Modulation instability and rogue wave solutions with polarisation force

3.1 Introduction

In this work, we present the formation of Rogue wave, also known as Peregrine soliton or envelope soliton, in a dense dusty plasma with polarization force. The initial nonlinear theory has already been developed in the last Chapter. We however now assume a more generalised electron and ion distribution function based on the $q$-nonextensive
3.2 The $q$-nonextensive distribution

distribution function [62]. The familiar Maxwellian distribution function can be thought to be subset of the $q$-nonextensive distribution function, which itself can be thought to be a subset of a more generalised class of distribution function — the Cairns-Tsallis distribution function [63].

Non-Maxwellian or non-thermal distribution functions appear when particles in a system can not relax to a thermal equilibrium or do not have enough time to relax. Examples of non-thermal velocity distribution can be found both in astrophysical and laboratory environments. The non-thermal velocity distribution for plasma’s particles in near-earth plasmas is confirmed from the observations of Freja satellite and Viking spacecraft [64, 65]. Many analyses related to these plasmas have used the Cairn’s model [63]. Verheest and Pillay [66] investigated large amplitude dust-acoustic solitary waves in plasmas with negatively charged cold dust and either non-thermally distributed ions or electrons. Verheest and Hellberg [67] examined large ion acoustic solitary waves and double layers in plasmas with positive ions and non-thermal electrons.

On the other hand, we have also seen in the last Chapter that, polarisation force can affect the propagation of DAW and DIAW significantly in laboratory plasmas. As laboratory plasmas usually do not have long enough time to relax to a thermally equilibrated velocity distribution, it will be interesting to investigate the formation of large amplitude solitary waves in such plasmas, which have high energy tail in the distribution and can thus be modeled by a non-thermal distribution such as $q$-nonextensive distribution.

3.2 The $q$-nonextensive distribution

The $q$-nonextensive distribution, also known as the Tsallis distribution, arises from the deviation of the Boltzmann-Gibbs-Shannon (BGS) entropic measure [68, 62]. The parameter $q$ represents the degree of nonextensivity and $q \rightarrow 1$ corresponds to the
standard extensive BGS statistics and we recover the Maxwell-Boltzmann velocity distribution. The one-dimensional $q$-nonextensive distribution function can be written as

$$f(v) = C_q \left[ 1 - (q - 1) \left( \frac{mv^2}{2T} + \frac{e\phi}{T} \right) \right]^{1/(q-1)},$$  \hspace{1cm} (3.1)

where $m$ is the mass of the particles, $T$ is the temperature expressed in the energy unit, and $e$ is the charge. $C_q$ is the normalisation factor,

$$C_q = \begin{cases} 
  n \frac{\Gamma \left( \frac{1}{1-q} \right)}{\Gamma \left( \frac{1}{1-q} - 1 \right)} \sqrt{\frac{m(1-q)}{2\pi T}}, & \text{for } -1 < q < 1 \\
  n \left( \frac{1+q}{2} \right) \frac{\Gamma \left( \frac{1}{1-q} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{1-q} \right)}, & \text{for } q > 1
\end{cases}$$  \hspace{1cm} (3.2)

where $n$ is the density of particles. We would like to point out that ‘$e$’ is used here as the charge of the particle, not electronic charge. So, $e \to -e$ for electrons $e \to +e$ for a proton. In the limit $q \to 1$, one recovers the well-known Maxwellian distribution function

$$f(v) = n \left( \frac{m}{2\pi T} \right)^{1/2} \exp \left( -\frac{1}{2} \frac{mv^2}{T} - \frac{e\phi}{T} \right).$$  \hspace{1cm} (3.3)

We should however note that the distribution function exhibits a thermal cutoff on the velocity of the particles $v \leq v_{\text{max}}$,

$$v_{\text{max}} = \left( \frac{2T}{m(q-1)} + \frac{2e\phi}{m} \right)^{1/2}.$$  \hspace{1cm} (3.4)
An integration over the velocity space yields the total density,

\[
\begin{align*}
3.3 & \text{ The plasma model} \\
\int_{-v_{\text{max}}}^{+v_{\text{max}}} f(v) \, dv, & \quad q > 1 \\
\int_{-v_{\text{max}}}^{+v_{\text{max}}} f(v) \, dv, & \quad 1 < q < 1 \\
\int_{-v_{\text{max}}}^{+v_{\text{max}}} f(v) \, dv, & \quad q > 1 \\
\int_{-v_{\text{max}}}^{+v_{\text{max}}} f(v) \, dv, & \quad q > 1
\end{align*}
\]

\[
= n \left[ 1 - (q - 1) \frac{e\phi}{T} \right]^{1/(q-1)+1/2}.
\]

\[\text{(3.5)}\]

\[
3.3 \text{ The plasma model}
\]

The basic equations including polarization force, governing the dynamics of dust particles, are the equations of continuity and momentum, followed by the Poisson equation,

\[
\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x}(n_d u_d) = 0,
\]

\[\text{(3.6)}\]

\[
m_d n_d \frac{du_d}{dt} + \frac{\partial p_d}{\partial x} = -Q_d n_d (1 - R) \frac{\partial \phi_p}{\partial x},
\]

\[\text{(3.7)}\]

\[
\epsilon_0 \frac{\partial^2 \phi_p}{\partial x^2} = -Q_d n_d + \epsilon(n_e - n_i),
\]

\[\text{(3.8)}\]

where \(Q_d = q_d z_d\) is the charge on the dust particle and \(d/dt \equiv \partial/\partial t + u_d \partial/\partial x\). The fraction of polarization force \(R\) is given by,

\[
R = \frac{1}{4} \beta_T \left( 1 - \frac{T_{i0}}{T_{e0}} \right) = \frac{1}{16\pi \epsilon_0 \lambda_D T_{i0}} \left( 1 - \frac{T_{i0}}{T_{e0}} \right),
\]

\[\text{(3.9)}\]

where \(\beta_T\) is the ratio of the Coulomb radius of interaction between the ions and dust particles and Debye length. The rest of the symbols have their usual meanings. The electron and ion densities are assumed to be functions of the plasma potential \(\phi_p\) only, \(n_{e,i} \equiv F(\phi_p)\). An example of such a function could be the Boltzmannian electron and ion distributions \(n_{e,i} = n_{e,i0} e^{\pm e\phi_p/T_{e,i0}}\), with ’\pm’ sign respectively for electrons and ions. We however, at this point retain the possibility of having more exotic distribution
functions such as the $q$-nonextensive distribution, which might be relevant in these plasmas. The temperature is expressed in energy unit. The Debye shielding length is defined as,

$$
\lambda_D = \frac{\lambda_{D_e} \lambda_{D_i}}{\sqrt{\lambda_{D_e}^2 + \lambda_{D_i}^2}} \equiv \left( \frac{\epsilon_0 T_{\text{eff}}}{n_d Q_d^2} \right)^{1/2},
$$

(3.10)

where $n_{d0}$ is the equilibrium dust density and $Q_d = -e z_d$, assuming negatively charged dust particles with dust charge number $z_d$. Note that the dust charge $Q_d$ is related to the relative dust potential $\varphi_d$ as (assuming a spherical dust grain) $Q_d \equiv C \varphi_d$, $\varphi_d = \phi_d - \phi_p$, where $C$ is the capacitance of the spherical dust grain, $\phi_p$ is the absolute grain potential (with respect to zero). The effective temperature $T_{\text{eff}}$ is defined as,

$$
T_{\text{eff}} = z_{d0} \left( \frac{T_{\text{eff}}}{\mu_e T_{\text{eff}} + \mu_i T_{\text{eff}}} \right),
$$

(3.11)

where we have expressed the ratios of densities as $\mu_{i,e} = n_{i0,e0} / (z_d n_{d0})$. The subscript ’0’ refers to equilibrium quantities. In terms of these normalized densities, the charge neutrality condition is given by $\mu_i = 1 + \mu_e$.

We have chosen to normalize length by $\lambda_D$ and time by $\omega_d^{-1}$, the inverse of dust plasma frequency. These also define the dust-acoustic velocity $c_d = \omega_d \lambda_D = (T_{\text{eff}}/m_d)^{1/2}$, $\omega_d = (n_{d0} Q_d^2 / \epsilon_0 m_d)^{1/2}$. The electron and ion densities are normalized by $(z_{d0} n_{d0})$, dust density by $n_{d0}$, electrostatic potential by $T_{\text{eff}}/(e z_d)$, velocities by $c_d$. The dust thermal pressure can be expressed through density $p_d \propto n_d$ (note that $\gamma = 1$, in this case, which is also consistent with the assumption of constant temperatures) and normalized by the equilibrium value $p_{d0} = n_{d0} T_{d0}$. With these normalizations,
Eqs. (3.6-3.8) become,

\[
\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x} (n_d u_d) = 0, \quad (3.12)
\]

\[
n_d \frac{du_d}{dt} = z_d n_d (1 - \mathcal{R}) \frac{\partial \phi_p}{\partial x} - \sigma \frac{\partial n_d}{\partial x}, \quad (3.13)
\]

\[
\frac{\partial^2 \phi_p}{\partial x^2} = z_d n_d - n_i + n_e. \quad (3.14)
\]

The quantity \( \mathcal{R} \) denotes the normalized equilibrium dust temperature \( T_{d0}/T_{e0} \). It is to be noted that \( \mathcal{R} \) is an equilibrium quantity and does not get evolved during a perturbation. We express the polarization factor as \( \mathcal{R} = R_0 z_d^2 n_d^{1/2} \), with

\[
R_0 = \frac{1}{16 \pi \epsilon_0} \frac{e|Q_d|}{\lambda_{D0} T_{i0}} (1 - \beta), \quad (3.15)
\]

where \( \lambda_{D0} \) is the Debye length corresponding to the equilibrium dust density \( n_{d0} \) and \( \beta = T_{i0}/T_{e0} \).

### 3.3.1 Dust charge and polarization

We assume that the net current to the surface of a dust particle remains zero and consider the current balance equation \( I_i + I_e = 0 \), \( I_{i,e} \) being the currents due to the ions and electrons. For negatively charged spherical dust grains of radius \( r_d \), we have the following expressions for \( I_{i,e} \) (un-normalized),

\[
I_i = \pi r_d^2 e n_i \left( \frac{8 T_i}{\pi m_i} \right)^{1/2} \left( 1 - \frac{e \varphi_d}{T_{i0}} \right), \quad I_e = -\pi r_d^2 e n_e \left( \frac{8 T_{e0}}{\pi m_e} \right)^{1/2} \exp \left( \frac{e \varphi_d}{T_{e0}} \right). \quad (3.16)
\]

We note that appropriate distributions of electrons and ions are required to exactly determine these currents as they involve the densities \( n_{i,e} \). However, as in a thermal plasma in equilibrium, the electron and ion distribution functions always relax to a Maxwellian, without loss of any generality we can assume the electron and ion
distributions to be Maxwellian for the purpose of determining the dust currents and assume that deviation of these distribution functions from a Maxwellian will not affect the dust currents appreciably. Under these assumptions, the normalized ion and electron densities are respectively given by \( n_{e,i} = \mu_{e,i} e^{\nu_{e,i} \phi_p} \), where \( \nu_e = \beta/(\mu_i + \mu_e \beta) \), and \( \nu_i = -1/(\mu_i + \mu_e \beta) \). With these expressions, the normalized current balance equation can be written as,

\[
-\delta_m \mu_e e^{\nu_e (\varphi_d + \phi_p)} + \mu_i \beta^{1/2} e^{\nu_i \phi_p} (1 + \nu_i \varphi_d) = 0, \quad (3.17)
\]

where \( \delta_m = \sqrt{m_i/m_e} \approx 43 \). The general solution of Eq. (3.17) can be written in terms of Lambert W function,

\[
\varphi_d = -\frac{1}{\nu_i} - \frac{1}{\nu_e} W \left[ \delta_m \beta^{1/2} \mu_e \frac{\mu_e}{\mu_i} e^{\beta - \nu_i (1+\beta) \phi_p} \right], \quad (3.18)
\]

where we need to take only the principal value. For spherical dust particles, the dust charge \( q_d z_d = r_d \varphi_d \), which can be used to express the dust charge number \( z_d \) as a function of dust potential \( \varphi_d \) relative to the dust charge number at zero potential

\[
z_d(\phi_p) = \frac{\varphi_d(\phi_p)}{\varphi_{d0}}, \quad (3.19)
\]

where \( \varphi_{d0} = \varphi_d(\phi_p)|_{\phi_p=0} \). This expression for \( z_d \) needs to be taken into account in while evaluating the polarization effect in momentum equation, Eq.(3.13).
3.3 The plasma model

3.3.2 Nonlinear expansions

In order to investigate the modulation instability, we use the reductive perturbation method. Following the standard procedure, we stretch the dependent variables as

\[ \xi = \epsilon(x - vt), \quad \tau = \epsilon^2 t. \] \hspace{1cm} (3.20)

We now consider the following nonlinear expansion

\[ F(x, t, \xi, \tau) = F_0 + \sum_{j=1}^{J} \epsilon^j \sum_{l=-L}^{L} f_{jl}(\xi, \tau) e^{i(kx-\omega t)}, \] \hspace{1cm} (3.21)

where

\[ \begin{aligned}
    F &= (n_d, u_d, \phi_p)', \\
    f &= (n, u, \phi_p)', \\
    n_{d0} &= 1, \\
    u_{d0} &= \phi_{p0} = 0,
\end{aligned} \] \hspace{1cm} (3.22)

The space and time derivatives are now mapped to the new set of derivatives as

\[ \begin{aligned}
    \frac{\partial}{\partial t} &\rightarrow \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau} - \epsilon v \frac{\partial}{\partial \xi}, \\
    \frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi}, \\
    \frac{\partial^2}{\partial x^2} &\rightarrow \frac{\partial^2}{\partial x^2} + 2 \epsilon \frac{\partial^2}{\partial x \partial \xi} + \epsilon^2 \frac{\partial^2}{\partial \xi^2}.
\end{aligned} \] \hspace{1cm} (3.23, 3.24, 3.25)

We assume that the electron and ion densities \((n_i - n_e)\) can be expanded into a power series in the plasma potential \(\phi_p\).

\[ n_i - n_e \simeq 1 + c_1 \phi_p + c_2 \phi_p^2 + c_3 \phi_p^3, \] \hspace{1cm} (3.26)
where the constants of expansion $c_j$s depend on particular distribution of densities which we use.

The first term on the right hand side of Eq. (3.13), which represents the polarization effect, is

$$z_d n_d (1 - R) \frac{\partial \phi_p}{\partial x} = z_d n_d \left( 1 - R_0 z_d^{2 \mu_e^{1/2}} \right) \frac{\partial \phi_p}{\partial x},$$

(3.27)

where $z_d$ is now given by Eq. (3.19) as,

$$z_d = \frac{W \left( \vartheta e^{\vartheta \phi_p} \right) - \beta}{W(\vartheta) - \beta},$$

(3.28)

$$\vartheta = \frac{1 + \beta}{\beta \mu_e + \mu_i}, \quad \vartheta = \delta_{m} \beta^{1/2} e^{\beta \mu_e} \mu_i.$$

(3.29)

The Lambert $W$ function has the following expansion near zero,

$$W(x) \approx x - x^2 + \frac{3}{2} x^3 + O(x^4),$$

(3.30)

from which we have an expansion for $z_d(\phi_p)$ near $\phi_p = 0$ as,

$$z_d \approx 1 + d_1 \phi_p + d_2 \phi_p^2 + d_3 \phi_p^3 + O\left(\phi_p^4\right),$$

(3.31)

where

$$d_1 = \frac{W}{(1 + W)(W - \beta) (\beta \mu_e + \mu_i)},$$

(3.32)

$$d_2 = \frac{W - \beta}{2W(1 + W)} d_1^2,$$

(3.33)

$$d_3 = \frac{2}{3} (1 - 2W) d_1^3 d_2.$$
In the above expressions \( W \equiv W(\vartheta) \). Finally, the nonlinear coefficient of \( d\phi_p/dx \) in Eq.(3.27) can be written as

\[
R_0 z_d^3 \left( 1 + \frac{3}{2} \hat{n}_d + \frac{3}{8} \hat{n}_d^2 - \frac{1}{16} \hat{n}_d^3 \right) \quad (3.35)
\]

up to the 3rd order, where

\[
\hat{n}_d = n_d - 1 = \sum_{j=1}^{J} \epsilon^j \sum_{l=-L}^{L} n_{j,l}(\xi, \tau) e^{i(kx-\omega t)} \quad (3.36)
\]

is the perturbed part of the dust density [see Eq.(3.21)].

Replacing the original derivatives of Eqs.(4.15-4.20) with those given in expressions (3.23-3.25), after substituting the expansion from Eq.(3.21), we obtain the full set of nonlinear equivalent of the continuity, momentum, and Poisson equation. In what follows, we consider the different and distinct coefficients of the exponential terms along with different powers of \( \epsilon \) to eliminate the higher order \( f_{j,l} \) terms (higher than \( f_{1,1} \)) and finally obtain the Nonlinear Schrödinger Equation (NLSE).

### 3.4 Perturbative reduction

#### 3.4.1 \( f_{1,1} \) terms

We now consider the nonlinear terms for \( (J = 1, L = 1) \). Collecting the coefficients of \( \epsilon \) of Eqs.( 4.15-4.20) which are the coefficients of a distinct exponential expression, we get the following equations for \( f_{1,1} \),

\[
-\omega n_{1,1} + ku_{1,1} = 0, \quad (3.37)
\]
\[
k \left[ \sigma n_{1,1} - (1 + R_0) \phi_{1,1} \right] - \omega u_{1,1} = 0, \quad (3.38)
\]
\[
n_{1,1} + \left( k^2 - c_1 + d_1 \right) \phi_{1,1} = 0. \quad (3.39)
\]
Solving for \((n, u)_{1,1}\) from Eqs.(3.37-3.38) in terms of \(\phi_{1,1}\), we get

\[
\begin{align*}
n_{1,1} &= \frac{k^2 \zeta}{k^2 \sigma - \omega^2 \phi_{1,1}}, \\
u_{1,1} &= \frac{k \omega \zeta}{k^2 \sigma - \omega^2 \phi_{1,1}}.
\end{align*}
\] (3.40)  

Substituting for \(n_{11}\) from the above in Eq.(3.39), we get the linear dispersion equation,

\[
\omega^2 = k^2 \left( \sigma - \frac{\zeta}{\delta} \right),
\] (3.42)

where

\[
\zeta = 1 + R_0, \quad \delta = c_1 - d_1 - k^2.
\] (3.43)

### 3.4.2 \(f_{1,0}, f_{1,2}, \text{and} f_{1,3} \) terms

In order to determine the \(f_{1,(0,1,2)}\) terms, we set \((J = 1, L = 3)\) in the expansion. Considering the coefficients of \(\epsilon^2\) of the resultant Eq.(4.15), we obtain the following set of equations,

\[
\begin{align*}
k n_{1,3} u_{1,3} &= 0, \\
n_{1,3} u_{1,2} + n_{1,2} u_{1,3} &= 0, \\
n_{1,3} u_{1,1} + n_{1,1} u_{1,3} + n_{1,2} u_{1,2} &= 0,
\end{align*}
\] (3.44)

which show that both \((n, u)_{1,2}\) and \((n, u)_{1,3}\) are arbitrary and can be set to 0. The coefficient of \(\epsilon^2\) yield one more equation,

\[
n_{1,0} u_{1,-3} + n_{1,-3} u_{1,0} + n_{1,-1} u_{1,-2} + n_{1,-2} u_{1,-1} = 0,
\] (3.45)

which shows that both \((n, u)_{1,0}\) are arbitrary and can be set to 0. We should note that the terms with negative \(l\)-index are complex conjugates of the corresponding positive \(l\)-index terms i.e. \(f_{j,-l} = f_{j,l}^*\), where ‘*’ denotes complex conjugate.
that Eqs.(3.44) or Eq.(3.45) are not the only unique equations. Through an inspection of the string of different terms belonging to different and distinct exponential terms of Eq.(4.15), we can find similar set of equivalent equations.

From the coefficients of $\epsilon^2$ from Eq.(3.13), we find

$$
\begin{align*}
3kz_0n_{1,1}\phi_{1,3} &= 0, \\
2kz_0n_{1,2}\phi_{1,2} &= 0,
\end{align*}
$$

(3.46)

which shows that both $\phi_{1,(2,3)}$ are arbitrary and can be set to 0. A similar expression resulting from coefficients of $\epsilon^2$ of Eq.(4.20) we find that $\phi_{1,0}$ are arbitrary and can be set to 0.

### 3.4.3 $f_{2,1}$ terms

We set $(J = 2, L = 1)$ and consider the coefficients of $\epsilon^2$ of Eq.(4.15-4.20), from which get the following set of equations for the $f_{2,1}$ components,

$$
\begin{align*}
-i\omega n_{2,1} + ik u_{2,1} &= \frac{v}{\xi} \frac{\partial n_{1,1}}{\partial \xi} - \frac{\partial u_{1,1}}{\partial \xi}, \\
-i\omega u_{2,1} + ik\sigma n_{2,1} - ikz_0\zeta \phi_{2,1} &= \frac{v}{\xi} \frac{\partial u_{1,1}}{\partial \xi} - \sigma \frac{\partial n_{1,1}}{\partial \xi} + \zeta \frac{\partial \phi_{1,1}}{\partial \xi}, \\
n_{2,1} - \delta \phi_{2,1} &= 2ik \frac{\partial \phi_{1,1}}{\partial \xi},
\end{align*}
$$

(3.47) - (3.49)

However, the resultant solutions of Eqs.(3.47-3.49) blow up to $\infty$ when used with the compatibility condition Eq.(3.42) unless $\phi_{2,1} = 0$. Letting $\phi_{2,1} = 0$, Eqs.(3.47-3.49) can now be solved for $(n, u)_{2,1}$,

$$
n_{2,1} = A_n \frac{\partial \phi_{1,1}}{\partial \xi}, \quad u_{2,1} = A_u \frac{\partial \phi_{1,1}}{\partial \xi},
$$

(3.50)
where
\[ A_n = \frac{2i\omega (\omega - kv)\delta^2}{k^3 \zeta}, \quad A_u = A_n \frac{k}{\omega} \left( \frac{\sigma - \zeta}{2\delta} \right). \] (3.51)

Substituting the values of \( f_{2,1} \) components in the Poisson equation now gives the second compatibility condition, which is an expression for the velocity \( v \),
\[ v = \frac{\omega}{k} - \frac{k^3 \zeta}{\omega \delta^2}. \] (3.52)

### 3.4.4 \( f_{2,2} \) terms

We now set \((J = 2, L = 2)\) and consider the coefficients of \( \epsilon^2 \) of Eqs.(4.15-4.20), from which we get the following set of equations for the \( f_{2,2} \) terms,
\[ -\omega n_{2,2} + ku_{2,2} = -kn_{1,1}u_{1,1}, \] (3.53)
\[ -2\omega u_{2,2} + 2k\sigma n_{2,2} - 2k\zeta \phi_{2,2} = \omega n_{1,1}u_{1,1} - ku_{1,1}^2 - k \left( 1 + \frac{3}{2} R_0 \right) n_{1,1} \phi_{1,1} \]
\[ + kd_1 (1 + 3 R_0) \phi_{1,1}^2, \] (3.54)
\[ n_{2,2} + (3k^2 - \delta) \phi_{2,2} = -d_1 n_{1,1} \phi_{1,1} + (c_2 - d_2) \phi_{1,1}^2. \] (3.55)

Solving these equations, we get the following solutions,
\[ (n, u, \phi)_{2,2} = B_{n,u,\phi} \phi_{1,1}^2. \] (3.56)

The coefficients \( B_{n,u,\phi} \) have lengthy expressions and are listed in the Appendix A.
3.4.5 \( f_{2,0} \) terms

In order to find out the \( f_{2,0} \) terms, we set \((J = 3, L = 1)\) and consider the coefficient of \( \epsilon^3 \) of Eq.(4.15), from which we get,

\[
v_n^{2,0} - u_{2,0} = 2k^3 \frac{\omega \epsilon^2}{(\omega^2 - k^2 \sigma)^2} |\phi_{1,1}|^2. \tag{3.57}
\]

From the coefficient of \( \epsilon^2 \) terms of the Poisson equation, Eq.(4.20), we get another equation,

\[
n_{2,0} - c_1 \phi_{2,0} = 2(c_2 - d_2)|\phi_{1,1}|^2 - d_1 \left( n_{1,1} \phi_{1,1}^* + n_{1,1}^* \phi_{1,1} \right), \tag{3.58}
\]

where \((*)\) denotes complex conjugate.

However, we need one more equation to uniquely determine all the \( f_{2,0} \) terms. The remaining equation can be found out by setting \((J = 3, L = 0)\) terms in Eq.(3.13) and considering the coefficients of the terms with \( \epsilon^3 \),

\[
\sigma n_{2,0} - vu_{2,0} - \zeta \phi_{2,0} = 0. \tag{3.59}
\]

The solutions now can be written as,

\[
(n, u, \phi)_{2,0} = C_{n,u,\phi}|\phi_{1,1}|^2, \tag{3.60}
\]

where

\[
C_n = \frac{1}{\eta} \left[ \zeta k (c_2 - \delta d_1 - d_2) + \delta^2 v \omega (c_1 - d_1) \right], \tag{3.61}
\]

\[
C_u = \frac{\delta^2 \omega}{k} - vC_n, \tag{3.62}
\]

\[
C_\phi = \frac{1}{\zeta} (vC_u + C_n \sigma), \tag{3.63}
\]
where

\[ \eta = k \left[ (c_1 - d_1)(v^2 - \sigma) + \zeta \right]. \tag{3.64} \]

### 3.4.6 The NLSE

Finally, to reduce all the expressions to an NLSE, we set \( J = 2, L = 1 \) and substitute the expressions for \( f_{2,1}, f_{2,2}, \) and \( f_{2,0} \) in the coefficient of \( \epsilon^3 \) terms of Eq.(4.15),

\[
i \frac{\partial \phi_{1,1}}{\partial \tau} + C_1 \frac{\partial^2 \phi_{1,1}}{\partial \xi^2} + C_2 |\phi_{1,1}|^2 \phi_{1,1} = 0. \tag{3.65} \]

The coefficients \( C_{1,2} \) are given by,

\[
C_1 = \frac{(\omega - kv)(\omega^2 - 2kv\omega + k^2\sigma)}{k^2(\omega^2 - k^2\sigma)}, \tag{3.66} \\
C_2 = t_5\omega^5 + t_4\omega^4 + t_2\omega^2 + t_1\omega + t_0 \tag{3.67}
\]

where

\[
t_5 = \frac{2\delta^2}{\zeta \eta k^3}, \tag{3.68} \\
t_4 = kv t_5, \tag{3.69} \\
t_2 = k^2 \zeta (d_1 - c_1) t_4 / \psi, \tag{3.70} \\
t_1 = -\frac{2\delta k}{\zeta \eta} \left[ c_1^2 \zeta (d_1 \sigma + \zeta) \right] + \frac{2\delta k}{\zeta \eta} c_1 \left( c_2 \sigma - d_2 \sigma + 2d_1 \psi \right) + \frac{2\delta k}{\zeta \eta} c_1 \zeta \left( \zeta + k^2 \sigma \right) - c_2 \sigma \psi \frac{2\delta k}{\zeta \eta} \left( d_1 + k^2 \right) \left[ d_1 \sigma (\zeta + \psi) + \zeta^2 \right] + \frac{d_2 \sigma \psi}{\zeta \eta} \frac{2\delta k}{\zeta \eta}, \tag{3.71} \\
t_0 = -k^4 \sigma (\delta \sigma - 2\zeta) t_4 / \delta, \tag{3.72} \\
\psi = -c_2 + \delta d_1 + d_2, \tag{3.73} \\
\bar{\psi} = d_1 \sigma + 2\zeta + k^2 \sigma. \tag{3.74} \]
Eq. (3.65) is in its general form with polarization effect and with an arbitrary electron-ion distributions.

### 3.5 Modulation instability

Following standard procedure, we can find out the domain of modulation instability (MI) from the sign of coefficients $C_{1,2}$ [40]. MI would set in whenever $C_1 C_2 > 0$ and $k < k_c$, where $k_c$ is the critical wave number $k_c = \sqrt{4C_2/C_1|\psi_0|}$, where $\psi_0$ is the amplitude of the carrier wave. The growth rate of MI is given by

$$
\gamma_{MI} = \text{Im}(\omega) = \frac{1}{2} |C_1| k^2 \left( \frac{k_c^2}{k^2} - 1 \right)^{1/2},
$$

which reaches its maximum value $\gamma_{\text{max}} = |C_2| \psi_0^2$ for $k = k_c/\sqrt{2}$.

#### 3.5.1 $q$-nonextensive particles

This case is characterized by $q$-nonextensive (Tsallis) distributions for electron and ion densities (normalized), as described at the beginning of this Chapter,

$$
n_j = \mu_j \left[ 1 + (q - 1)\nu_j \phi \right]^{1/(q-1)+1/2},
$$

where $j = i, e$ and $q (> -1)$ is the nonextensive parameter. The coefficients $c_{1,2}$ are given by,

$$
c_1 = -\frac{1}{2} (1 + q),
$$

$$
c_2 = \frac{(3-q)(1+q)}{8(1 + \mu_e + \beta \mu_e)^2} \left[ 1 + (1 - \beta)^2 \mu_e \right].
$$

Note that the other coefficients are not needed at this point. The stability regime is shown in Figs.3.1,3.2, and 3.3.
Figure 3.1 The stability regime in the $k$-$q$ plane for different $\mathcal{R}_0$. The arrow shows the contour line for $C_1 C_2 = 0$. The other parameters are $\sigma = 0.1, \beta = 0.1, \mu_e = 9.0$. 
Figure 3.2 The unstable regime in the $k$-$\mu_e$ plane for different for $q \neq 1$ (top) and $q = 1$ (bottom). The shaded region is where $C_1 C_2 > 0$ and $k < k_c$. Different parameters are calculated self-consistently for $T_e = 1$ eV, $T_i = 0.03$ eV, $T_d = 0.01$ eV, $z_d = 10^5$, and $n_{d0} = 10^6$ m$^{-3}$. 
Figure 3.3 The dependence of the critical wave number $k_c$ with $C_1C_2$ (top) and $k$ (bottom) is shown for the stability domain. Different parameters are same as in Fig.3.2. The numbers in the figure are the corresponding values of $q$. 
3.5 Modulation instability

Figure 3.4 The first (first row) and second order (third row) rogue wave solutions $\psi_{1,2}(\xi, \tau)$ along with the cross-sections of the amplitudes and FWHMs. See text for details.
What we have seen from these figures are that though the polarisation factor $R_0$ does not change the instability domain, it affects the way the instability develops. The first figure shows the domains of instability in the $k$-$q$ plane for two values of $R_0$, while the second one shows the domain in the $k$-$\mu_e$ for different values of $q$. The polarisation factor is self-consistently calculated in the second case from the basic plasma parameters. The third figure shows the contours of the critical wave number $k_c$ as it changes with $C_1 C_2$ and wavenumber $k$ in the unstable regime. So, in the unstable regime i.e. when $C_1 C_2 > 0$, the instability develops only when $k < k_c$. So, in the top panel of Fig.3.3, the MI develops when the $k$ value lies below the $k_c$ curves for each corresponding $q$. What we can see that the system is unconditionally unstable to MI as $C_1 C_2 \rightarrow 0$. In the lower panel of the same figure, we see that the instability region lies above each curve for corresponding $q$. So for $q = -1$, the system is unconditionally unstable.

### 3.6 Rogue wave solutions

The standard NLSE is a well studied equation and it has a family of rational solutions, which are localized both in the $(\tau, \xi)$ plane [69, 70]. The family of rational solutions can be represented as,

$$
\psi_j(\xi, \tau) = \psi_0 \left[ (-1)^j + \frac{P_j(\xi, \tau) + i\tau Q_j(\xi, \tau)}{R_j(\xi, \tau)} \right] e^{i\tau},
$$

(3.77)
where $\tilde{\tau} = C_1 \tau$. The terms $P_j, Q_j, R_j$, up to the second order are given by,

\begin{align}
P_1 & = \frac{1}{2} Q_1 = 4, \quad R_1 = 1 + 4(\xi^2 + \tilde{\tau}^2), \\
P_2 & = \frac{3}{8} - \xi^2 \left(3 + 2\xi^2\right) - \tilde{\tau}^2 \left(9 + 10\tilde{\tau}^2\right) - 12\xi^2\tilde{\tau}^2, \\
Q_2 & = \frac{15}{4} + 2\xi^2 \left(3 - 2\xi^2\right) - 2\tilde{\tau}^2 \left(1 + 2\tilde{\tau}^2\right) - 8\xi^2\tilde{\tau}^2, \\
R_2 & = \frac{1}{8} \left[\frac{3}{4} + \xi^2 \left(9 + 4\xi^2 + \frac{16}{3}\xi^4\right) + \tilde{\tau}^2 \left(33 + 36\tilde{\tau}^2 + \frac{16}{3}\tilde{\tau}^4\right) \\
& \quad - 8\xi^2 \left(\tilde{\tau}^2 + 2\xi^2 + 2\tilde{\tau}^2\right)\right].
\end{align}

The first order solution $\psi_1(\xi, \tilde{\tau})$ is known as the Peregrine soliton. This solution is localised in space and time, which can also be termed as the rogue waves, similar to those appear in ocean. The second order solution $\psi_2(\xi, \tilde{\tau})$ is known as the super-rogue wave solution within the unstable region of the MI. These solutions are shown in Fig.3.4, where the first (first row) and second order (third row) rogue wave solutions $\psi_{1,2}(\xi, \tilde{\tau})$ along with the cross-sections of the amplitudes and FWHMs are shown. The first (left panel, second row) and second order (left panel, fourth row) solutions are shown against $\tilde{\tau}$ while $\xi = 0$ for different values of $R_0$ (inscribed). The panels on the right (second and fourth rows) show the FWHM of the wave against $R_0$. The other parameters are taken as $\sigma = 0.1, \beta = 0.1, \mu_e = 9.0, q = 0$ at $k = 1$. The effect of the polarisation can be seen from the FWHM curves, which show that a factor of $R_0$ cause the wave (both first order and second order) steeper by about 30%. As pointed out earlier, as the parameters are very well realisable in laboratory plasmas, in suitable conditions, this steepness due to polarisation should be able to be detected.