Chapter 1:

Introduction
1. INTRODUCTION

1.1 Statement of the Research Problem

1.1.1 The Inventory Problem

By inventory we mean a stock of goods or services held to meet demand arising in future. Since inventory corresponds to locked-up capital, there is a need to minimize inventory. At the same time, if demand arising cannot be completely met and cannot be fully backlogged till the time the stock is replenished, we will have to incur shortage costs. More than the shortage costs involved, the organization holding the stock runs the risk of losing its reputation and goodwill in terms of ability to meet demand. To this is added the problem of excess cost associated with a stock that exceeds the demand. Thus a balancing between these two types of costs and working out an optimum stock size becomes an interesting problem in operations management. Sometimes a stock does not receive an instantaneous replenishment immediately after the order is placed. This may necessitate preponing the order to a point of time ahead of the period during which demand does arise. It is also true that whatever is ordered to replenish the stock may not be received from the vendor. This randomness in supply against any order quantity may complicate the problem of inventory management.

The decision variable commonly involved in a simple inventory problem is the order quantity to start with or at periodic intervals. There could be other decision variables in specific cases like the order preponement time, the instance of time when the order had to be placed and the like. In situations where payments made against orders can be
delayed with or without some penalty and where the stock holding organization can continue meeting demand and earning profit even during the period before the first instalment of payment to the vendor, we have quite a few decision variables to be controlled by both the vendor and the stock holding organization.

Inventory models may be classified with respect to whether it is a production or purchase inventory model, whether deterministic or stochastic demand is considered, whether the inventory considered is of a single item or of multiple items, whether a lead time is involved for the procurement of the order quantity or not, whether payment is instantaneous or deferred, whether unmet demand during a period is backlogged or lost, etc.

Most often demand during a future period cannot be exactly predicted and is treated as a random variable following a certain probability distribution. In some cases, as mentioned earlier, supply against an order quantity may also have to be treated as a random variables with a probability distribution that would generally depend on the order quantity. An inventory model generally links up the total cost associated with an ordering policy with the decision variables, given values of the given variables like costs or given distributions like those of demand and supply, etc.

Solving an inventory problem implies deriving values of the decision variables viz., quantities to be ordered, times to order, etc. Sometimes some constraints may be imposed and have to be taken into account while solving such a model.
Considering the common situation characterized by a single decision variable, viz., the order quantity, one tries to consider the total inventory cost or the total profit associated with the order quantity as a measure of effectiveness. Since this cost or profit will be a random variable in a probabilistic inventory model, one would usually like to minimize a summary measure like the expected value of this cost (profit) distribution. Classical solutions to inventory problems have used the expected total cost (profit) as the objective function to determine the optimal order quantity. Subsequently, robustness of this optimal order quantity against possible deviations in demand distribution from the one assumed in the model is examined.

1.1.2 The Problem Undertaken

In this work we have confined our attention to the simplest inventory problem, viz., classical newspaper boy problem illustrating a static probabilistic inventory model. Our focus is to examine comprehensively different statistical problems associated with this model and its solution. In this connection we have considered the situation where supply against an order quantity varies randomly with a distribution depending on the order quantity. It may be pointed out in this connection that this simple inventory problem provides an interesting platform for examining various probabilistic and statistical issues explored in this work. We have also considered a natural extension this model to the case of a two-item inventory where the two items held in stock are either supplementary or complementary to each other resulting in the fact that demands for the two items follow a joint probability distribution.
Throughout the investigation we have tried to examine the impact of random supply on the optimal order quantity and the associated cost. Several forms of demand distributions, some over a finite and others with infinite range, some symmetrical and some other skewed have been taken into account.

Traditional solutions to such models as also of other models yield an optimal order quantity in terms of the parameters of the demand distribution besides the cost coefficients. Data on demands are not difficult to obtain and can be utilized to estimate the optimal order quantity. Both classical as well as Bayesian methods of estimation with different forms of loss function can be attempted to yield optimal order quantities based on sample observations on demand. And, this has been a major task taken up in the present work.

Other than the expected total cost, as the objective function, for obtaining the optimal order quantity, we have considered the variance, co-efficient of variation, penalized mean and mode of the total cost. We have also obtained the optimal order quantity by minimizing the probability that total cost exceeds a specified high value. Clearly, the last two approaches require the explicit distribution of total cost. Derivation of the cost distribution and of its different characteristic properties has been another aspect of the research problem.

### 1.2 Review of Relevant Literature

In the field of random supply some works have already been carried out by Panda [16] and also previously by Silver [21]. Silver obtained the optimal order quantity in a situation when quantity received from the supplier deviates from the quantity ordered.
He has shown that the best order quantity depends upon the means and the standard deviations of the amount received. Panda [21] modified the work of Silver by introducing planning period of infinite length with a known and constant demand. He also assumed that the entire supply arrives in a single lot. Panda [17] also dealt with a static inventory model considering uncertainty in supply. In this work he has established the optimum time to order and also the optimum order quantity considering both demand and supply as continuous random variables. Panda [18] also considered a situation where a certain portion of the quantity ordered is damaged or lost because of some reason. He developed two pricing policies under this situation. Santosh Kumar U and Pakkala [20] considered a stochastic inventory model for deteriorating items under the situation of random supply. They captured the uncertainty in the quantity received with respect to an order quantity. They have considered a continuous review of \((s, S)\) inventory system for a single item under Poisson demand distribution with items having independent exponentially distributed life times. Chandra [3] also considered the situation of random supply where the supply depends on the order quantity – either it varies around the order quantity or is a random fraction of the order quantity. Chakrabarty [2] has taken up the study of robustness of the total cost in a static risk inventory model. He investigated the inference robustness of the optimal order level under the said model by considering a variety of demand distributions and cost parameters. He compared the minimum total expected cost corresponding to the true optimum order level with that corresponding to the optimum order level of some other demand distribution. Mukherjee and Dasgupta [14] obtained the optimal order quantity under the consideration of a random lead time. Their study was carried out under the situations when demand is time dependent but known, random one point demand, and, constant demand rate with random lead time.
Mukherjee and Dasgupta [15] have revisited the inventory problem by defining a new way of working out the optimal order quantity. They worked out the EOQ by minimizing the mode of the total cost. In this work, Mukherjee and Dasgupta derived the distribution of the total cost associated with an order quantity under the general setup assuming an arbitrary continuous distribution for demand. Then, under classical newsboy setup, they have obtained the optimal order quantity by minimizing the modal cost. Analytical results were worked out for Gamma and Beta demand distributions. This work was extended by Chandra and Mukherjee [4] by proposing a number of alternative approaches for obtaining the optimal order quantity under classical newsboy setup. They altogether proposed five alternative approaches. The optimal order quantity was worked out by minimizing the standard deviation, coefficient of variation and standard deviation penalized mean of the total cost. Further, they have taken up the mode minimizing solution where the cost distribution was worked out explicitly. Moving further, they have obtained the optimal order quantity by minimizing the probability that the total cost exceeds a specified high value so that the optimal order quantity safeguards against the cost. They have extended their work by considering the supply distribution to be random in addition to the demand. A comparison was also done between these new approaches and the classical mean minimizing approach through numerical illustrations.

Roger [8] considered a Bayesian approach to estimate the parameters in the context of a single period inventory model. He has considered a uniform prior for the demand distribution. Kevork [9] also taken up estimation of the parameters of demand distribution using Monte Carlo simulation. The demand distribution was taken as normal and he has worked under classical newsboy setup. Mukherjee and Chandra [12] have given both parametric and non-parametric estimates of the optimal order quantity under classical newsboy setup considering an exponential demand distribution. They have also
taken up Bayesian estimation of the optimal order quantity considering Gamma prior for demand distribution. Bayes’ risk was also worked out under squared error and linex loss functions. Going further, empirical Bayes’ estimates were also worked out. The sensitivity of the parameters was also studied through numerical examples. Chandra and Mukherjee [5] have extended the estimation procedure in a more realistic framework when the demand varies over a finite range. The Bayes’ estimation was obtained under a rectangular prior. A general formulation of the EOQ for a static one point inventory model was also worked out.

Considering demand to have prior distribution, Mukherjee and Chandra [12] have proposed some alternative formulation of the optimal order quantity under classical newsboy setup. The have worked out the unconditional expected total cost with respect to the prior as well as posterior distributions. The optimal order quantity was worked out by minimizing this unconditional expected total cost. In their work, in addition to demand, supply was also considered as random following a rectangular and a triangular distribution varying around the order quantity.

1.3 Preview of the Present Work

A brief chapter-wise account of the features mentioned above is given below:

Chapter 2: Situation of Random Supply

In this chapter we have considered the situation where the supply, whatever is ordered, is not same as the order quantity, but, vary randomly. Here two distinct situations have been taken up – supply varies randomly around the order quantity and supply is a random fraction of the order quantity.

A variety of demand distributions have been considered both varying over finite and infinite range. As far as finite range demand distribution is considered, here rectangular and beta demands (both right triangular and left triangular) have been taken into account and exponential demand has been taken up as an infinite range demand distribution.

In each of the above situations, we have considered supply to be varying around the order quantity and supply to be a random fraction of the order quantity. As choice of supply varying around the order quantity, we have considered rectangular distribution varying over the range (q-a, q+a), q being the quantity ordered and also a triangular distribution with mid-point coinciding with ‘q’. As the random fraction, a beta distribution with specified parameters has been considered. In this case, both right triangular and left triangular beta distributions have been considered. For the
distributions of the random fraction, the corresponding supply distribution has also been worked out. Here in each situation, the optimal order quantity is derived by minimizing the expected total cost.

In case the demand and supply, both having rectangular distributions, it has been observed that the optimal order quantity does not involve the supply distribution parameter, although, the associated optimal cost involves this. Further, it has been observed in this case that, if the shortage and excess costs are same, optimal order quantity comes out simply as the mean of the demand distribution.

It has also been observed that for rectangular supply and right triangular beta demand, the optimal order quantities are less than the corresponding optimal order quantities with lesser values of ‘a’ although the corresponding optimal costs have shown the reverse behaviour. On the other hand, for rectangular supply and left triangular demand, the optimal order quantities as well as the corresponding costs have been seen to take lesser values than the corresponding optimal order quantities and the associated costs with smaller values of ‘a’.

It has also been observed that the optimal order quantities are same for rectangular and triangular supply distributions in case the demand distribution is rectangular. However, the associated optimal costs are different.

In case supply is a random fraction of the order quantity, it was observed that the optimal order quantities and the corresponding costs have higher values for a left triangular
distribution of the fraction than the case when it is right triangular irrespective of the demand distributions.

Chapter 3: *Alternative Criteria for Obtaining Optimal Order Quantity*

Existing solutions to stochastic models in Inventory Management, and even in other areas of Operational Research, have been generally based on the expectation-minimisation criterion. The total cost associated with any order quantity is a random variable whose distribution is not required to minimize the expected cost. However, the order quantity that minimizes just the expected cost may not be desirable in all situations. In fact, the classical optimum order quantity may be associated with a very high variance and in some repetition of the ordering situation the cost may be quite high. In fact, it is possible that the variance of cost corresponding to an order quantity that does not minimize the expected cost may be smaller and, that way, regarded as better to a decision-makers who like to avoid large costs. To an risk-averter, the probability of incurring a given high cost or a higher cost should be minimized, even at the cost of a larger order quantity resulting in a larger expected cost. Further, the most probable cost corresponding to any order quantity has a natural appeal to a decision-maker in repetitive ordering situations. Thus the mode-minimising order quantity which is conceptually analogous to the mini-max cost decision should attract our attention.

One reason for sticking to the minimum expected cost criterion is the natural appeal of the mean. And, as pointed out earlier, avoiding the somewhat involved derivation of the cost distribution adds to its appeal. The present author has derived the cost distribution in each of the simple inventory models treated in this dissertation and this has enabled the application of the following alternative criteria for obtaining the optimal order quantity.
Incidentally, three alternative ways to solve stochastic optimization problems are the E-rule, the V-rule and the P-rule. The criteria examined in the present dissertation to get alternative optimal order quantities are

- Minimization of variance of cost
- Minimisation of penalized mean cost
- Minimisation of coefficient of variation of cost
- Minimisation of modal cost and
- Minimisation of probability that cost exceeds a given value

Taken along with the classical mean-minimising solution, we can offer six alternative optimal order quantities. A pertinent question to ask will be: which one to accept as the best. The answer is that such a question cannot be directly answered. A comparison of the six alternative order quantities will call for a yardstick for comparison and it may not be quite rational to compare these in terms of the corresponding expected cost. In fact, the situation in which one alternative is preferred to the others differs and probably dictates the choice.

Chapter Three examines this broad view of an optimal order quantity and considers two distinct scenarios viz. when supply equals order quantity (Section 3.2) and when supply varies randomly around the order quantity (Section 3.3). Under each section, demand has been assumed to follow the exponential and the Beta distribution (with several parameter combinations for a skewed pattern). Section 3.2.1 provides the classical minimizing solutions, while Section 3.2.2 is devoted to the alternative solutions. Section 3.2.3 offers some illustrative calculation to show how optimal order quantities vary across optimality
criteria. Section 3.3 extends the analysis to the care of random supply along with the same demand distribution and Section 3.3.2 offers a glimpse into variations in order quantities.

It comes out that in the exponential setup, the optimal order quantity using the Standard Deviation or the Penalized Mean as the optimality criterion comes out to be larger than the corresponding quantity in the classical expectation minimizing setup and this is possibly expected. However, the EOQ with coefficient of variation as the optimality criterion comes out to be lower than classical EOQ in all the cases considered. However, this comparative pattern does not hold in case of Beta demand – the optimal order quantity as with coefficient of variation as the criterion turns out to be larger than the classical EOQ.

Chapter 4: Estimation of the Optimal Order Quantity

In general, the optimal order quantities, obtained in any stochastic model, are based on completely specified distributions of the decision variables (demand, supply, lead time, etc.) in terms of the parameters. However, in real life situations, these parameters and consequently the optimal order quantity remains unknown. Hence, in this situation, it is required to estimate the parameters involved in the EOQ. In this chapter we have taken up the problem of estimation of optimal order quantity.

Here we have considered demand distributions varying over both finite and infinite range. Specifically, we have considered rectangular demand in the finite case and exponential demand for the infinite case. We have worked out the optimal order quantities in both the situations under a classical newsboy setup. Evidently, these EOQs
involve parameters of the rectangular demand distribution in the first case and that of exponential distribution in the second. Assuming that we have a random sample from the demand distribution, we have worked out both parametric and non-parametric estimates of the EOQ. As far as parametric estimates are concerned, we have considered both method of moments and the method of maximum likelihood. In the case of non-parametric estimation, the asymptotic distributions of the estimates have been reported. In parametric estimation, for exponential demand, the exact distribution of the estimated EOQ has been worked out along with its mean and variance. In case of rectangular demand as well, the mean and variance of the estimate has been worked out and the sampling distribution of the estimate is reported.

Going beyond, we have considered the situation when the parameters of the demand distributions are not fixed, but vary randomly according to a prior distribution. In that situation, the EOQ remains valid for not only a specific demand distribution, but, for a wide class of demand distributions.

For exponential demand, we have taken a Gamma prior. Deriving the posterior distribution, we have worked out the Bayes’ estimate of the EOQ under squared error loss function and linex loss function. The corresponding Bayes’ risks have also been worked out. Equating the moments of the prior distribution with those of the sample of demand, we have further obtained the empirical Bayes’ estimate of the optimal order quantity.

In the case of rectangular demand, a uniform prior has been considered. Here also, we have worked out the posterior distribution and obtained the Bayes’ estimate of the EOQ under squared error loss. The empirical Bayes’ estimate was also obtained. In this case it
has been observed that empirical Bayes’ estimate of the EOQ exceeds the estimate found out by the method of moments.

To study the sensitivity of the parameters, hyper-parameters and the cost components, a number of numerical illustrations were considered. It was revealed that for exponential demand, the classical solution yields the highest value of the optimal order quantity. As far as Bayes’ estimates are concerned, the estimated optimal order quantities under squared error loss are slightly less that those under linear loss. The optimal costs also show similar behaviour. On the other hand, for rectangular demand, optimal order quantity obtained by classical method is found to less that the Bayes’ estimate for comparable parameters of conditional and prior distributions, that is, when the means and the variances of these distributions are equal.

Chapter 5: *Use of Prior and Posterior Demand Distributions*

In inventory models, the optimal order quantity is, in general, obtained by minimizing the expected total cost for specified distributions of the decision variables like demand, supply, lead time, etc. However, this expected total cost is conditional in nature in the sense that it is assumed that the parameters involved in these distributions are fixed. In a realistic situation it is possible that these parameters have a prior distribution. Hence, it is essential that the optimal order quantity should account for the variations in the parameters involved in the distributions of the decision variable.

In this chapter we have obtained the optimal order quantity in a classical newsboy setup by minimizing the unconditional expected total cost with respect to prior and posterior distributions. The use of posterior distribution makes the EOQ dependent not only on the
hyper-parameters, but also, on the sample data on demand. Going beyond, we have taken into account the more realistic situation when the supply corresponding to an order quantity is random.

Specifically, in this chapter we have considered exponential and rectangular demand distributions. The distributions of supply have been considered as rectangular and triangular varying around the order quantity ‘q’. The prior distributions considered are Gamma for exponential demand and uniform for rectangular demand.

In the first place, we have worked out the conditional expected total costs and consequently the optimal order quantities in case supply is same as the order quantity and when it varies randomly around the order quantity.

We then take the prior distributions into consideration and obtain the expected total cost with respect to these prior distributions, which are unconditional in nature. The optimal order quantity is then obtained by minimizing this unconditional expected total cost. This problem was taken up both in the cases when supply is same as the order quantity and when it varies randomly around the order quantity following a rectangular or a triangular distribution.

When we have sample data on demand at our disposal, we can think of an EOQ that depends on this data. In this situation, we have derived the posterior distribution for the demand distribution parameter and worked out the expected total cost with respect to the posterior distribution. The optimal order quantity was obtained consequently. This problem was also carried out considering supply to be same as the order quantity as well as when it is random.
It has been derived that, for exponential demand, the prior expectation minimizing EOQ (when supply is same as the order quantity) is approximately same as the corresponding expected conditional EOQ, the expectation being taken over the prior distribution of demand parameter. Also, in the same situation, the posterior expectation minimizing EOQ is approximately equal to the corresponding expected conditional EOQ, the expectation being taken over the posterior distribution.

For rectangular demand, it was seen that the conditional EOQs are same irrespective of the fact whether supply is same as the order quantity or supply varies randomly around the order quantity following a uniform or a triangular distribution. Although, the corresponding conditional expected total costs are different in different situations. It was similarly seen that the prior expectation minimizing EOQs are equal as well as the posterior expectation minimizing EOQs are equal. In these cases also, the corresponding costs are different.

The study of sensitivity of the parameters and the cost components was carried out through numerical illustrations. It was observed that, for exponential demand, the EOQs (and the associated costs) for classical situation, that is, when no prior distribution is considered, are less than the corresponding prior expectation minimizing EOQs (and the associated costs) and also posterior expected minimizing EOQs (and the associated costs). Further, the prior expectation minimizing EOQs (and the associated costs) are more than the posterior expectation minimizing EOQs (and the associated costs).

On the other hand, for rectangular demand, we found that The EOQs for classical situation, that is, when no prior distribution is considered, are more than the corresponding prior expectation minimizing EOQs and those, in turn, are more than the posterior expected minimizing EOQs. Also, the optimal costs associated with the
classical EOQs are less than the corresponding costs for prior expectation minimizing EOQs while the optimal costs associated with the classical EOQs are more than the corresponding costs for posterior expectation minimizing EOQs. It was further observed that the optimal costs associated with the prior expectation minimizing EOQs are more than the corresponding costs for posterior expectation minimizing EOQs.

Chapter 6: Case of Bivariate Demand

In this chapter we have considered the joint demand distribution of two items, supplementary or complementary and worked out the expected total cost. Optimal order quantities were obtained by minimizing the expected total cost. The joint demand distribution was obtained from the marginal distributions using the Farlie – Gumble – Morgenstern approach introducing an association parameter. The more realistic situation where the supplies vary over the order quantities was also considered in terms of a joint supply distribution of the two items. In this case also, the optimal order quantities were derived. Some numerical examples were also considered.

1.4 Contribution in the Present Work

The focus of the present research work is not to develop new inventory models to embrace more general situations and on their classical solutions. However, one contribution in the present work towards building and solving inventory models have been the consideration of possible random variations in supply against any order quantity. Two characteristically different situations in this context have been dealt with. In the first, supply varies randomly – symmetrically or otherwise – around the order
quantity, while in the second situation, the effective supply is a fraction of the quantity supplied. The latter is illustrated by the real life situation where the items supplied are inspected and classified as defective to be rejected or non-defective to be accepted, the effective inventory in such a case is in terms of non-defective items. The consequence is that, in the second situation, supply can never exceed demand.

Of the two major contributions in this dissertation, the first relates to the use of several different optimality criteria in deriving the optimal order quantity. In fact, most cost distributions are asymmetrical and have not well represented by their expectations only. There could be decision makers who are risk-averse and would like to guard themselves against committing very high costs, rather than focussing on the minimization of the expected cost. In fact, the basic problem being one of stochastic optimization involving a random objective function, three established approaches, viz., the E-Rule, the V-Rule and the P-Rule have been fully taken into account in the present work. The idea of using a standard deviation penalized mean of the cost distribution as the objective function can be regarded as a combination of both the E-Rule and the V-Rule.

The other major contribution rests in estimating the optimal order quantity in terms of sample data on demand by using the classical methods of moments or maximum likelihood as also the Bayesian method involving both the squared error and the linex loss function. A somewhat related result has been the derivation of a posterior expected cost minimizing solution which is not the same as the estimate of the optimal solution considering a prior distribution of the demand parameter and sample data.

While multi-item inventory models have been discussed earlier, the case of correlated demands has not found much of a consideration, especially when supplies against the respective order quantities are random. Optimal order quantities for both the items using
a joint demand distribution, both when supplies equal the order quantities and when these supplies vary randomly around the respective order quantities have been worked out.

1.5 Scope for Future Work

The present work is definitely limited to a very simple inventory model taken along with random variations in supply. This can possibly be generalized to other inventory models, though derivations of results would be quite involved. In the problem of estimation the present work has not come out with sampling distributions of the estimated order quantities in all the cases and confidence intervals have not been worked out.

Notations and Abbreviations Used:

(a) **PDF:** Probability Density Function

It is a function that represents the probability distribution of a continuous random variable.

(b) **CDF:** Cumulative Density Function

If $X$ be a random variable and $x$ be a realised value of $X$, the CDF $F(x)$ is defined as

$$F(x) = P(X \leq x).$$

(c) **SD:** Standard Deviation

It is a measure of absolute dispersion of a probability distribution, defined as

$$SD(X) = \sqrt{E[(X - E(X))^2]},$$

$E(X)$ being the Expectation.

(d) **CV:** Coefficient of Variation

It is a measure of relative dispersion of a probability distribution, defined as
\[ CV(X) = \frac{SD(X)}{E(X)} \]

(c) **MLE: Maximum Likelihood Estimator**

Let \( X_1, X_2, \ldots, X_n \) be a random sample drawn from some distribution characterised by some unknown parameter \( \theta \) varying over a parametric space \( \Theta \).

Let \( x_1, x_2, \ldots, x_n \) be the realized values of \( X_1, X_2, \ldots, X_n \). The likelihood function of \( \theta \) is the joint probability of the realised values from the distribution indicated as \( L(\varphi) \). The MLE of \( \theta \), say, \( \hat{\theta} \) is that value of \( \theta \) for which
\[ L(\hat{\theta}) = \sup_{\theta \in \Theta} L(\theta). \]