CHAPTER - VIII
exactly true, does not exist in actual universe. For actual universe (general space) Einstein (1950) observed that a tensor
\[ G^i_j - \frac{1}{2} g^i_j c \text{ exists whose divergence vanishes identically and he identified this tensor with the material energy tensor. This gave us the law of gravitation for actual universe. When there is no material energy tensor, it reduces to} \]

(59.2) \[ G^i_i = 0. \]

From equation (59.2) we see that the geometry of the empty region of the world is not of the most general Riemannian type but is limited. The general Riemannian geometry corresponds to the metric

(59.3) \[ ds^2 = g_{ij} dx^i dx^j, \]

where \( g_{ij} \) are the functions of the coordinates \( (x^i) \) and can vary from point to point. Comparing the derivatives of the tensor \( g_{ij} \) with those of the Newtonian potential we find that \( g_{ij} \) play the same role as Newtonian potential in the classical theory and may therefore be regarded as gravitational potentials of the Einstein’s theory.
In this chapter we propose to consider the metric tensor $g_{ij}$ (for a Finsler space) not only as a function of coordinates but also of their derivatives, i.e.

\[(59.4) \quad g_{ij} = g_{ij}(x^l, x^i)\]

and extend the law of gravitation for this space.

This can be justified by the fact that as the value of the gravitational potential $g_{ij}$ changes, from point to point, the direction of the gravitational field also changes, particularly in the problems of gravitational radiation from collapsing bodies where the matter undergoes a change in its volume due to gravitational contraction, pulsation, super eruption or collapse (Robinson, Schild etc. 1963). In view of the above in equation (59.4) we have taken the derivatives of the coordinates corresponding to the change of direction as in a Finsler space (Rund, 1959).

60. THE FIELD EQUATIONS

The Riemann curvature tensor in a Riemannian space is
One can also see that the divergence of material energy tensor has to vanish and a verification for the case of special relativity (Minkowskian system) is simple.

We readily find that the material tensor $T_{jk} = 0$, except when

$$T_{44} = \rho$$

(the density of the matter).

$$T_0 = \rho_0 \frac{dx^i}{ds} \frac{dx^j}{ds}$$

and

$$\rho = \frac{\rho_0}{(1 - \frac{V^2}{a^2})}$$

In a Minkowskian system

$$ds^2 = dt^2 - c^{-2} dx^2 - c^{-2} dy^2 - c^{-2} dz^2,$$

i.e.

$$\frac{ds}{dt} = 1 - c^{-2} v^2$$

where

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$
Hence

\[(60.11) \quad \rho = \rho_0 \left( \frac{dt}{ds} \right)^2 \]

therefore in Minkowskian system

\[(60.12) \quad T^{ij} = \rho \frac{dx^i}{dt} \frac{dx^j}{dt}, \]

which implies

\[(60.13) \quad T_{ij} = 0 \]

or

\[T_{i,j} = 0, \]

which results from lowering the suffix i. In other words, the divergence of the energy tensor vanishes.

To show that the divergence of the tensor $G_k^* - \frac{1}{2} g^i_k G^*$ vanishes identically, we introduce coordinates which will be approximately Minkowskain in a small region round a selected
point, \( g_{ij} \) not being constant but stationary with respect to position as well as direction.

i.e.

\[
(60.14) \quad \partial_k g_{ij} = 0
\]

and

\[
\dot{\partial}_k g_{ij} = 0
\]

This amounts to identifying the curved space time with the osculating flat space time in the neighbourhood of the point. Therefore in view of above we have

\[
(60.15) \quad G^{*i}_{j,i} - \frac{1}{2} (g^i_j G^*)_{,j} = \partial_i G^*_{j} - \frac{1}{2} g^i_j \partial_j G^* \\
= g^{ir} g^{pm} [\partial_i G^*_{(pj)m} - \frac{1}{2} \partial_j G^*_{ipm}] \\
= g^{ir} g^{pm} [\partial_i (\partial_m \Gamma^*_{(pj)} - \partial_j \Gamma^*_{ipm})] \\
- \frac{1}{2} \partial_j (\partial_m \Gamma^*_{ipm} - \partial_i \Gamma^*_{ipm})]
\]
or

\[(60.16)\]

\[G_{j,i}^{*} = g^{ij} g^{pm} \left\{ \partial_i \left\{ \partial_m \left( \gamma_{pj} - C_{jph} \partial_i G^h \right) + 
\right. \]

\[+ C_{fph} \partial_i G^h - C_{fph} \partial_i G^h \right\} - \]

\[\left. - \partial_j \left( \gamma_{fpm} - C_{fph} \partial_i G^h + C_{fph} \partial_p G^h - c_{inh} \partial_i G^h \right) \right\} - \]

\[\left. - \frac{1}{2} \partial_j \left\{ \partial_m \left( \gamma_{ipm} - C_{iph} \partial_i G^h + C_{iph} \partial_i G^h \right) \right\} - \right. \]

\[\left. - C_{iph} \partial_i G^h \right\} - \]

\[\left. - \partial_i \left( \gamma_{ipm} \partial_i G^h + c_{inh} \partial_i G^h - c_{iph} \partial_i G^h \right) \right\}.\]

In equation (60.16) all the further derivatives involved have been taken with respect to coordinates, therefore, the variation with respect to direction will remain unchanged, so all the terms involving \(c_{iph}\) with vanish.

Hence equation (60.16) reduces to

\[(60.17)\]

\[G_{j,i}^{*} = g^{ij} g^{pm} \left\{ \partial_i \left\{ \partial_m \partial_j g^p - \partial_m \partial_p g^j \right\} - \right. \]

\[\left. - \partial_i \left( \gamma_{ipm} \partial_i G^h + c_{inh} \partial_i G^h - c_{iph} \partial_i G^h \right) \right\}.\]
\[-\partial_\ell \partial_i g_{p\ell} + \partial_j \partial_p g_{i\ell} \}\) -
\[-\partial_j \{ \partial_m \partial_r g_{rp} - \partial_m \partial_r g_{i\ell} \} -
\[-\partial_r \partial_i g_{p\ell} + \partial_r \partial_p g_{i\ell} \}\),

where we have used equation (60.4),

By double interchange of dummy indices \(i\) for \(m\) and \(\ell\) for \(p\), the above expression vanishes.

Hence accordingly we set

\[(60.18) \quad K_1T^{ij} = G^{*ij} - \frac{1}{2} g^{ij}G^* .\]

We thus arrive at the law of gravitation in the Finsler space. When there is no material energy tensor this gives

\[(60.19) \quad G^{*ij} = 0,\]

which is equivalent to the Einstien's law of gravitation for empty space.
61. CONCLUSION:

The physical basis of the theory of gravitation is the law of equality of inertial and gravitational mass and the mathematical basis is the hypothesis that space time possesses a Finsler type metric in which 'distances', 'actions', etc. depend not only on position but also on direction. Therefore the fundamental tensor $g_{ij}$ determines the nature of space-time geometry in its material aspect and motion and direction of the gravitational field in its gravitational aspects.