(n) \right) \left\| x_{\sigma^n(n)} \right\| \right]^{\frac{1}{k}} \rightarrow 0 \text{ as } k \rightarrow \infty$$

Let $t_k = k^{-s} \left[ q \left( \sigma^k(n) \right) \left\| x_{\sigma^n(n)} \right\| \right]^{\frac{1}{k}} \rightarrow 0$ and $\lambda_k = \frac{p_k}{r_k}$.

Since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$.

Take $0 < \lambda > \lambda_k$.

Define $u_k = t_k \ (t_k \geq 1)$; $u_k = 0 \ (t_k < 1)$; and

$v_k = 0 \ (t_k \geq 1)$; $v_k = t_k \ (t_k < 1)$; $t_k = u_k + v_k t_k^\lambda_k + v_k^\lambda$. Now it follows that

$$u_k^\lambda \leq t_k \text{ and } v_k^\lambda \leq v_k^\lambda$$

(i.e) $t_k^\lambda \leq t_k + v_k^\lambda$ by (4)
\[ k^{-s} \left[ q \left( \sigma^k(n)! \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{1/k} \right] \leq k^{-s} \left[ q \left( \sigma^k(n)! \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{1/k} \right] \]

\[ k^{-s} \left[ q \left( \sigma^k(n)! \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{1/k} \right] \leq k^{-s} \left[ q \left( \sigma^k(n)! \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{1/k} \right] \]

But \[ k^{-s} \left[ q \left( \sigma^k(n)! \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{1/k} \right] \to 0 \text{ as } k \to \infty \text{ by (3)} \]

\[ k^{-s} \left[ q \left( \sigma^k(n)! \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{1/k} \right] \to 0 \text{ as } k \to \infty . \]

Hence

\[ x \in \chi(p, \sigma, q, s) \quad (5) \]

From (2) and (5) we get \( \chi(r, \sigma, q, s) \subset \chi(p, \sigma, q, s) \). Hence the proof.

**Proof (Proof of (ii))**

It is routine verification. Therefore the proof is omitted.

**Theorem 3.2.5**

The space \( \chi(p, \sigma, q, s) \) is solid and as such is monotone.

**Proof**

Let \( (x_k) \in \chi(p, \sigma, q, s) \) and \( (\alpha_k) \) be a sequence of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \). Then
\[
\begin{align*}
\kappa^{-s} \left[ \left( \sigma^k(n) \, |\alpha_k x_{\sigma(n)}| \right)^{\frac{1}{k}} \right]^{p_k} & \leq \kappa^{-s} \left[ \left( \sigma^k(n) \, |\alpha_k x_{\sigma^k(n)}| \right)^{\frac{1}{k}} \right]^{p_k} \\
\text{for all } k \in \mathbb{N}.
\end{align*}
\]

This completes the proof.

**Theorem 3.2.6**

The space \( \chi(p, \sigma, q, s) \) are not convergence free in general.

**Proof**

The proof follows from the following example.

**Example**

Let \( s = 0 \); \( p_k = 1 \) for \( k \) even and \( p_k = 2 \) for \( k \) odd.

Let \( X = C \), \( q(x) = |x| \) and \( \sigma(n) = n + 1 \) for \( n \in \mathbb{N} \).

Then we have \( \sigma^2(n) = \sigma(\sigma(n)) = \sigma(n+1) = (n+1)+1 = n+2 \) and

\[
\sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n+2) = (n+2)+1 = n+3.
\]

Therefore, \( \sigma^k(n) = (n + k) \) for all \( n, k \in \mathbb{N} \). Consider the sequences \( (x_k) \) and \( (y_k) \)

defined as \( x_k = \left( \frac{1}{k} \right)^{k} \times \frac{1}{k!} \) and \( y_k = k^k \times \frac{1}{k!} \) for all \( k \in \mathbb{N} \).

(i.e.) \( |x_k|^k = \frac{1}{k^k} \times \frac{1}{k!} \) and \( |y_k|^k = \frac{1}{k^k} \times \frac{1}{k!} \) for all \( k \in \mathbb{N} \).
Hence \( \left( \frac{1}{n+k} \right)^{n+k} p_k \to 0 \) as \( k \to \infty \). Therefore \( (x_k) \in \chi(p, \sigma) \).

But \( \left( \frac{1}{n+k} \right)^{n+k} p_k \to 0 \) as \( k \to \infty \). Hence \( (y_k) \notin \chi(p, \sigma) \). Hence the space \( \chi(p, \sigma, q, s) \) are not convergence free in general. This completes the proof.