Chapter 4

Projection of Cyclic Codes over $R_\infty$ onto Cyclic Codes over $R_i$
Projection of Cyclic Codes over $R_\infty$ onto Cyclic Codes over $R_i$

Cyclic, negacyclic and constacyclic codes are widely studied in coding theory due to their rich algebraic structure. They can be encoded efficiently using shift registers. Thus, they are preferred in engineering. We are investigating cyclic codes over finite chain rings over finite fields in this research work. We have to find all the cyclic codes of given length over a finite chain ring over a finite field and this is our basic problem. Every ideal of the ring $\frac{R_i[x]}{<x^n-1>}$ corresponds to a cyclic code of length $n$ over the finite chain ring $R_i$ over a finite field $\mathbb{F}$. So if we can determine the total number of ideals of the ring $\frac{R_i[x]}{<x^n-1>}$ and the structure of ideals of the same ring, it will give us the answer of our problem. The structure of ideals of the ring $\frac{R_i[x]}{<x^n-1>}$ are given by the generators of the ideals. The general idea to determine the ideals of the ring $\frac{R_i[x]}{<x^n-1>}$ is to determine the structure of the ideals of the ring $\frac{R_\infty[x]}{<x^n-1>}$. Since the formal power series ring $R_\infty$ over a finite field $\mathbb{F}$ is the natural extension of the finite chain ring $R_i$, we can take projection of the coefficients of the elements of the ring $\frac{R_\infty[x]}{<x^n-1>}$ onto the ring $\frac{R_i[x]}{<x^n-1>}$. Thus any property satisfied by the ring $\frac{R_\infty[x]}{<x^n-1>}$ will also be satisfied by the ring $\frac{R_i[x]}{<x^n-1>}$ under the projection mapping. Hence at first we determine the generators of the ideals of the ring $\frac{R_\infty[x]}{<x^n-1>}$ and then by taking projection we can determine the generators of the ideals of the ring $\frac{R_i[x]}{<x^n-1>}$. In some cases we can determine the total number of ideals of
the ring \( \frac{R_\infty[x]}{<x^m - 1>_{<x^m - 1>}} \) also. Here we have taken a different approach to determine the structure of the ideals of the ring \( \frac{R_\infty[x]}{<x^m - 1>_{<x^m - 1>}} \). In the second section of this chapter, we have constructed an isomorphism between \( \frac{R_\infty[x]}{<x^m - 1>_{<x^m - 1>}} \) and \( \frac{R_n[u]}{<x^m - 1>_{<x^m - 1>}} \) and proved that cyclic codes of composite length \( mn \) over the formal power series ring \( R_\infty \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_n[x]}{<x^m - 1>_{<x^m - 1>}} \). Then by taking projection we have proved that cyclic codes of length \( mn \) over the finite chain ring \( R_i \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_i[x]}{<x^m - 1>_{<x^m - 1>}} \).

We have determined the types of ideals of the ring \( \frac{R_\infty[x]}{<x^m - 1>_{<x^m - 1>}} \) as well as the ring \( \frac{R_i[x]}{<x^m - 1>_{<x^m - 1>}} \) that will give us cyclic codes over \( R_\infty \) and \( R_i \) respectively.

### 4.1 Polynomial Rings over \( R_\infty \) and \( R_i \)

In this section, at first we prove that for any two elements in the formal power series ring over a finite field there exists a greatest common divisor. Then we shall discuss some properties of polynomials over \( R_\infty \) and \( R_i \).

#### Lemma 4.1.1[27]
If \( a \) and \( b \) are any two elements of \( R_\infty \) such that both not zero, then the greatest common divisor \( gcd(a, b) \) exists.

**Proof.** Let us assume, \( a = 0 \) and \( b \neq 0 \). Since any nonzero element \( c \) of \( R_\infty \) can be written as \( c = u\gamma^m \), where \( u \) is a unit and \( m \geq 0 \), therefore \( b = d\gamma^l \) where \( d \) is a unit and \( l \geq 0 \). Then,

\[
gcd(a, b) = gcd(0, d\gamma^l) = \gamma^l
\]

Now we assume \( a \neq 0 \) and \( b \neq 0 \). Thus \( a = a_1\gamma^i \) and \( b = b_1\gamma^j \), where \( a_1, b_1 \) are units and \( i \leq j \). Then \( \gamma^i|a \) and \( \gamma^j|b \). Let \( c \in R_\infty \) such that \( c|a \) and \( c|b \), then \( c \neq 0 \) and we have \( a = cc' \) for some \( c' \in R_\infty \). Thus we have

\[
\gamma^i = aa_1^{-1} = (cc')a_1^{-1} = c(c'a_1^{-1})
\]

which implies \( c|\gamma^i \). Hence \( gcd(a, b) = \gamma^i \).

By induction we can prove the following corollary.

#### Corollary 4.1.2[27]
If \( a_1, a_2, \ldots, a_n \in R_\infty \) such that \( a_j \neq 0 \) for some \( 0 \leq J \leq n \), then the greatest common divisor \( gcd(a_1, a_2, \ldots, a_n) \) exists. If \( a_j \) is a unit for some \( j \), then,
4.1 Polynomial Rings over $R_\infty$ and $R_i$

$$\gcd(a_1, a_2, \ldots, a_n) = 1.$$  

Let us consider the polynomial ring over $R_\infty$ given by

$$R_\infty[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in R_\infty, n \geq 0\}$$

Now $R_\infty[x]$ is a domain as $R_\infty$ is a domain\cite{27}. Let us take a polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in R_\infty[x]$$

We have the following mapping:

$$\psi_j : R_\infty[x] \to R_j[x], \quad f(x) \mapsto \psi_j(f(x)) \quad (4.2)$$

where

$$\psi_j(f(x)) = \psi_j(a_0) + \psi_j(a_1) x + \cdots + \psi_j(a_n) x^n \in R_j[x]$$

Thus we got the ring of polynomials over $R_j$ from the ring of polynomials over $R_\infty$ by projecting the coefficients of the elements in $R_\infty[x]$ onto the coefficients of the elements in $R_j[x]$.

Let us consider

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in R_j[x]$$

Again we have a mapping as follows:

$$\psi_i^j : R_j[x] \to R_i[x], \quad f(x) \mapsto \psi_i^j(f(x)) \quad (4.3)$$

where

$$\psi_i^j(f(x)) = \psi_i^j(a_0) + \psi_i^j(a_1) x + \cdots + \psi_i^j(a_n) x^n \in R_i^j[x]$$

**Definition 4.1.1**\cite{27} If $f(x) \in R_\infty[x]$ such that $\deg(f(x)) > 0$ and $\gcd(a_1, a_2, \ldots, a_n) = 1$, then $f(x)$ is called a primitive element.

**Lemma 4.1.3**\cite{27} If $f(x) \in R_\infty[x]$ such that $\deg(f(x)) > 0$, then $f(x)$ is a primitive polynomial iff $\psi_i(f(x)) \neq 0 \forall i < \infty$. 
**Proof.** If for some $i$, $\psi_i(f(x)) = 0$, then the nonzero coordinates of $f(x)$ must have the form $a_j = b_j \gamma^i$ with $l_j \geq i$. Thus we have $gcd(a_1, a_2, \ldots, a_n) = \gamma^m$ for some $m \geq i$. Which implies $f(x)$ is not a primitive polynomial.

Conversely, let $f(x) \in R_\infty[x]$ is not a primitive polynomial. Then $gcd(a_1, a_2, \ldots, a_n) = \gamma^i$ for some $i$. Thus $\psi_i(f(x)) = 0$. \(\square\)

**Theorem 4.1.4**[27] If $f(x) \in R_\infty[x]$ such that $deg(f(x)) > 0$, then there exist a unique $s$ and a primitive polynomial $g(x)$, such that $f(x) = \gamma^s g(x)$

**Proof.** Let us consider a polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R_\infty[x]$. We can express any nonzero $a_j \in R_\infty$ as $a_j = \gamma^j b_j$, where $j \geq 0$ and $b_j$ is a unit. Let

$$s = \min \{ l_j | 0 \neq a_j = \gamma^j b_j \} \tag{4.4}$$

Then

$$f(x) = \gamma^s (\gamma^{0-s}b_0 + \gamma^{1-s}b_1x + \cdots + b_s x^s + \cdots + \gamma^{n-s}b_n x^n). \tag{4.5}$$

Let

$$g(x) = \gamma^{0-s}b_0 + \gamma^{1-s}b_1x + \cdots + b_s x^s + \cdots + \gamma^{n-s}b_n x^n. \tag{4.6}$$

Thus, $f(x) = \gamma^s g(x)$.

Since $b_s \in R_\infty^*$ therefore, we have

$$gcd(\gamma^{0-s}b_0, \gamma^{1-s}b_1, \ldots, b_s, \ldots, \gamma^{n-s}b_n) = 1 \tag{4.7}$$

Which implies that $g(x)$ is a primitive polynomial. \(\square\)

**Definition 4.1.2**[27] If $< f(x) > + < g(x) >= R_i[x]$, then the polynomials $f(s), g(x) \in R_i[x]$ are called coprime, where $i < \infty$ or equivalently, if there exists $u(x), v(x) \in R_i[x]$ such that $f(x)u(x) + g(x)v(x) = 1$, then the polynomials $f(s), g(x) \in R_i[x]$ are called coprime.

### 4.2 Cyclic and Negacyclic Codes over the Ring $R_\infty$

In our study $p$ is the characteristic of the finite field $\mathbb{F}$. Thus $p$ is prime. We assume that $n$ is relatively prime to $p$. 
Let \( \lambda \) be an arbitrary unit of \( R_{\infty} \) and let
\[
\frac{R_{\infty}[x]}{< x^n - \lambda >} = \{ f(x) + < x^n - \lambda > \mid f(x) \in R_{\infty}[x] \}
\]

Let
\[
f(x) + < x^n - \lambda >, \quad g(x) + < x^n - \lambda > \quad \in \quad \frac{R_{\infty}[x]}{< x^n - \lambda >},
\]
such that \( 0 \leq \text{deg}(f(x)), \text{deg}(g(x)) < n \), and \( f(x) + < x^n - \lambda > = g(x) + < x^n - \lambda > \). Then, we have \( f(x) - g(x) \in < x^n - \lambda > \). Which implies that \( f(x) = g(x) \) as \( R_{\infty} \) is a domain. Hence, for each \( f(x) + < x^n - \lambda > \in \frac{R_{\infty}[x]}{< x^n - \lambda >} \), there is a unique \( f(x) \) with \( \text{deg}(f(x)) < n \). We can identify each coset \( f(x) + < x^n - \lambda > \) with its unique representative polynomial \( f(x) \), where \( \text{deg}(f(x)) < n \). That is,
\[
\frac{R_{\infty}[x]}{< x^n - \lambda >} = \{ f(x) + < x^n - \lambda > \mid \text{where } \text{deg}(f(x)) < n \text{ or } f(x) = 0 \}.
\]

Let us define a mapping
\[
P\lambda : R_{\infty}^n \rightarrow \frac{R_{\infty}[x]}{< x^n - \lambda >}
\]
given by
\[
(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + < x^n - \lambda > \quad (4.9)
\]

Putting \( \lambda = 1 \) and \( \lambda = -1 \) we get \( P_1 \) and \( P_{-1} \) as follows:
\[
P_1 : R_{\infty}^n \rightarrow \frac{R_{\infty}[x]}{< x^n - 1 >}
\]
given by
\[
(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + < x^n - 1 > \quad (4.10)
\]
and
\[
P_{-1} : R_{\infty}^n \rightarrow \frac{R_{\infty}[x]}{< x^n + 1 >}
\]
given by
\[
(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + < x^n - 1 > \quad (4.11)
\]

Let \( C \) be an arbitrary subset of \( R_{\infty}^n \). We denote the image of \( C \) under the map \( P\lambda \) by \( P\lambda(C) \). We use \( a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \) to denote the image of \( (a_0, a_1, \ldots, a_{n-1}) \) under \( P\lambda, P_1 \) and \( P_{-1} \) respectively.[27]
**Definition 4.2.1** [27] Let $C$ be a linear code of length $n$ over $R_\infty$. The code $C$ is called a $\lambda$-cyclic code over $R_\infty$, if

$$c = (c_0, c_1, \ldots, c_{n-1}) \in C \implies (\lambda c_{n-1}, c_0, \ldots, c_{n-2}) \in C$$

(4.14)

If $\lambda = 1$ then $C$ is called a cyclic code and if $\lambda = -1$, then $C$ is called a negacyclic code, otherwise, it is called a constacyclic code. Thus

$$P_\lambda(C) = \{c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + x^n - \lambda \mid (c_0, c_1, \ldots, c_{n-1}) \in C\}$$

Now the following lemma can be easily proved.

**Lemma 4.2.1** [27] A linear code $C$ of length $n$ over $R_\infty$ is a $\lambda$-cyclic code iff $P_\lambda(C)$ is an ideal of $\frac{R_\infty[x]}{<x^n-\lambda>}$

From **Lemma 4.2.1** we get the following corollary:

**Corollary 4.2.2** [27] Assuming the notations given above

(i) A linear code $C$ of length $n$ over $R_\infty$ is a cyclic code iff $P_1(C)$ is an ideal of $\frac{R_\infty[x]}{<x^n-1>}$;

(ii) A linear code $C$ of length $n$ over $R_\infty$ is a negacyclic code iff $P_{-1}(C)$ is an ideal of $\frac{R_\infty[x]}{<x^n+1>}.$

Let us consider the following ring homomorphism:

$$\psi_i : \frac{R_\infty[x]}{<x^n-1>} \rightarrow \frac{R_i[x]}{<x^n-1>}$$

(4.15)

given by

$$f(x) \mapsto \psi_i(f(x))$$

(4.16)

Since $\psi_i$ is a ring homomorphism, therefore if $I$ is an ideal of $\frac{R_\infty[x]}{<x^n-1>}$, then $\psi_i(I)$ is an ideal of $\frac{R_i[x]}{<x^n-1>}$.

**Theorem 4.2.3** [27] If $C$ is a cyclic code over $R_\infty$, then, $\psi_i(C)$ is a cyclic code over $R_i$ for all $i < \infty$.

Now we are going to establish an important result which is the central result of this chapter. Let $\mathbb{F}$ be a finite field and $p$ be the characteristic of $\mathbb{F}$. Thus $p$ is a prime. Let

$$R_\infty = \mathbb{F}[[\gamma]] = \{\sum_{l=0}^{\infty} a_l \gamma^l \mid a_l \in \mathbb{F}\}$$

be the formal power series ring over $\mathbb{F}$, where $\gamma$ is the
indeterminate. Let $\lambda$ be an arbitrary unit of $R_\infty$. If we consider $m$ and $n$ to be two positive integers relatively prime to $p$, then we have the following result:

**Theorem 4.2.4[37]** Assuming the notations given above we have

$$\frac{R_\infty [u]_{<u^m-\lambda>} [x]}{<x^m - u>} \cong \frac{R_\infty [x]}{<x^{mn} - \lambda>}$$

**Proof.** We shall prove it by constructing an isomorphism between $\frac{R_\infty [u]_{<u^m-\lambda>} [x]}{<x^m - u>}$ and $\frac{R_\infty [x]}{<x^{mn} - \lambda>}$. Let us consider the mapping

$$\Phi: \frac{R_\infty [u]_{<u^m-\lambda>} [x]}{<x^m - u>} \rightarrow \frac{R_\infty [x]}{<x^{mn} - \lambda>}$$

(4.17)

Which is given by

$$\Phi(x) = x, \quad \Phi(u) = x^m, \quad \text{and} \quad \Phi(a_{i,j}) = a_{i,j}, \quad \forall \ i, j \quad \text{where} \quad a_{i,j} \in R_\infty$$

(4.18)

**Claim:** $\Phi$ is an isomorphism.

Let us consider

$$(a_{0,0} + a_{1,0}u + \cdots + a_{n-1,0}u^{n-1}) + (a_{0,1} + a_{1,1}u + \cdots + a_{n-1,1}u^{n-1})x$$

$$+ \cdots + (a_{0,m-1} + a_{1,m-1}u + \cdots + a_{n-1,m-1}u^{n-1})x^{m-1} \in \frac{R_\infty [u]_{<u^m-\lambda>} [x]}{<x^m - u>}$$

$$\Rightarrow \left( \sum_{i=0}^{n-1} a_{i,0}u^i \right) + \left( \sum_{i=0}^{n-1} a_{i,1}u^i \right)x + \cdots + \left( \sum_{i=0}^{n-1} a_{i,m-1}u^i \right)x^{m-1} \in \frac{R_\infty [u]_{<u^m-\lambda>} [x]}{<x^m - u>}$$

$$\Rightarrow \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j}u^j \right)x^j \in \frac{R_\infty [u]_{<u^m-\lambda>} [x]}{<x^m - u>}$$

(4.19)

Now for

$$a_{0,0} + a_{0,1}x + \cdots + a_{0,m-1}x^{m-1} + a_{1,0}x^m + a_{1,1}x^{m+1} + \cdots + a_{1,m-1}x^{2m-1}$$

$$+ \cdots + a_{n-1,0}x^{m(n-1)} + \cdots + a_{n-1,m-1}x^{mn-1} \in \frac{R_\infty [x]}{<x^{mn} - \lambda>}$$
There exists
\[ \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \in R_\infty \left\langle \frac{\left< u^e - \lambda \right>}{x^m - u} \right\rangle \]
Such that
\[ \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \right) = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} x^m \right) x^j \]
\[ = \sum_{j=0}^{m-1} \left( a_{0,j} x^0 + a_{1,j} x^m + \cdots + a_{n-1,j} x^{m(n-1)} \right) x^j \]
\[ = a_{0,0} + a_{0,1} x + \cdots + a_{0,m-1} x^{m-1} + a_{1,0} x^m + a_{1,1} x^{m+1} \]
\[ + \cdots + a_{1,m-1} x^{2m-1} + \cdots + a_{n-1,0} x^{m(n-1)} + \cdots + a_{n-1,m-1} x^{mn-1} \]
\[ \in R_\infty \left\langle \frac{x^m - \lambda}{x^{mn} - \lambda} \right\rangle \]
Therefore the mapping \( \Phi \) is onto.

To prove \( \Phi \) is one-one, we take
\[ \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \right) = \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i \right) x^j \right) \tag{4.20} \]
\[ \implies \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} x^m \right) x^j = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} x^m \right) x^j \]
\[ \implies \sum_{j=0}^{m-1} \left( a_{0,j} x^0 + a_{1,j} x^m + \cdots + a_{n-1,j} x^{m(n-1)} \right) x^j \]
\[ = \sum_{j=0}^{m-1} \left( b_{0,j} x^0 + b_{1,j} x^m + \cdots + b_{n-1,j} x^{m(n-1)} \right) x^j \]
\[ \implies a_{0,0} + a_{0,1} x + \cdots + a_{0,m-1} x^{m-1} + a_{1,0} x^m + a_{1,1} x^{m+1} + \cdots \]
\[ + a_{1,m-1} x^{2m-1} + \cdots + a_{n-1,0} x^{m(n-1)} + \cdots + a_{n-1,m-1} x^{mn-1} \]
\[ = b_{0,0} + b_{0,1} x + \cdots + b_{0,m-1} x^{m-1} + b_{1,0} x^m + b_{1,1} x^{m+1} + \cdots \]
\[ + b_{1,m-1} x^{2m-1} + \cdots + b_{n-1,0} x^{m(n-1)} + \cdots + b_{n-1,m-1} x^{mn-1} \]
\[ \implies a_{0,0} = b_{0,0}, \ a_{0,1} = b_{0,1}, \ldots, \ a_{n-1,m-1} = b_{n-1,m-1} \tag{4.21} \]
\[ \implies \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i \right) x^j \tag{4.22} \]
Thus $\Phi$ is one-one and hence it is a bijection.

Now for
\[
\sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i x^j \right) + \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i x^j \right) \in \frac{R_{n[u]}[x]}{<x^m-u>}
\]

\[
\Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i x^j \right) \right) + \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i x^j \right) \right) = \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} \left( a_{i,j} + b_{i,j} \right) u^i x^j \right) \right)
\]

\[
\implies \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i x^j \right) \right) + \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i x^j \right) \right) = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} \left( a_{i,j} + b_{i,j} \right) u^i x^j \right)
\]

\[
\implies \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i x^j \right) \right) + \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i x^j \right) \right) = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i x^j \right) + \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i x^j \right)
\]

Hence $\Phi$ preserves addition.

Let us consider
\[
a_{i,j} u^i x^j, \ b_{r,s} u^r x^s \in \frac{R_{n[u]}[x]}{<x^m-u>}
\]

Now we have
\[
a_{i,j} u^i x^j \cdot b_{r,s} u^r x^s = a_{i,j} b_{r,s} u^{i+r} x^{j+s} \in \frac{R_{n[u]}[x]}{<x^m-u>},
\]

where $x^m = u$, $u^n = \lambda$ and $(a_{i,j}, b_{r,s}) \in R_\infty$

\[
\therefore \Phi(a_{i,j} u^i x^j) = a_{i,j} (x^m)^i x^j \text{ and } \Phi(b_{r,s} u^r x^s) = b_{r,s} (x^m)^r x^s
\]

\[
\therefore \Phi(a_{i,j} u^i x^j) \cdot \Phi(b_{r,s} u^r x^s) = a_{i,j} b_{r,s} x^{m(i+r)+j+s} \quad (4.23)
\]

\[
\Phi(a_{i,j} u^i x^j \cdot b_{r,s} u^r x^s) = \Phi(a_{i,j} b_{r,s} u^{i+r} x^{j+s}) = a_{i,j} b_{r,s} x^{m(i+r)+j+s} \quad (4.24)
\]

Hence from 4.23 and 4.24
\[
\Phi(a_{i,j} u^i x^j \cdot b_{r,s} u^r x^s) = \Phi(a_{i,j} u^i x^j) \cdot \Phi(b_{r,s} u^r x^s) \quad (4.25)
\]

This implies that $\Phi$ preserves multiplication.

Thus it is proved that $\Phi$ is an isomorphism.
Hence proved.

**Corollary 4.2.5[37]** Assuming the notations given above we have

\[
\frac{R_u[x]}{<u^m-1>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

\[
\frac{R_u[x]}{<u^m-\lambda>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

**Proof.** We know that

\[
\frac{R_u[x]}{<u^m-\lambda>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

Putting \( \lambda = 1 \) in 4.26, we get

\[
\frac{R_u[x]}{<u^m-1>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

Hence proved.

**Corollary 4.2.6[37]** Assuming the notations given above we have

\[
\frac{R_u[x]}{<u^m+1>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

**Proof.** We know that

\[
\frac{R_u[x]}{<u^m-\lambda>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

Putting \( \lambda = -1 \) in 4.27, we get

\[
\frac{R_u[x]}{<u^m+1>}[x] \cong \frac{R_\infty[x]}{<x^m-u>}
\]

Hence proved.

Thus we have established that cyclic codes of composite length \( mn \) over the formal power series ring \( R_\infty \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_u[x]}{<u^m-1>} \). Similarly, negacyclic codes of composite length \( mn \) over the formal power series ring \( R_\infty \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_u[x]}{<u^m+1>} \). Now by taking projection we are going to prove that \( \lambda \)-cyclic codes of length \( mn \) over the finite chain ring \( R_i \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_i[x]}{<u^m-\lambda>} \).
Theorem 4.2.7[37] Assuming the notations given above we have

\[
\frac{R_{|u|}}{<w^L - \lambda> [x]} \sim \frac{R_i}{<x^m - u>}
\]

Proof. Let us consider

\[
(a_{0,0} + a_{1,0}u + \cdots + a_{n-1,0}u^{n-1})x
\]
\[
+ \cdots + (a_{0,m-1} + a_{1,m-1}u + \cdots + a_{n-1,m-1}u^{n-1})x^{m-1}
\in \frac{R_{|u|}}{<w^L - \lambda> [x]} \sim \frac{R_i}{<x^m - u>}
\]

Now we have

\[
\psi_i((a_{0,0} + a_{1,0}u + \cdots + a_{n-1,0}u^{n-1}) + (a_{0,1} + a_{1,1}u + \cdots + a_{n-1,1}u^{n-1})x
\]
\[
+ \cdots + (a_{0,m-1} + a_{1,m-1}u + \cdots + a_{n-1,m-1}u^{n-1})x^{m-1})
\]
\[
= (\psi_i(a_{0,0}) + \psi_i(a_{1,0})u + \cdots + \psi_i(a_{n-1,0})u^{n-1}) + (\psi_i(a_{0,1}) + \psi_i(a_{1,1})u + \cdots + \psi_i(a_{n-1,1})u^{n-1})x
\]
\[
+ \cdots + (\psi_i(a_{0,m-1}) + \psi_i(a_{1,m-1})u + \cdots + \psi_i(a_{n-1,m-1})u^{n-1})x^{m-1}
\in \frac{R_{|u|}}{<w^L - \lambda> [x]} \sim \frac{R_i}{<x^m - u>}
\]
\[
\therefore \psi_i(a_{0,0}), \psi_i(a_{1,0}), \ldots, \psi_i(a_{n-1,m-1}) \in R_i \subset R_{i+1} \subset \cdots \subset R_{\infty}
\]
\[
\therefore \Phi(\psi_i(a_{0,0})) = \psi_i(a_{0,0}), \Phi(\psi_i(a_{1,0})) = \psi_i(a_{1,0}), \ldots, \Phi(\psi_i(a_{n-1,m-1})) = \psi_i(a_{n-1,m-1})
\]
\[
\therefore \Phi^{-1}(\psi_i(a_{0,0})) = \psi_i(a_{0,0}), \Phi^{-1}(\psi_i(a_{1,0})) = \psi_i(a_{1,0}), \ldots, \Phi^{-1}(\psi_i(a_{n-1,m-1})) = \psi_i(a_{n-1,m-1})
\]

Thus we have

\[
\Phi(\psi_i((a_{0,0} + a_{1,0}u + \cdots + a_{n-1,0}u^{n-1}) + (a_{0,1} + a_{1,1}u + \cdots + a_{n-1,1}u^{n-1})x
\]
\[
+ \cdots + (a_{0,m-1} + a_{1,m-1}u + \cdots + a_{n-1,m-1}u^{n-1})x^{m-1}) = \psi_i(a_{0,0}) + \psi_i(a_{1,1})x
\]
\[
+ \cdots + \psi_i(a_{0,m-1})x^{m-1} + \psi_i(a_{1,0})x^m + \psi_i(a_{1,1})x^{m+1} + \cdots + \psi_i(a_{1,m-1})x^{2m-1}
\]
\[
+ \cdots + \psi_i(a_{n-1,0})x^{m(n-1)} + \cdots + \psi_i(a_{n-1,m-1})x^{mn-1}
\in \frac{R_i}{<x^m - u>}
\]

Again we have

\[
\psi_i(a_{0,0} + a_{0,1}x + \cdots + a_{0,m-1}x^{m-1} + a_{1,0}x^m + a_{1,1}x^{m+1} + \cdots + a_{1,m-1}x^{2m-1} +
\]
\[
\cdots + a_{n-1,0}x^{m(n-1)} + \cdots + a_{n-1,m-1}x^{mn-1}) = \psi_i(a_{0,0}) + \psi_i(a_{0,1})x + \cdots +
\]
\[
\psi_i(a_{0,m-1})x^{m-1} + \psi_i(a_{1,0})x^m + \psi_i(a_{1,1})x^{m+1} + \cdots + \psi_i(a_{1,m-1})x^{2m-1} + \\
\cdots + \psi_i(a_{n-1,0})x^{m(n-1)} + \cdots + \psi_i(a_{n-1,m-1})x^{mn-1} \in \frac{R_i[x]}{<x^{mn} - \lambda>}
\]

\[
\implies \Phi^{-1}(\psi_i(a_{0,0} + a_{0,1}x + \cdots + a_{0,m-1}x^{m-1} + a_{1,0}x^m + a_{1,1}x^{m+1} + \cdots + a_{1,m-1}x^{2m-1} + \cdots + a_{n-1,0}x^{m(n-1)} + \cdots + a_{n-1,m-1}x^{mn-1}))
\]

\[
= \Phi^{-1}(\psi_i(a_{0,0}) + \psi_i(a_{0,1})x + \cdots + \psi_i(a_{0,m-1})x^{m-1} + \psi_i(a_{1,0})x^m + \psi_i(a_{1,1})x^{m+1} + \cdots + \psi_i(a_{1,m-1})x^{2m-1} + \cdots + \psi_i(a_{n-1,0})x^{m(n-1)} + \cdots + \psi_i(a_{n-1,m-1})x^{mn-1})
\]

\[
= (\psi_i(a_{0,0}) + \psi_i(a_{1,0})u + \cdots + \psi_i(a_{n-1,0})u^{n-1}) + (\psi_i(a_{0,1}) + \psi_i(a_{1,1})u + \cdots + \psi_i(a_{n-1,1})u^{n-1})x + \cdots + (\psi_i(a_{0,m-1}) + \psi_i(a_{1,m-1})u)
\]

\[
\cdots + (\psi_i(a_{n-1,m-1})u^{n-1})x^{m-1} \in \frac{R_i[u]}{<u^m - \lambda> \cdot x^m - u >}
\]

Thus \(\Phi\) is a bijection between \(\frac{R_i[u]}{<u^m - \lambda> \cdot x^m} \) and \(\frac{R_i[x]}{<x^{mn} - \lambda> \cdot x^m - u >}\). Again \(\Phi\) preserves addition and multiplication between \(\frac{R_i[u]}{<u^m - \lambda> \cdot x^m} \) and \(\frac{R_i[x]}{<x^{mn} - \lambda> \cdot x^m - u >}\) as it preserves both the operations between \(\frac{R_i[u]}{<u^m - \lambda> \cdot x^m} \) and \(\frac{R_i[x]}{<x^{mn} - \lambda> \cdot x^m - u >}\).

\[
\implies \frac{R_i[u]}{<u^m - \lambda>} \approx \frac{R_i[x]}{<x^{mn} - \lambda>}
\]

Hence proved. \(\square\)

Putting \(\lambda = 1\) and \(\lambda = -1\), we get the following two corollaries:

**Corollary 4.2.8** Assuming the notations given above we have

\[
\frac{R_i[u]}{<u^m - 1>} \approx \frac{R_i[x]}{<x^{mn} - 1>}
\]

**Corollary 4.2.9** Assuming the notations given above we have

\[
\frac{R_i[u]}{<u^m + 1>} \approx \frac{R_i[x]}{<x^{mn} + 1>}
\]
4.2 Cyclic and Negacyclic Codes over the Ring $R_{\infty}$

**Theorem 4.2.10** [37] Let $m$ and $n$ are two odd numbers and $\gcd(m, p) = 1$, $\gcd(n, p) = 1$. Then

$$\frac{R_{\infty}[u]}{<x^{m-u}>} \cong \frac{R_{\infty}[u]}{<x^{m-n}>} \cong \frac{R_{\infty}[u]}{<x^{n-u}>}$$

**Proof.** Since $m$ and $n$ both are odds, $mn$ is also odd. Again $\gcd(m, p) = 1$ and $\gcd(n, p) = 1$. Therefore $\gcd(mn, p) = 1$. We define the map

$$\eta : \frac{R_{\infty}[x]}{<x^{mn}+1>} \rightarrow \frac{R_{\infty}[x]}{<x^{mn}-1>}$$

Given by

$$f(x) + <x^{mn}+1> \rightarrow f(-x) + <x^{mn}-1>$$

Now if

$$f(x) + <x^{mn}+1> = g(x) + <x^{mn}+1>$$

Then we have

$$f(x) - g(x) \in <x^{mn}+1>$$

$$\therefore f(x) - g(x) = (x^{mn}+1)q(x) \text{ for some } q(x)$$

$$\therefore f(-x) - g(-x) = (-x)^{mn}q(-x) = (-x^{mn}+1)q(-x) = (x^{mn}-1)(q(-x)) \in <x^{mn}-1>$$

This implies that

$$\eta(f(x) + <x^{mn}+1>) = f(-x) + <x^{mn}-1> = g(-x) + <x^{mn}-1> = \eta(g(x) + <x^{mn}+1>).$$

Thus the correspondence $\eta$ is a well-defined map. It can be easily proved that $\eta$ is an isomorphism.

$$\therefore \frac{R_{\infty}[u]}{<x^{m-u}>} \cong \frac{R_{\infty}[u]}{<x^{m-n}>} \cong \frac{R_{\infty}[u]}{<x^{n-u}>}$$

Hence proved. $\square$

**Theorem 4.2.11** [37] A linear code $C$ of length $mn$ over $R_{\infty}$ is a $\lambda-$cyclic code if and only if $\Phi^{-1}(P_{\lambda}(C))$ is an ideal of $\frac{R_{\infty}[u]}{<x^{m-u}>}$.
Proof. From Theorem 4.2.1 we know that, a linear code \( C \) of length \( mn \) over \( R_\infty \) is a \( \lambda \)-cyclic code if and only if \( P_\lambda(C) \) is an ideal of \( \frac{R_\infty[x]}{<x^{mn}\lambda>} \). Again \( \Phi \) is an isomorphism between \( \frac{R_\infty[x]}{<x^{mn}\lambda>} \) and \( \frac{R_\infty[x]}{<x^{m\lambda}>} \). Thus \( \Phi^{-1} \) is an isomorphism. So \( \Phi^{-1}(P_\lambda(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{m\lambda}>} \) if and only if \( (P_\lambda(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{mn}\lambda>} \). Thus A linear code \( C \) of length \( mn \) over \( R_\infty \) is a \( \lambda \)-cyclic code if and only if \( \Phi^{-1}(P_\lambda(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{m\lambda}>} \). \( \square \)

Corollary 4.2.12[37] Assuming the notations given above we have

(i) A linear code \( C \) of length \( mn \) over \( R_\infty \) is a cyclic code if and only if \( \Phi^{-1}(P_1(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{m-u}>} \).

(ii) A linear code \( C \) of length \( mn \) over \( R_\infty \) is a negacyclic code if and only if \( \Phi^{-1}(P_{-1}(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{m-u}>} \).

Theorem 4.2.13[37] If \( C \) is a cyclic code of length \( mn \) over \( R_\infty \), then \( \Phi^{-1}(\psi_i(P_i(C))) \) is an ideal of \( \frac{R_i[x]}{<x^{m-i}>} \).

Proof. From Theorem 4.2.3 we know that if \( C \) is a cyclic code over \( R_\infty \), then \( \psi_i(C) \) is a cyclic code over \( R_i \) for all \( i < \infty \). Thus if \( C \) is a cyclic code of length \( mn \) over \( R_\infty \) then \( \psi_i(P_1(C)) \) is an ideal of \( \frac{R_i[x]}{<x^{m-i}>} \). As \( \Phi \) is an isomorphism between \( \frac{R_i[x]}{<x^{m-i}>} \) and \( \frac{R_\infty[x]}{<x^{m-u}>} \), \( \Phi^{-1} \) is an isomorphism between \( \frac{R_i[x]}{<x^{m-i}>} \) and \( \frac{R_\infty[x]}{<x^{m-u}>} \). Hence \( \psi_i(P_1(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{m-u}>} \) if and only if \( \Phi^{-1}(\psi_i(P_1(C))) \) is an ideal of \( \frac{R_i[x]}{<x^{m-i}>} \). Thus if \( C \) is a cyclic code of length \( mn \) over \( R_\infty \), then \( \Phi^{-1}(\psi_i(P_i(C))) \) is an ideal of \( \frac{R_i[x]}{<x^{m-i}>} \). \( \square \)

4.3 Conclusion

In [27] Dougherty and Liu proved that corresponding to every cyclic code of odd length \( n \) over \( R_\infty \) there exists a negacyclic code of same length over \( R_\infty \). Here we have considered both \( m \) and \( n \) as odd numbers and proved that \( u \)-constacyclic codes of length \( m \) over \( \frac{R_\infty[x]}{<x^{m+1}>} \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_\infty[x]}{<x^{m+1}>} \). Neither a counter example have been found to disprove that \( u \)-constacyclic codes of length \( m \) over \( \frac{R_\infty[x]}{<x^{m+1}>} \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_\infty[x]}{<x^{m+1}>} \), nor any isomorphism has been constructed between \( \frac{R_\infty[x]}{<x^{m+1}>} \) and \( \frac{R_\infty[x]}{<x^{m+1}>} \) to prove that \( u \)-constacyclic codes of length \( m \) over \( \frac{R_\infty[x]}{<x^{m+1}>} \) corresponds to \( u \)-constacyclic codes of length \( m \) over \( \frac{R_\infty[x]}{<x^{m+1}>} \), when at least one of \( m \) or \( n \) is
even. Hence still the problem whether \( \frac{R_0}{<x^m-u>} \) is isomorphic to \( \frac{R_0}{<x^m-u>} \) or not is unsolved, when at least one of \( m \) or \( n \) is even.