CHAPTER - VII
52. **INTRODUCTION:**

G-curves in a subspace of a Finsler space have been defined and studied by Rastogi and Trivedi (1972) and their differential equations have been obtained in the following form

\[(52.1) \quad G^\alpha \overset{\text{def}}{=} p^\alpha - K_h^\alpha \sec \phi_{2(\mu)} \{ t^\alpha_{(\mu)} \cos \chi_{(\mu)} -
- \frac{du^\alpha}{ds} \cos \phi_{1(\mu)} - \sum_r \xi^\alpha_{(r)} \cos \phi_{r(\mu)} + \]

\[+ Q_{(\mu)\gamma}^\alpha \frac{du^\gamma}{ds} \sin \chi_{(\mu)} \frac{1}{K_h^\alpha} \} = 0,\]

where $\phi_{1(\mu)}$, $\phi_{2(\mu)}$ and $\phi_{r(\mu)}$ are the angles which the unit vector $\omega_{(\mu)}^i$ makes with $dx^i/ds$, $q$ and $\eta_{(i)}^i$, respectively and $\chi_{(\mu)}$ is the angle between $\omega_{(\mu)}^i$ and $\lambda_{(\mu)}^i$. 
Writing

\begin{equation}
D^a_{\beta y} = g_{\beta y} \cdot K^*_h \sec \phi_{2(\mu)} t^*_{(\mu)} \cos \chi^*_{(\mu)} - \frac{du^a}{ds} \cos \phi_{(\mu)} + Q^*_{(\mu)\delta} \frac{du^\delta}{ds} \sin \chi^*_{(\mu)}/K^*_h - \sum_r \xi^a_r \cos \phi_{r(\mu)}.
\end{equation}

and

\begin{equation}
R^a_{\beta y} = \Gamma^a_{\beta y} - D^a_{\beta y}.
\end{equation}

Rastogi and Trivedi (1973) defined the \( G \)-covariant derivative of a mixed tensor \( T^\alpha_{\beta_1 \beta_2} \) as follows:

\begin{equation}
T^\alpha_{\beta_1 \beta_2} = \partial_\gamma T^\alpha_{\beta_1 \beta_2} - (\partial_\delta T^\alpha_{\beta_1 \beta_2}) \partial^\gamma T^\delta
\end{equation}

In the present chapter we have used equation (52.4) to define the \( G \)-Lie derivative of a mixed tensor in a Finsler
space $F_m$. We have also defined $G$-Lie derivative of the coefficient of connection $\Gamma^\alpha_{\beta\gamma}$ and the coefficient of connection $R^\alpha_{\beta\gamma}$. We have obtained some commutation formulae in a Finsler space. We have also defined and studied $G$-motion, $G$-correspondence, $G$-conformal motion, $G$-affine motion, special affine motion and special $G$-affine motion in a Finsler space.

53. **G-LIE DERIVATIVE**

Let us consider an infinitesimal coordinate transformation of the type

\[(53.1) \quad \bar{u}^\alpha = u^\alpha + v^\alpha(u)dt,\]

and

\[(53.2) \quad \bar{u}^{\alpha\beta} = u^{\alpha\beta} + (\partial_\beta v^\alpha)u^\beta dt.\]

Corresponding to these the Lie-derivative of the metric tensor is given by (Rund, 1959),

\[(53.3) \quad D^L g_{\alpha\beta} = g_{\alpha\beta\gamma} v^\gamma + (\partial_\delta g_{\alpha\beta}) v^\delta_{\beta} u^\gamma + g_{\alpha\gamma} v^\gamma_{\beta} + g_{\alpha\beta} v^\gamma_{\alpha}.\]
where

\begin{equation}
(53.4) \quad g_{\alpha \beta \gamma} = \partial_\gamma g_{\alpha \beta} - (\partial_\delta g_{\alpha \beta})(\partial_\gamma G^{\delta})
\end{equation}

\[ -\Gamma^*_{\alpha \gamma} g_{\delta \beta} - \Gamma^*_{\gamma \beta} g_{\alpha \delta}. \]

since we know that

\begin{equation}
(53.5) \quad g_{\alpha \beta \gamma} = \partial_\gamma g_{\alpha \beta} - (\partial_\delta g_{\alpha \beta})(\partial_\gamma G^{\delta})
\end{equation}

\[ -R^*_{\alpha \gamma} g_{\delta \beta} - R^*_{\gamma \beta} g_{\alpha \delta}, \]

therefore using (52.3), (53.4) and (53.5) in (53.3) we obtain on simplification

\begin{equation}
(53.6) \quad D g_{\alpha \beta} = v^\gamma (g_{\alpha \beta \gamma} - g_{\delta \beta} D^*_{\alpha \gamma} - g_{\alpha \delta} D^*_{\gamma \beta}) + \\
+ (\partial^\gamma g_{\alpha \beta}) u^\gamma (v^\delta + v^e D_{\epsilon \gamma}^* \gamma) + \\
+ (v^\gamma + v^e D_{\epsilon \gamma}^* \gamma) g_{\alpha \gamma} + (v^\gamma + v^e D_{\epsilon \alpha}^* \gamma) g_{\beta \gamma},
\end{equation}

which when simplified, yields

\begin{equation}
(53.7) \quad D g_{\alpha \beta} = v^\gamma (g_{\alpha \beta \gamma} + (\partial^\gamma g_{\alpha \beta}) u^\gamma (v^\delta + v^e D_{\epsilon \gamma}^* \gamma) \\
+ g_{\alpha \gamma} v^\gamma + g_{\beta \gamma} v^\gamma.
\end{equation}
Now, we define the G-Lie derivative of the metric tensor as follows:

\[
\overline{\partial}_{\dot{L}} g_{\alpha\beta} \overset{\text{def}}{=} D_{\dot{L}} g_{\alpha\beta} - v^\gamma u^\alpha D_\gamma^* (\partial_\delta g_{\alpha\beta}),
\]

which can also be written as

\[
\overline{\partial}_{\dot{L}} g_{\alpha\beta} \overset{\text{def}}{=} v^\gamma g_{\alpha\beta\gamma} + (\partial_\delta g_{\alpha\beta}) u^\gamma v^\delta_\beta + g_{\alpha\gamma} v^\gamma_\beta - g_{\gamma\beta} v^\gamma_\alpha.
\]

Now, we define the G-Lie derivative of the covariant vector \( \omega_\alpha \) and contravariant vector \( y^\alpha \) as follows:

\[
\overline{\partial}_{\dot{L}} \omega_\alpha \overset{\text{def}}{=} \omega_{\alpha\beta} v^\beta + \omega_\beta v^\beta_\alpha + \quad (\partial_\delta \omega_\alpha) u^\gamma v^\delta_\beta
\]

and

\[
\overline{\partial}_{\dot{L}} y^\alpha \overset{\text{def}}{=} y^\alpha_{\alpha\beta} v^\beta - y^\beta v^\beta_\alpha + (\partial_\delta y^\alpha) u^\gamma v^\delta_\beta.
\]

On the basis of these definitions we define the G-Lie derivative of a mixed tensor
\[ T^{a_1, a_r}_{\beta_1, \beta_r} \quad \text{as follows} \]

\[
(53.12) \quad \overline{D} T^{a_1, a_r}_{\beta_1, \beta_r} \overset{\text{def}}{=} T^{a_1, a_r}_{\beta_1, \beta_r} v^\gamma + \\
+ \left( \partial_\delta T^{a_1, a_r}_{\beta_1, \beta_r} \right) v^\delta_{\beta_r} u^\gamma - \\
- \sum_{p=1} T^{a_1, a_{p+1} a_r}_{\beta_1, \beta_r} v^{a_p}_{\beta_p} + \\
+ \sum_{q=1} T^{a_1, a_r}_{\beta_1, \beta_q} \delta_{\beta_q \beta_1} v^{\delta}_{\beta_q}
\]

Now, to define the G-Lie derivative of the coefficient of connection \( \Gamma^{a}_{\beta r}(u, u') \), we shall use the definition of its Lie derivative (Rund, 1959),

\[
(53.13) \quad D \Gamma^{a}_{\beta r} = v^\alpha_{\beta r} + K^a_{\beta r \delta} v^\delta + \left( \partial_\delta \Gamma^a_{\beta r} \right) v^\delta_{\beta_r} G^e - \\
\]

where

\[
(53.14) \quad K^a_{\beta r \delta} = \partial_\delta \Gamma^a_{\beta r} - \left( \partial_\delta \Gamma^a_{\beta r} \right) \partial_\delta G^e - \\
- \partial_\delta \Gamma^a_{\beta r} + \left( \partial_\delta \Gamma^a_{\beta r} \right) \partial_\delta G^e + \\
+ \Gamma^*_{\beta r} G^e - \Gamma^*_{\beta r} \Gamma^*_{\beta r} \delta e
\]

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and

\[(53.15) \quad v^\alpha_{\beta\gamma} = \partial_\beta \partial_\gamma v^\alpha + (\partial_\gamma \Gamma^\alpha_{\beta\delta}) v^\delta + \Gamma^\alpha_{\delta\beta} \partial_\gamma v^\delta - (\delta_e \Gamma^\alpha_{\delta\beta}) \Gamma^e_{\alpha\gamma} u^\theta v^\delta + \]

\[+ \Gamma^\alpha_{\beta\gamma} \partial_\beta v^\delta - \Gamma^\alpha_{\delta\gamma} \partial_\delta v^\alpha + \]

\[+(\Gamma^\alpha_{\delta\gamma} \Gamma^e_{\beta\delta} - \Gamma^\alpha_{\delta\gamma} \Gamma^e_{\beta\delta}) v^\delta.\]

since equation (53.15) can be expressed as

\[(53.16) \quad v^\alpha_{\beta\gamma} = v^\alpha_{\beta\gamma} + v^\delta D^\alpha_{\delta\gamma} + v^\delta D^\alpha_{\delta\beta} + \]

\[+ v^\delta (D^\alpha_{\delta\beta} \Gamma^\delta_{\epsilon\gamma} - D^\alpha_{\delta\epsilon} \Gamma^\delta_{\epsilon\beta} - D^\alpha_{\delta\epsilon} \Gamma^\delta_{\epsilon\beta} ) - D^\alpha_{\delta\epsilon} \Gamma^\delta_{\epsilon\beta} - v^\delta D^\delta_{\epsilon\gamma} + v^\epsilon D^\epsilon_{\epsilon\gamma} D^\alpha_{\delta\beta},\]

therefore by virtue of equations (53.13) and (53.16) we define

\[(53.17) \quad \begin{array}{c}
\overline{D} \Gamma^\alpha_{\beta\gamma} \quad \text{def} \quad \overline{D} \Gamma^\alpha_{\beta\gamma} - v^\delta D^\alpha_{\delta\beta\gamma} - v^\delta D^\alpha_{\delta\beta\gamma} - v^\delta D^\alpha_{\delta\beta\gamma} - \\
\end{array}

\[+ v^\delta D^\delta_{\epsilon\gamma} + v^\delta D^\delta_{\epsilon\gamma} - v^\delta D^\delta_{\epsilon\gamma} D^\alpha_{\delta\beta} - \]

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\[-v^\delta D^e_{\delta \beta} \Gamma^e_{\epsilon} + v^\theta D^e_{\theta \epsilon} u^e_\delta \Gamma^e_{\beta} +
\]

\[+ v^\delta D^e_{\epsilon \beta} \Gamma^e_{\delta} + v^\delta D^e_{\epsilon \alpha} \Gamma^e_{\delta} ,
\]

which can be written as

(53.18) \[\overline{D}_{\beta} \Gamma^e_{\epsilon} \overset{\text{def}}{=} v^\alpha_{\beta \epsilon} + K^\alpha_{\beta \delta} v^\delta + (\partial^\delta_{\beta} \Gamma^e_{\epsilon}) v^\delta_{\epsilon} u^e_{\epsilon} .\]

Now, using equations (52.3)(53.12) and (53.18) we define

(53.19) \[\overline{D}^e_{\epsilon} \overset{\text{def}}{=} v^\alpha_{\beta \epsilon} + K^\alpha_{\beta \delta} v^\delta +
\]

\[+(\partial^\delta_{\beta} \Gamma^e_{\epsilon}) v^\delta_{\epsilon} u^e_{\epsilon} - D^e_{\epsilon \beta} v^\delta +
\]

\[+ D^e_{\epsilon} v^\alpha_{\beta \epsilon} - D^e_{\epsilon \beta} v^\alpha_{\beta \epsilon} -
\]

\[- D^e_{\epsilon} v^\alpha_{\beta \epsilon} - (\partial^\alpha_{\beta} D^e_{\epsilon}) u^e_{\epsilon} v^\epsilon_{\epsilon} .\]

54. **SOME COMMUTATION FORMULAE** :

Now, we shall study some commutation formulae arising by the G-Lie derivative and other derivatives. Differentiating

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equation (53.11) partially with respect to \( u^e \), we get

\[
(54.1) \quad \partial_e^i (D_L y^a) = (\partial_e^i y^a) v^\beta \iiota (\partial_e^i \partial_\beta y^a) v_\beta^\delta u^\delta + \]

\[
+ (\partial_\delta^i y^a) v_\beta^\delta (\partial_e^i u^\beta) - v_\beta^\alpha \partial_e^i y^\alpha.
\]

The G-Lie derivative of \((\partial_e^i y^a)\) is given by

\[
(54.2) \quad \overline{D}_L (\partial_e^i y^a) = v^\beta (\partial_e^i y^a) \iota_{\beta} + (\partial_\delta^i \partial_e^i y^a) v_\beta^\delta u^\delta
\]

\[
+ (\partial_\delta^i y^a) v_\beta^\delta - v_\beta^\alpha (\partial_e^i y^\alpha).
\]

From equations (54.1) and (54.2) we obtain

\[
(54.3) \quad \partial_e^i (D_L y^a) - \overline{D}_L (\partial_e^i y^a) = v^\beta [(\partial_e^i y^a) \iota_{\beta} - (\partial_e^i y^a) \iota_{\beta}]
\]

\[
= v^\beta [(\partial_\delta^i y^a) R_\iota^\delta + y^\gamma \partial_e^i R_\iota^\gamma]
\]

Hence,

**Theorem (54.1)**

The commutation formula arising from the partial derivative and G-Lie derivative for the contravariant vector \( y^a \) is given by (54.3).
Similarly for a covariant vector \( \omega_\alpha \) we obtain

**Theorem (54.2)**

The commutation formula for the covariant vector \( \omega_\alpha \) is given by

\[
(54.4) \quad \partial_e (\overline{D} \omega_\alpha) - \overline{D} (\partial_e \omega_\alpha) = \nu^\beta [(\partial_\delta \omega_\alpha) R_{\epsilon \beta}^\delta - \omega_\delta \partial_\epsilon R_{\alpha \beta}].
\]

Let \( B(u,u') \) be an arbitrary covariant vector field such that its inner product with the tensor \( T^{\omega}(u,u') \) is given by

\[
(54.5) \quad y^\alpha (u,u') \overset{\text{def}}{=} T^{\alpha \beta}(u,u')B_\beta(u,u').
\]

Substituting for \( y^\alpha \) from equation (54.5) in equation (54.3) and using equation (54.4) we obtain

\[
(54.6) \quad B_\beta \left[ \partial_e (\overline{D} T^{\alpha \beta}) - \overline{D} (\partial_e T^{\alpha \beta}) - T^{\alpha \delta} \nu^\tau \partial_\epsilon R_{\delta \tau}^\beta - \nu^\tau (\partial_\delta T^{\alpha \beta}) R_{\epsilon \gamma}^\delta - \nu^\tau T^{\gamma \delta} \partial_\epsilon R_{\alpha \delta}^\gamma \right] = 0.
\]
since $B_{\beta}$ is arbitrary, we have

**Theorem (54.3)**

The commutation formula for a second order contravariant tensor $T^{a\beta}$ is given by

\[
\partial_e^c (\bar{D}T^{a\beta}) - \bar{D}(\partial_e^c T^{a\beta})
= [T^\alpha{}^\delta \partial_e^c R^{*\beta}_{\delta\gamma} + (\partial_e^c T^{a\beta}) R^{*\delta}_{e\gamma} + \\
+ T^{s\beta} \partial_e^c R^{*a}_{\delta\gamma}] \nu^\gamma.
\]

Similarly for a covariant tensor we have

**Theorem (54.4)**

The commutation formula for the covariant tensor $T_{a\beta}$ of order 2 is given by

\[
\partial_e^c (\bar{D}T_{a\beta}) - \bar{D}(\partial_e^c T_{a\beta})
= \nu^\gamma [(\partial_e^c T_{a\beta}) R^{*\delta}_{e\gamma} - T_{b\delta} \partial_e^c R^{*\delta}_{a\gamma} - \\
- T_{a\delta} \partial_e^c R^{*b}_{e\gamma}].
\]

[192]
These formulae can easily be extended for covariant and contravariant tensors of any order. Thus in consequence of these theorems we may establish

**Theorem (54.5)**

The commutation formula for the mixed tensor $T_{\beta_1...\beta_r}^{\alpha_1...\alpha_r} (u,u')$ is given by

\[(54.9) \quad \partial_\gamma^L (\overline{D} T_{\beta_1...\beta_r}^{\alpha_1...\alpha_r}) - \overline{D} (\partial_\gamma^L T_{\beta_1...\beta_r}^{\alpha_1...\alpha_r}) = V^\gamma [(\partial_\delta^L T_{\beta_1...\beta_r}^{\alpha_1...\alpha_r}) R_{\gamma \delta}^* + 

\sum_{p=1}^r T_{\beta_1...\beta_r}^{\alpha_1...\alpha_p} \delta_{\gamma \delta}^p + \cdots ].

\[\partial_\gamma^L R_{\gamma \delta}^* - \sum_{q=1}^s T_{\beta_1...\beta_r}^{\alpha_1...\alpha_q} \delta_{\gamma \delta}^q \partial_\gamma^L R_{\gamma \delta}^*].\]

When the right hand side of equation (54.9) vanishes, the partial derivative and G-Lie derivative of a mixed tensor commute.

Similarly we may obtain commutation formulae arising from the G-covariant derivative and G-Lie derivative.
thus we have

**Theorem (54.6)**

The commutation formula arising from the $G$-covariant derivative and $G$-Lie derivative of a mixed tensor is given by

\begin{equation}
(54.10) \quad \overline{D}(T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_s}}) - (\overline{D}T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_s}})_{\underline{\beta}} \\
= v^\gamma [\sum_{p=1}^r T_{\overset{\alpha_1,\ldots,\alpha_p,\ldots,\alpha_r}{\beta_1,\ldots,\beta_p}} J_{\underline{\beta\gamma\delta}}^p - \\
- (\partial_0 T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_r}}) J_{\underline{\beta\gamma\delta}}^0 u^\gamma - \\
- \sum_{q=1}^s T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_q,\ldots,\beta_s}} J_{\underline{\beta\gamma\delta}}^q v^\gamma - \\
\hspace{1cm} + v_{\underline{\beta\gamma\delta}}^0 u_{\underline{\beta\gamma\delta}} [\partial_0 T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_r}} R_{\underline{\beta\gamma\delta}}^0 + \\
\hspace{1cm} + \sum_{p=1}^r T_{\overset{\alpha_1,\ldots,\alpha_p,\ldots,\alpha_r}{\beta_1,\ldots,\beta_p}} \partial_0 R_{\underline{\beta\gamma\delta}}^{p\gamma} - \\
\hspace{1cm} - \sum_{q=1}^s T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_q,\ldots,\beta_s}} \partial_0 R_{\underline{\beta\gamma\delta}}^{q\gamma} + \\
\hspace{1cm} + \sum_{p=1}^r T_{\overset{\alpha_1,\ldots,\alpha_p,\ldots,\alpha_r}{\beta_1,\ldots,\beta_p}} v_{\underline{\beta\gamma\delta}}^p - \\
\hspace{1cm} - \sum_{q=1}^s T_{\overset{\alpha_1,\ldots,\alpha_r}{\beta_1,\ldots,\beta_q,\ldots,\beta_s}} v_{\underline{\beta\gamma\delta}}^q ]
\end{equation}

[194]
where

\[(54.11) \quad J^\alpha_{\beta\gamma} = (\partial_\gamma R^\alpha_{\rho\gamma} - \partial^\rho e R^{\alpha\rho}_{\beta\gamma} \partial^e_\delta G^e) -
\]

\[ - (\partial_\gamma R^\alpha_{\beta\delta} - \partial^\delta e R^{\alpha\delta}_{\beta\gamma} \partial^e_\gamma G^e) \]

\[ + R^{\alpha\delta}_{\epsilon\delta} R^{\epsilon\gamma}_{\beta\gamma} - R^{\alpha\delta}_{\epsilon\gamma} R^{\epsilon\gamma}_{\beta\gamma}. \]

55. **G-MOTION IN A FINSLER SPACE**:

The point transformations (53.1) and (53.2) are said to define a G-motion in a Finsler space if and only if

\[(55.1) \quad \bar{D}g_{\alpha\beta} = 0. \]

Since we know that for such transformation to be a motion (Rund, 1959);

\[(55.2) \quad Dg_{\alpha\beta} = 0. \]

therefore by virtue of equation (53.8) we obtain

**Theorem (55.1)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be a motion as well as
G-motion is that

\[(\partial^\gamma_{\alpha} g_{\alpha\beta}) v^\gamma D^\delta_{\gamma e} u^e = 0.\]

We shall call a vector \(v\) to be a G-killing if it satisfies

\[v^\gamma g_{\alpha\beta} + (\partial^\gamma_{\alpha} g_{\alpha\beta}) u^\gamma v^{\delta}_{\beta} + g_{\alpha\gamma} v^\gamma_{\beta} + g_{\gamma\beta} v^\gamma_{\alpha} = 0.\]

The equation (55.4) will be called G-killing equation.

From the definition of a G-killing vector and G-motion in a Finsler space we obtain

**Theorem (55.2)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be a G-motion in a Finsler space \(F_m\) is that the vector \(v\) be a Killing vector.

Using equation (53.3) and (53.9) in (53.8) we get

\[v^\gamma g_{\alpha\beta} + (\partial^\gamma_{\alpha} g_{\alpha\beta}) u^\gamma v^{\delta}_{\beta} + g_{\alpha\gamma} v^\gamma_{\beta} + g_{\gamma\beta} v^\gamma_{\alpha} =
\]

\[= g_{\alpha\beta} v^\gamma + (\partial^\gamma_{\alpha} g_{\alpha\beta}) v^{\delta}_{\beta} u^\gamma + g_{\alpha\gamma} v^\gamma_{\beta} +
\]

\[+ g_{\gamma\beta} v^\gamma_{\alpha} - v^\gamma u^e D^\delta_{\gamma e} (\partial^\gamma_{\alpha} g_{\alpha\beta}).\]
which on simplification yields

**Theorem (55.3)**

The necessary and sufficient condition for a vector \( u \) to be a killing or a G-Killing vector is given by

\[
(55.6) \quad v^\gamma g_{\alpha \beta |\gamma} + 2C_{\alpha \beta \delta} v^\delta u^{\gamma} + g_{\alpha \gamma} v^\gamma_{|\beta} +
+ g_{\gamma \beta} v^\gamma_{|\alpha} + 2C_{\alpha \beta \delta} D^\delta_{\gamma \epsilon} v^\gamma u^{\epsilon} = 0
\]

or

\[
(55.7) \quad v_{\alpha \beta} + v_{|\beta \alpha} + 2C_{\alpha \beta \gamma} v^\gamma_{|\delta} u^{\delta} -
- 2C_{\alpha \beta \gamma} v^\delta u^{\epsilon} D^\gamma_{\delta \epsilon} = 0,
\]

respectively, where \( C_{\alpha \beta \gamma} \overset{\text{def}}{=} \frac{1}{2} \partial^{\gamma} g_{\alpha \beta} \).

The condition that the infinitesimal point transformation along the vector \( \rho v \) to be a G-motion in \( F_m \) is that

\[
\rho v^\gamma g_{\alpha \beta |\gamma} + (\partial^\gamma_\delta g_{\alpha \beta}) u^{\gamma} (\rho v^\delta)_{|\gamma} + g_{\alpha \gamma} (\rho v^\gamma)_{|\beta} +
+ g_{\alpha \beta} (\rho v^\gamma)_{|\alpha} = 0,
\]

[197]
which on simplification reduces to

\[(55.8) \quad \rho (v^\gamma g_{\alpha \beta \gamma} + 2 C_{\alpha \beta \delta} u^\gamma v^\delta + g_{\alpha \gamma} v^\gamma_{|\beta} + g_{\alpha \beta} v^\gamma_{|\gamma}) +
+ 2 C_{\alpha \beta \delta} u^\gamma \rho_{|\gamma} v^\delta + \rho_{|\beta} v^\alpha + \rho_{|\alpha} v^\beta = 0.\]

If the infinitesimal point transformation along the vector \(v\) is also a G-motion, equation (55.7) reduces to

\[(55.9) \quad 2 C_{\alpha \beta \delta} u^\gamma \rho_{|\gamma} v^\delta + \rho_{|\beta} v^\alpha + \rho_{|\alpha} v^\beta = 0.\]

One of the solutions of (55.8) is \(\rho = \text{constant}\). Hence in analogy with a result by Yano, (1955, p. 49), we can easily state that two infinitesimal G-motion cannot have the same trajectories.

56. **G-CORRESPONDENCE**:

If \(F_m\) and \(\bar{F}_m\) be two Finsler spaces, we say that they are in G-correspondence if they satisfy

\[(56.1) \quad R^*_{\beta \gamma} = R^*_{\beta \gamma} + \frac{1}{2} A^a_{\beta} \partial_\gamma (\log \phi) + \frac{1}{2} A^a_{\gamma} \partial_\beta (\log \phi),\]

where \(\phi\) is a scalar quantity, \(R^*_{\beta \gamma}\) and \(\bar{R}^*_{\beta \gamma}\) are G-connections in \(F_m\) and \(\bar{F}_m\) respectively and \(A^a_\beta = \partial_\beta u^a\).
Since equation (53.9) can be written as

\[
(56.2) \quad \bar{D}_L g_{\alpha \beta} = \nu^\gamma (\partial_\gamma g_{\alpha \beta} - \partial_\delta g_{\alpha \beta} \partial_\gamma G^\delta - g_{\alpha \beta} R^*_{\alpha \gamma} - g_{\alpha \delta} R^*_{\gamma \beta})
\]

\[+ (\partial^\gamma g_{\alpha \beta}) u^\nu (\partial_\gamma v^\delta - (\partial_\epsilon v^\delta)(\partial^\gamma G^\epsilon) + v^\epsilon R^*_{\epsilon \gamma}) + \]

\[+ g_{\alpha \beta} (\partial_\nu v^\gamma - (\partial_\epsilon v^\gamma)(\partial_\nu G^\epsilon) + v^\epsilon R^*_{\epsilon \nu}) \]

\[+ g_{\nu \beta} (\partial_\alpha v^\gamma - (\partial_\epsilon v^\gamma)(\partial_\alpha G^\epsilon) + v^\epsilon R^*_{\epsilon \alpha}), \]

therefore by virtue of equation (56.1) it reduces to

\[
(56.3) \quad \bar{D}_L g_{\alpha \beta} = \nu^\gamma (\partial_\gamma g_{\alpha \beta}) - \nu^\gamma (\partial_\delta g_{\alpha \beta}) (\partial_\gamma G^\delta) + 
\]

\[+ (\partial^\gamma g_{\alpha \beta}) u^\nu (\partial_\gamma v^\delta - (\partial_\epsilon v^\delta)(\partial^\gamma G^\epsilon) + v^\epsilon R^*_{\epsilon \gamma}) + \]

\[+ (\partial_\nu v^\gamma) g_{\alpha \gamma} - (\partial_\epsilon v^\gamma)(\partial_\nu G^\epsilon) g_{\alpha \gamma} + (\partial_\alpha v^\gamma) g_{\nu \beta} \]

\[- (\partial_\epsilon v^\gamma)(\partial_\alpha G^\epsilon) g_{\nu \beta} + v^\epsilon (\partial^\gamma g_{\alpha \beta}) u^\nu \]

\(- \frac{1}{2} A^\delta_\epsilon \partial_\gamma (\log \phi) - \frac{1}{2} A^\delta_\epsilon \partial_\epsilon (\log \phi). \)

Equation (56.3) on simplification yields

\[
(56.4) \quad \bar{D}_L g_{\alpha \beta} = \frac{1}{\phi} \bar{D}_L g_{\alpha \beta}, 
\]

[199]
Hence,

**Theorem (57.1)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be conformal as well as G-conformal motion in $F_m$ is that

$$v^\gamma u^e D_{\gamma e}^\delta (\partial_\delta g_{\alpha \beta}) = (\phi - \psi) g_{\alpha \beta}.$$ 

from (53.8) and (57.2) we have

**Theorem (57.2)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be G-conformal motion in $F_m$ is that the Lie derivative of $g_{\alpha \beta}$ is given by

$$D^L_L g_{\alpha \beta} = \psi g_{\alpha \beta} + v^\gamma u^e D_{\gamma e}^\delta (\partial_\delta g_{\alpha \beta}).$$

Also from (53.8) and (57.1) we obtain

**Theorem (57.3)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be conformal motion in

[200]
$F_m$ is that

\[ (57.5) \quad \overline{\nabla}_L g_{\alpha \beta} = \phi g_{\alpha \beta} - v^\gamma u^\epsilon D_{\gamma \epsilon} (\overline{\nabla}_L g_{\alpha \beta}). \]

**Definition (57.1)**

We call the point transformations (53.1) and (53.2) to be G-projective motion in $F_m$ if they transform G-curves of $F_m$ into the same system.

A necessary and sufficient condition for the point transformations (53.1) and (53.2) to be G-projective motion in $F_m$ is that

\[ (57.6) \quad DR^*_\beta = A^\epsilon_{\beta} - A^\gamma_{\gamma} \omega_\beta, \]

where $\omega_\gamma$ is a covariant vector field.

Equation (57.6) easily yields

\[ (57.7) \quad DR^*_\beta = (m+1)\omega_\beta, \]

which by virtue of (57.6) implies

\[ (57.8) \quad (m+1) DR^*_\beta = A^\epsilon_{\beta} DR^*_\gamma + A^\gamma_{\gamma} DR^*_\epsilon. \]
Now, by virtue of equation (57.11) and (57.12) we have

**Theorem (57.4)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be G-affine as well as G-projective motion is that

\[
A^\alpha_\beta (v^e_\gamma + \partial^*_{\delta} R^e_\gamma v^\delta_\beta u^{\gamma_0} + J^e_\gamma v^\delta_\beta) + A_\gamma^\alpha (v^e_\beta + \\
\partial^*_{\delta} R^e_\beta v^\delta_\gamma u^{\gamma_0} + J^e_\beta v^\delta_\gamma) = 0
\]

If we put

\[
(57.13) \quad M^{*a}_{\beta\gamma} \equiv R^{*a}_{\beta\gamma} - \frac{1}{m+1} (A^a_\beta R^e_\gamma + A^a_\gamma R^e_\beta)
\]

we can easily prove

**Theorem (57.5)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be G-projective motion in \( F_m \) is that

\[
D M^{*a}_{\beta\gamma} = 0.
\]
\[ \partial_\gamma \Gamma^\ast_\beta_\gamma + v^\gamma D^\ast_\gamma \Gamma^\ast_\alpha \epsilon + v^\gamma D^\ast_\alpha \Gamma^\ast_\epsilon \beta = 0. \]

The infinitesimal point transformations (53.1) and (53.2) will be called special G-affine motion if and only if

\[ (58.4) \quad \overline{\text{DR}}^\ast_\nu = 0. \]

Using value of \( R^\ast_\nu \) from equation (52.3) in (58.4) we get

\[ (58.5) \quad \overline{\text{DF}}^\ast_\nu - \overline{\text{DD}}^\ast_\nu = 0. \]

Thus we have

**Theorem (58.2)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be special affine as well as G-affine motion is given by

\[ (58.6) \quad \overline{\text{DD}}^\ast_\nu = 0. \]
Using the definition of G-Lie derivative we can easily write

\[(58.7) \quad \bar{D}^{\alpha}_{\beta\gamma} = D^{\alpha}_{\beta\gamma} - (\partial^{\alpha}_{\beta\gamma})D^{\alpha}_{\theta\delta}u^{\delta}v^{\theta}.\]

Now, equation (58.7) by virtue of (52.3) and (53.17) implies

\[(58.8) \quad \bar{R}^{\alpha}_{\beta\gamma} = R^{\alpha}_{\beta\gamma} - v^{\delta}_{\delta\beta}D^{\alpha}_{\theta\delta} - v^{\delta}_{\delta\gamma}D^{\alpha}_{\theta\delta} - \]

\[-v^{\delta}_{\delta\beta}D^{\alpha}_{\theta\delta} + v^{\alpha}_{\theta\beta}D^{\alpha}_{\theta\beta} - v^{\delta}_{\delta\gamma}D^{\alpha}_{\theta\delta} - \]

\[-v^{\delta}_{\delta\gamma}D^{\alpha}_{\theta\delta} + v^{\theta}_{\theta\delta}D^{\alpha}_{\theta\delta} - \]

\[+v^{\delta}_{\delta\beta}D^{\alpha}_{\theta\delta} + v^{\delta}_{\delta\gamma}D^{\alpha}_{\theta\delta} + \]

\[+(\partial^{\alpha}_{\theta\gamma})D^{\alpha}_{\theta\delta}u^{\delta}v^{\theta}.\]

Hence:

**Theorem (58.3)**

The necessary and sufficient condition for the point transformations (53.1) and (53.2) to be special G-affine motion
REFERENCES
