CHAPTER - 5

INTERIOR OPERATORS OVER BIPOLAR INTUITIONISTIC M- FUZZY PRIME GROUP, ANTI M- FUZZY PRIME GROUP AND SPECIAL OPERATOR OVER BIPOLAR INTUITIONISTIC M- FUZZY GROUP AND ANTI M- FUZZY GROUP.

Abstract

In this chapter the concept of a bipolar intuitionistic M- fuzzy prime group, M- fuzzy group and anti M- fuzzy prime group, anti M- fuzzy group is a new algebraic structure of a bipolar intuitionistic M- fuzzy prime subgroup of a M- fuzzy prime group, M- fuzzy group and anti M- fuzzy prime group, anti M- fuzzy group are defined, related operators and interior operators, special operators are investigated. The relation between operation of interior operators and special operator of bipolar intuitionistic M- fuzzy prime group, M- fuzzy group and bipolar intuitionistic anti M- fuzzy prime group, anti M- fuzzy group are established.

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5.1. Introduction

In this chapter we define interior operators, $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$ and $G_{\alpha,\beta}$ operators of bipolar intuitionistic $M$-fuzzy group and anti $M$-fuzzy group and we discuss various properties of bipolar intuitionistic $M$-fuzzy prime group, anti $M$-fuzzy prime group and special of bipolar intuitionistic $M$-fuzzy group and anti $M$-fuzzy group.

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5.2. Interior operators over bipolar intuitionistic M-Fuzzy prime group.

**Definition: 5.2.1**

Let $A$ be a bipolar intuitionistic fuzzy set of $E$, then the interior operator “$I$” is defined by

$$I(A^+) = \{ <x, \min(\mu^+_A(y)), \max(\nu^+_A(y))> \mid x, y \in E \}$$

and

$$I(A^-) = \{ <x, \max(\mu^-_A(y)), \min(\nu^-_A(y))> \mid x, y \in E \}.$$ 

**Definition: 5.2.2**

A bipolar fuzzy subgroup $A$ of a $M$-fuzzy group of $G$ is said to be bipolar intuitionistic $M$-fuzzy prime group or bipolar intuitionistic anti $M$-fuzzy group of $G$, if for all $x, m \in G$ either

$$\mu^+_A(mx) = \mu^+_A(m), \nu^+_A(mx) = \nu^+_A(m) \text{ and}$$

$$\mu^-_A(mx) = \mu^-_A(m), \nu^-_A(mx) = \nu^-_A(m) \text{ or else}$$

$$\mu^+_A(mx) = \mu^+_A(x), \nu^+_A(mx) = \nu^+_A(x) \text{ and}$$

$$\mu^-_A(mx) = \mu^-_A(x), \nu^-_A(mx) = \nu^-_A(x).$$

**Example: 5.2.3**

Let $G = \{1, 2, 3, 5, 7\}$ and $A = \{1, 2, 5, 7\}$.

$$\mu^+_A(x) = \begin{cases} 1 & \text{if } x \in \{7\} \\ 0.5 & \text{if } x \in \{2\} \land \neg \{7\} \end{cases} \quad \nu^+_A(x) = \begin{cases} 0 & \text{if } x \in \{7\} \\ 0.4 & \text{if } x \in \{2\} \land \neg \{7\} \end{cases}$$

and

$$\mu^-_A(x) = \begin{cases} -1 & \text{if } x \in \{7\} \\ -0.5 & \text{if } x \in \{2\} \land \neg \{7\} \end{cases} \quad \nu^-_A(x) = \begin{cases} 0 & \text{if } x \in \{7\} \\ -0.4 & \text{if } x \in \{2\} \land \neg \{7\} \end{cases}$$
Theorem: 5.2.4

If \( A \) is a bipolar intuitionistic \( M \) fuzzy prime group of \( G \), then \( I(A) \) is a bipolar intuitionistic \( M \) fuzzy prime group of \( G \).

Proof:

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{I(A)}^+(mx) = \min (\mu_A^+(ab)) \)
\[ \geq \min (\mu_A^+(b)) \] (by defn.1.3.9)
\[ = \mu_{I(A)}^+(x). \]

Therefore, \( \mu_{I(A)}^+(mx) \geq \mu_{I(A)}^+(x) \) for some \( a, m \in M \). ....(5.2.5)

Consider \( \nu_{I(A)}^+(mx) = \max (\nu_A^+(ab)) \)
\[ \leq \max (\nu_A^+(b)) \] (by defn.1.3.9)
\[ = \nu_{I(A)}^+(x). \]

Therefore, \( \nu_{I(A)}^+(mx) \leq \nu_{I(A)}^+(x) \) for some \( a, m \in M \). ....(5.2.6)

Consider \( \mu_{I(A)}^-(mx) = \max (\mu_A^-(ab)) \)
\[ \leq \max (\mu_A^-(b)) \] (by defn.1.3.9)
\[ = \mu_{I(A)}^-(x). \]

Therefore, \( \mu_{I(A)}^-(mx) \leq \mu_{I(A)}^-(x) \) for some \( a, m \in M \). ....(5.2.7)

Consider \( \nu_{I(A)}^-(mx) = \min (\nu_A^-(ab)) \)
\[ \geq \min (\nu_A^-(b)) \] (by defn.1.3.9)
\[ = \nu_{I(A)}^-(x). \]

Therefore, \( \nu_{I(A)}^-(mx) \geq \nu_{I(A)}^-(x) \) for some \( a, m \in M \). ....(5.2.8)

Therefore, \( I(A) \) is a bipolar intuitionistic \( M \)-fuzzy prime group of \( G \). \( \square \)
Theorem: 5.2.9

If \( A \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \), then

\[ I ( I ( A ) ) = I ( A ) \]

is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \).

Proof:

Let \( x, m \in G \) then \( x, m \in A \).

Consider

\[ \mu_{I(A)}^+(mx) = \min ( \mu_{I(A)}^+(ab) ) \]

\[ = \min ( \min ( \mu_{A}^+(mx) ) ) \]

\[ = \min ( \mu_{A}^+(mx) ) \]

\[ \geq \min ( \mu_{A}^+(x) ) \] (by defn.5.2.2)

\[ = \mu_{I(A)}^+(x). \] (by defn.5.2.2)

Therefore, \( \mu_{I(A)}^+(mx) \geq \mu_{I(A)}^+(x) \) for some \( m \in M \). ..........(5.2.10)

Consider

\[ \nu_{I(A)}^+(mx) = \max ( \nu_{I(A)}^+(ab) ) \]

\[ = \max ( \max ( \nu_{A}^+(mx) ) ) \]

\[ = \max ( \nu_{A}^+(mx) ) \]

\[ \leq \max ( \nu_{A}^+(x) ) \] (by defn.5.2.2)

\[ = \nu_{I(A)}^+(x). \] (by defn.5.2.1)

Therefore, \( \nu_{I(A)}^+(mx) \leq \nu_{I(A)}^+(x) \) for some \( m \in M \). ..........(5.2.11)

Consider

\[ \mu_{I(A)}^-(mx) = \max ( \mu_{I(A)}^-(ab) ) \]

\[ = \max ( \max ( \mu_{A}^-(mx) ) ) \]

\[ = \max ( \mu_{A}^-(mx) ) \]

\[ \leq \max ( \mu_{A}^-(x) ) \] (by defn.5.2.2)

\[ = \mu_{I(A)}^-(x). \] (by defn.5.2.1)

Therefore, \( \mu_{I(A)}^-(mx) \leq \mu_{I(A)}^-(x) \) for some \( m \in M \). ..........(5.2.12)
Consider \( v_{I(A)}^{-}\)(\(^{x}\)) = \min (v_{I(A)}^{-}(ab))

= \min (\min (v_{A}^{-}(mx)))

= \min (v_{A}^{-}(mx))

\geq \min (v_{A}^{-}(x)) \quad \text{(by defn.5.2.1)}

= v_{I(A)}^{-}(x).

Therefore, \( v_{I(A)}^{-}\)(\(^{x}\)) \geq v_{I(A)}^{-}(x) \) for some \( m \in M \). \quad \text{.....(5.2.13)}

Therefore, \( I(I(A)) = I(A) \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \).

\[ \square \]

**Theorem: 5.2.14**

If \( A \) and \( B \) are bipolar intuitionistic \( M \)-fuzzy prime group of \( G \), then

\( I(A \cap B) = I(A) \cap I(B) \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \cap B \) implies \( x, m \in A \) and \( x, m \in B \).

Consider \( \mu_{I(A \cap B)}^{+}(\mu_{x}^{m}) = \min (\mu_{A \cap B}^{+}(ab)) \)

\[ = \min (\min (\mu_{A}^{+}(ab), \mu_{B}^{+}(ab))) \quad \text{(by defn.3.2.1)} \]

\[ \geq \min (\min (\mu_{A}^{+}(b), \mu_{B}^{+}(b))) \quad \text{(by defn.1.3.9)} \]

\[ = \min (\mu_{A}^{+}(b), \min (\mu_{B}^{+}(b))) \]

\[ = \min (\mu_{I(A)}^{+}(x), \mu_{I(B)}^{+}(x)) \quad \text{(by defn.5.2.1)} \]

\[ = \mu_{I(A \cap B)}^{+}(x). \]

Therefore, \( \mu_{I(A \cap B)}^{+}(\mu_{x}^{m}) \geq \mu_{I(A \cap B)}^{+}(x) \) for some \( a, m \in M \). \quad \text{.....(5.2.15)}

Consider \( v_{I(A \cap B)}^{+}(\mu_{x}^{m}) = \max (v_{A \cap B}^{+}(ab)) \)

\[ = \max (\max (v_{A}^{+}(ab), v_{B}^{+}(ab))) \quad \text{(by defn.3.2.1)} \]

\[ = \max (\max (v_{A}^{+}(ab), v_{B}^{+}(ab))) \quad \text{(by defn.1.3.9)} \]
\[ \leq \max \left( \max (v^*_A(b)), \max (v^*_B(b)) \right) \]
\[ = \max (v^*_I(A)(x), v^*_I(B)(x)) \quad \text{(by defn.5.2.1)} \]
\[ = v^*_I(A \cap I(B))(x). \]

Therefore, \( v^*_I(A \cap I(B))(mx) \leq v^*_I(A \cap I(B))(x) \) for some \( a, m \in M \). \hspace{1cm} \text{(5.2.16)}

Consider \( \mu^-_{I(A \cap B)}(mx) = \max (\mu^-_{A \cap B}(ab)) \)
\[ = \max (\max (\mu^-_A(ab), \mu^-_B(ab))) \quad \text{(by defn.3.2.1)} \]
\[ \leq \max (\max (\mu^-_A(b), \mu^-_B(b))) \quad \text{(by defn.1.2.9)} \]
\[ = \max (\mu^-_I(A)(x), \mu^-_I(B)(x)) \quad \text{(by defn.5.2.1)} \]
\[ = \mu^-_{I(A \cap I(B))}(x). \]

Therefore, \( \mu^-_{I(A \cap B)}(mx) \leq \mu^-_{I(A \cap I(B))}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{(5.2.17)}

Consider, \( v^-_{I(A \cap B)}(mx) = \min (v^-_{A \cap B}(ab)) \)
\[ = \min (\min (v^-_A(ab), v^-_B(ab))) \quad \text{(by defn.3.2.1)} \]
\[ = \min (\min (v^-_A(ab), \min (v^-_B(ab)))) \quad \text{(by defn.1.2.9)} \]
\[ \geq \min (\min (v^-_A(b), \min (v^-_B(b)))) \]
\[ = \min (v^-_I(A)(x), v^-_I(B)(x)) \quad \text{(by defn.5.2.1)} \]
\[ = v^-_{I(A \cap I(B))}(x). \]

Therefore, \( v^-_{I(A \cap B)}(mx) \geq v^-_{I(A \cap I(B))}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{(5.2.18)}

Therefore, \( I(A \cap B) = I(A) \cap I(B) \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \). \hspace{1cm} \Box

**Theorem: 5.2.19**

If \( A \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \), then
\[ \Box(I(A)) = I(\Box(A)) \]
is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \).

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Proof:

Let $x, m \in G$ then $x, m \in A$.

Consider $\mu_{\text{cl}(I(A))}^{+}(mx) = \mu_{I(A)}^{+}(mx)$

$= \min (\mu_{A}^{+}(ab))$

$\geq \min (\mu_{A}^{+}(b))$ (by defn.1.3.9)

$= \min (\mu_{A}^{+}(b))$ (by defn.3.2.2)

$= \mu_{I(A)}^{+}(b)$ (by defn.5.2.1)

Therefore, $\mu_{\text{cl}(I(A))}^{+}(mx) \geq \mu_{I(A)}^{+}(x)$ for some $a, m \in M$. ....(5.2.20)

Consider $\nu_{\text{cl}(I(A))}^{+}(mx) = 1 - \mu_{\text{cl}(I(A))}^{+}(mx)$

$= 1 - (\mu_{I(A)}^{+}(mx))$

$= 1 - \min (\mu_{A}^{+}(ab))$

$\leq 1 - \min (\mu_{A}^{+}(b))$ (by defn.1.3.9)

$= 1 - \min (\mu_{A}^{+}(b))$ (by defn.3.2.2)

$= 1 - (\mu_{I(A)}^{+}(x))$ (by defn.5.2.1)

$= \nu_{I(A)}^{+}(x)$.

Therefore, $\nu_{\text{cl}(I(A))}^{+}(mx) \leq \nu_{I(A)}^{+}(x)$ for some $a, m \in M$. ....(5.2.21)

Consider $\mu_{\text{cl}(I(A))}^{-}(mx) = \mu_{I(A)}^{-}(mx)$

$= \max (\mu_{A}^{-}(ab))$

$\leq \max (\mu_{A}^{-}(b))$ (by defn.1.3.9)

$= \max (\mu_{A}^{-}(b))$ (by defn.3.2.2)

$= \mu_{I(A)}^{-}(x)$ (by defn.5.2.1)

Therefore, $\mu_{\text{cl}(I(A))}^{-}(mx) \leq \mu_{I(A)}^{+}(x)$ for some $a, m \in M$. ....(5.2.22)
Consider \( u_{\mathcal{I}(A)}^{-}(mx) = -1 - \mu_{\mathcal{I}(A)}^{-}(mx) \)
\[ = -1 - (\mu_{\mathcal{I}(A)}^{-}(mx)) \]
\[ = -1 - \max(\mu_{\mathcal{A}}(ab)) \]
\[ \geq -1 - \max(\mu_{\mathcal{A}}^{-}(b)) \quad \text{(by defn.1.3.9)} \]
\[ = -1 - \max(\mu_{\mathcal{U}_{A}}^{-}(b)) \quad \text{(by defn.3.2.2)} \]
\[ = -1 - (\mu_{\mathcal{I}_{\mathcal{U}(A)}}^{-}(x)) \quad \text{(by defn.5.2.1)} \]
\[ = u_{\mathcal{I}_{\mathcal{U}(A)}}^{-}(x). \]

Therefore, \( u_{\mathcal{I}(A)}^{-}(mx) \geq u_{\mathcal{I}_{\mathcal{U}(A)}}^{-}(x) \) for some \( a, m \in M \). ....(5.2.23)

Therefore, \( \mathcal{I}(A) = I(\mathcal{I}(A)) \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \).

**Theorem: 5.2.24**

If \( A \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \), then \( \mathcal{I}(A) = I(\mathcal{I}(A)) \) is an bipolar intuitionistic \( M \)-fuzzy prime group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{\mathcal{I}(A)}^{+}(mx) = 1 - u_{\mathcal{I}(A)}^{+}(mx) \)
\[ = 1 - \max(\nu_{\mathcal{A}}^{+}(ab)) \]
\[ \geq 1 - \max(\nu_{\mathcal{A}}^{+}(b)) \quad \text{(by defn.1.3.9)} \]
\[ = 1 - \max(\nu_{\mathcal{U}_{A}}^{+}(b)) \quad \text{(by defn.3.2.3)} \]
\[ = 1 - (\nu_{\mathcal{I}_{\mathcal{U}(A)}}^{+}(x)) \quad \text{(by defn.5.2.1)} \]
\[ = \mu_{\mathcal{I}_{\mathcal{U}(A)}}^{+}(x). \]

Therefore, \( \mu_{\mathcal{I}(A)}^{+}(mx) \geq \mu_{\mathcal{I}_{\mathcal{U}(A)}}^{+}(x) \) for some \( a, m \in M \). ....(5.2.25)
Consider $\nu_{0_{I(A)}}(mx) = \nu_{I(A)}^+(mx)$

$$= \max(\nu_A^+(ab)) \quad \text{(by defn.5.2.1)}$$

$$\leq \max(\nu_A^+(b)) \quad \text{(by defn.1.3.9)}$$

$$= \max(\nu_{0_A}^+(b)) \quad \text{(by defn.3.2.3)}$$

$$= \nu_{I(\emptyset A)}^+(x).$$

Therefore, $\nu_{0_{I(A)}}^+(mx) \leq \nu_{I(\emptyset A)}^+(x)$ for some $a, m \in M$. ....(5.2.26)

Consider $\mu_{0_{I(A)}}(mx) = -1 - \nu_{I(A)}^-(mx)$

$$= -1 - \min(\nu_A^-(ab))$$

$$\leq -1 - \min(\nu_A^-(b)) \quad \text{(by defn.1.3.9)}$$

$$= -1 - \min(\nu_{0_A}^-(b)) \quad \text{(by defn.3.2.3)}$$

$$= -1 - (\nu_{I(\emptyset A)}^-(x)) \quad \text{(by defn.5.2.1)}$$

$$= \mu_{I(\emptyset A)}^-(x).$$

Therefore, $\mu_{0_{I(A)}}^-(mx) \leq \mu_{I(\emptyset A)}^-(x)$ for some $a, m \in M$. ....(5.2.27)

Consider $\nu_{0_{I(A)}}^-(mx) = \nu_{I(A)}^-(mx)$

$$= \min(\nu_A^-(ab)) \quad \text{(by defn.5.2.1)}$$

$$\geq \min(\nu_A^-(b)) \quad \text{(by defn.1.3.9)}$$

$$= \min(\nu_{0_A}^-(b)) \quad \text{(by defn.3.2.3)}$$

$$= \nu_{I(\emptyset A)}^-(x).$$

Therefore, $\nu_{0_{I(A)}}^-(mx) \geq \nu_{I(\emptyset A)}^-(x)$ for some $a, m \in M$. ....(5.2.28)

Therefore, $\Diamond (I(A)) = I(\Diamond(A))$ is an bipolar intuitionistic $M$-fuzzy prime group of $G$. □

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5.3. Interior operator over bipolar intuitionistic anti M-Fuzzy prime group.

Theorem: 5.3.1

If $A$ is an bipolar intuitionistic anti $M$-fuzzy prime group of $G$, then $I(A)$ is an bipolar intuitionistic anti $M$-fuzzy prime group of $G$.

Proof:

Let $x, m \in G$ then $x, m \in A$.

Consider

$$\mu_{I(A)}^+(mx) = \min (\mu_A^+(ab))$$

$$\leq \min (\mu_A^+(b))$$

(by defn.1.3.9)

$$= \mu_{I(A)}^+(x).$$

Therefore, $\mu_{I(A)}^+(mx) \leq \mu_{I(A)}^+(x)$ for some $a, m \in M$. .....(5.3.2)

Consider

$$\nu_{I(A)}^+(mx) = \max (\nu_A^+(ab))$$

$$\geq \max (\nu_A^+(b))$$

(by defn.1.3.9)

$$= \nu_{I(A)}^+(x).$$

Therefore, $\nu_{I(A)}^+(mx) \geq \nu_{I(A)}^+(x)$ for some $a, m \in M$. .....(5.3.3)

Consider

$$\mu_{I(A)}^-(mx) = \max (\mu_A^-(ab))$$

$$\geq \max (\mu_A^-(b))$$

(by defn.1.3.9)

$$= \mu_{I(A)}^-(x).$$

Therefore, $\mu_{I(A)}^-(mx) \geq \mu_{I(A)}^-(x)$ for some $a, m \in M$. .....(5.3.4)

Consider

$$\nu_{I(A)}^-(mx) = \min (\nu_A^-(ab))$$

$$\leq \min (\nu_A^-(b))$$

(by defn.1.3.9)

$$= \nu_{I(A)}^-(x).$$

Therefore, $\nu_{I(A)}^-(mx) \leq \nu_{I(A)}^-(x)$ for some $a, m \in M$. .....(5.3.5)
Therefore, \( I(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \). \( \blacksquare \)

**Theorem: 5.3.6**

If \( A \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \), then \( I(I(A)) = I(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider

\[
\mu^+_{I(I(A))}(mx) = \min(\mu^+_{I(A)}(ab))
\]

\[
= \min(\min(\mu^+_A(mx)))
\]

\[
= \min(\mu^+_A(mx))
\]

\[
\leq \min(\mu^+_A(x)) \quad \text{(by defn.5.2.2)}
\]

\[
= \mu^+_I(x). \quad \text{(by defn.5.2.1)}
\]

Therefore, \( \mu^+_{I(I(A))}(mx) \leq \mu^+_I(x) \) for some \( m \in M \). \( .....(5.3.7) \)

Consider

\[
\nu^+_{I(I(A))}(mx) = \max(\nu^+_{I(A)}(ab))
\]

\[
= \max(\max(\nu^+_A(mx)))
\]

\[
= \max(\nu^+_A(mx))
\]

\[
\geq \max(\nu^+_A(x)) \quad \text{(by defn.5.2.2)}
\]

\[
= \nu^+_I(x). \quad \text{(by defn.5.2.1)}
\]

Therefore, \( \nu^+_{I(I(A))}(mx) \geq \nu^+_I(x) \) for some \( m \in M \). \( .....(5.3.8) \)

Consider

\[
\mu^-_{I(I(A))}(mx) = \max(\mu^-_{I(A)}(ab))
\]

\[
= \max(\max(\mu^-_A(mx)))
\]

\[
= \max(\mu^-_A(mx))
\]

\[
\geq \max(\mu^-_A(x)) \quad \text{(by defn.5.2.2)}
\]
Therefore, \( \mu_{I(A)}(x) \) is a bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \).

\[ \mu_{I(A)}(x) = \mu_{I(A)}^{-}(x). \quad \text{(by defn.5.2.1)} \]

Therefore, \( \mu_{I(I(A))}(mx) \geq \mu_{I(A)}(x) \) for some \( m \in M \). \hspace{1cm} \text{(5.3.9)}

Consider \( \nu_{I(I(A))}^{-}(mx) = \min (\nu_{I(A)}(ab)) \)

\[ = \min (\min (\nu_{A}^{-}(mx))) \]
\[ = \min (\nu_{A}^{-}(mx)) \]
\[ \leq \min (\nu_{A}^{-}(x)) \quad \text{(by defn.5.2.2)} \]
\[ = \nu_{I(A)}^{-}(x). \quad \text{(by defn.5.2.1)} \]

Therefore, \( \nu_{I(I(A))}^{-}(mx) \leq \nu_{I(A)}^{-}(x) \) for some \( m \in M \). \hspace{1cm} \text{(5.3.10)}

Therefore, \( I(A) = I(I(A)) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \). \[ \blacksquare \]

**Theorem: 5.3.11**

If \( A \) and \( B \) are bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \), then \( I(A \cap B) = I(A) \cap I(B) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \cap B \) implies \( x, m \in A \) and \( x, m \in B \).

Consider \( \mu_{I(A \cap B)}^{+}(mx) = \min (\mu_{I(A \cap B)}^{+}(ab)) \)

\[ = \min (\min (\mu_{A}^{+}(ab), \mu_{B}^{+}(ab))) \quad \text{(by defn.3.2.1)} \]
\[ \leq \min (\mu_{A}^{+}(b), \mu_{B}^{+}(b)) \quad \text{(by defn.1.3.9)} \]
\[ = \min (\mu_{A}^{+}(b), \min (\mu_{B}^{+}(b))) \]
\[ = \min (\mu_{I(A)}^{+}(x), \mu_{I(B)}^{+}(x)) \quad \text{(by defn.5.2.1)} \]
\[ = \mu_{I(I(A \cap B))}^{+}(x). \]

Therefore, \( \mu_{I(A \cap B)}^{+}(mx) \leq \mu_{I(A \cap I(B))}^{+}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{(5.3.12)}
Consider \( \nu^+_{I(A \cap B)}(mx) = \max (\nu^+_{A \cap B}(ab)) \)
\[= \max (\max(\nu^+_A(ab), \nu^+_B(ab))) \quad \text{(by defn.3.2.1)} \]
\[\geq \max (\max(\nu^+_A(b), \nu^+_B(b))) \quad \text{(by defn.1.3.9)} \]
\[= \max (\nu^+_{I(A)}(x), \nu^+_{I(B)}(x)) \quad \text{(by defn.5.2.1)} \]
\[= \nu^+_{I(A \cap I(B)}(x). \]
Therefore, \( \nu^+_{I(A \cap B)}(mx) \geq \nu^+_{I(A \cap I(B)}(x) \) for some \( a, m \in M \). .....(5.3.13)

Consider \( \mu^-_{I(A \cap B)}(mx) = \max (\mu^-_{A \cap B}(ab)) \)
\[= \max (\max(\mu^-_A(ab), \mu^-_B(ab))) \quad \text{(by defn.3.2.1)} \]
\[\geq \max (\max(\mu^-_A(b), \mu^-_B(b))) \quad \text{(by defn.1.3.9)} \]
\[= \max (\mu^-_{I(A)}(x), \mu^-_{I(B)}(x)) \quad \text{(by defn.5.2.1)} \]
\[= \mu^-_{I(A \cap I(B)}(x). \]
Therefore, \( \mu^-_{I(A \cap B)}(mx) \geq \mu^-_{I(A \cap I(B)}(x) \) for some \( a, m \in M \). .....(5.3.14)

Consider \( \nu^-_{I(A \cap B)}(mx) = \min (\nu^-_{A \cap B}(ab)) \)
\[= \min (\min(\nu^-_A(ab), \nu^-_B(ab))) \quad \text{(by defn.3.2.1)} \]
\[= \min (\min(\nu^-_A(ab), \min(\nu^-_B(ab)))) \quad \text{(by defn.1.3.9)} \]
\[\leq \min (\min(\nu^-_A(b), \min(\nu^-_B(b)))) \]
\[= \min (\nu^-_{I(A)}(x), \nu^-_{I(B)}(x)) \quad \text{(by defn.5.2.1)} \]
\[= \nu^-_{I(A \cap I(B)}(x). \]
Therefore, \( \nu^-_{I(A \cap B)}(mx) \leq \nu^-_{I(A \cap I(B)}(x) \) for some \( a, m \in M \). .....(5.3.15)

Therefore, \( I(A \cap B) = I(A) \cap I(B) \) is an bipolar intuitionistic anti-M-fuzzy prime group of \( G \).
Theorem: 5.3.16

If \( A \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \), then \( \Box (I(A)) = I(\Box(A)) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \).

Proof:

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{\Box(I(A))}^+ (mx) = \mu_{I(A)}^+ (mx) \)

\[
= \min (\mu_A^+ (ab))
\]

\[
\leq \min (\mu_A^+ (b)) \quad \text{(by defn.1.3.9)}
\]

\[
= \min (\mu_{\Box A}^+ (b)) \quad \text{(by defn.3.2.2)}
\]

\[
= \mu_{I(\Box A)}^+ (x) \quad \text{(by defn.5.2.1)}
\]

Therefore, \( \mu_{\Box(I(A))}^+ (mx) \leq \mu_{I(\Box A)}^+ (x) \) for some \( a, m \in M \). \( \cdots (5.3.17) \)

Consider \( \nu_{\Box(I(A))}^+ (mx) = 1 - \mu_{\Box(I(A))}^+ (mx) \)

\[
= 1 - (\mu_{I(A)}^+ (mx))
\]

\[
= 1 - \min (\mu_{I(A)}^+ (ab))
\]

\[
\geq 1 - \min (\mu_A^+ (b)) \quad \text{(by defn.1.3.9)}
\]

\[
= 1 - \min (\mu_{\Box A}^+ (b)) \quad \text{(by defn.3.2.2)}
\]

\[
= 1 - (\mu_{I(\Box A)}^+ (x)) \quad \text{(by defn.5.2.1)}
\]

\[
= \nu_{I(\Box(A))}^+ (x).
\]

Therefore, \( \nu_{\Box(I(A))}^+ (mx) \geq \nu_{I(\Box A)}^+ (x) \) for some \( a, m \in M \). \( \cdots (5.3.18) \)

Consider \( \mu_{\Box(I(A))}^- (mx) = \mu_{I(A)}^- (mx) \)

\[
= \max (\mu_A^- (ab))
\]

\[
\geq \max (\mu_A^- (b)) \quad \text{(by defn.1.3.9)}
\]

\[
= \max (\mu_{\Box A}^- (b)) \quad \text{(by defn.3.2.2)}
\]
Therefore, \( \mu_{\tilde{I}(I(A))}(mx) \geq \mu_{\tilde{I}(I(A))}(x) \) for some \( a, m \in M \). ....(5.3.19)

Consider \( \nu_{\tilde{I}(I(A))}(mx) = -1 - \mu_{\tilde{I}(I(A))}(mx) \)

\[ = -1 - (\mu_{\tilde{I}(A)}(mx)) \]
\[ = -1 - \max (\mu_{\tilde{I}(A)}(ab)) \] (by defn.1.3.9)
\[ \leq -1 - \max (\mu_{\tilde{I}(A)}(b)) \] (by defn.3.2.2)
\[ = -1 - \max (\mu_{\tilde{I}(A)}(b)) \] (by defn.5.2.1)
\[ = -1 - \mu_{\tilde{I}(I(A))}(x)) \]
\[ = \nu_{\tilde{I}(I(A))}(x). \]

Therefore, \( \nu_{\tilde{I}(I(A))}(mx) \leq \nu_{\tilde{I}(I(A))}(x) \) for some \( a, m \in M \). ....(5.3.20)

Therefore, \( \Box(I(A)) = I(\Box(A)) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \). ■

**Theorem: 5.3.21**

If \( A \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \), then 
\( \Diamond(I(A)) = I(\Diamond(A)) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{\tilde{I}(I(A))}(mx) = 1 - \nu_{\tilde{I}(I(A))}(mx) \)

\[ = 1 - \max (\nu_{\tilde{I}(A)}(ab)) \]
\[ \leq 1 - \max (\nu_{\tilde{I}(A)}(b)) \] (by defn.1.3.9)
\[ = 1 - \max (\nu_{\tilde{I}_A}(b)) \] (by defn.3.2.3)
\[ = 1 - (\nu_{\tilde{I}(I(A))}(x)) \] (by defn.5.2.1)
\[ \mu_{\hat{I}(\partial A)}(x). \]

Therefore, \( \mu_{\hat{I}(\partial A)}(mx) \leq \mu_{\hat{I}(\partial A)}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{....(5.3.22)}

Consider \( v^*_{\hat{I}(\partial A)}(mx) = v^*_{I(A)}(mx) \)

\[ = \max (v^*_{\hat{A}}(ab)) \quad \text{(by defn.5.2.1)} \]

\[ \geq \max (v^*_{\hat{A}}(b)) \quad \text{(by defn.1.3.9)} \]

\[ = \max (v^*_{\hat{0}}(b)) \quad \text{(by defn.3.2.3)} \]

\[ = v^*_{I(\partial A)}(x). \]

Therefore, \( v^*_{\hat{I}(\partial A)}(mx) \geq v^*_{I(\partial A)}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{....(5.3.23)}

Consider \( \mu^*_{\hat{I}(\partial A)}(mx) = -1 - v^*_{I(A)}(mx) \)

\[ = -1 - \min (v^*_{\hat{A}}(ab)) \]

\[ \geq -1 - \min (v^*_{\hat{A}}(b)) \quad \text{(by defn.1.3.9)} \]

\[ = -1 - \min (v^*_{\hat{0}}(b)) \quad \text{(by defn.3.2.3)} \]

\[ = -1 - (v^*_{\hat{I}(\partial A)}(x)) \quad \text{(by defn.5.2.1)} \]

\[ = \mu^*_{I(\partial A)}(x). \]

Therefore, \( \mu^*_{\hat{I}(\partial A)}(mx) \geq \mu^*_{I(\partial A)}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{....(5.3.24)}

Consider \( v^-_{\hat{I}(\partial A)}(mx) = v^-_{I(A)}(mx) \)

\[ = \min (v^-_{\hat{A}}(ab)) \quad \text{(by defn.5.2.1)} \]

\[ \leq \min (v^-_{\hat{A}}(b)) \quad \text{(by defn.1.3.9)} \]

\[ = \min (v^-_{\hat{0}}(b)) \quad \text{(by defn.3.2.3)} \]

\[ = v^-_{I(\partial A)}(x). \]

Therefore, \( v^-_{\hat{I}(\partial A)}(mx) \leq v^-_{I(\partial A)}(x) \) for some \( a, m \in M \). \hspace{1cm} \text{....(5.3.25)}

Therefore, \( \hat{\partial}(I(A)) = I(\hat{\partial}(A)) \) is an bipolar intuitionistic anti \( M \)-fuzzy prime group of \( G \) \hspace{1cm} \boxed{\text{\star}}

****
5.4. Special operators $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$ over bipolar intuitionistic $M$-Fuzzy group.

Definition: 5.4.1

Let $A$ be an bipolar intuitionistic fuzzy set of $E$, then the operator is defined by

$$P_{\alpha,\beta}(A^+) = \{ x \in A | \mu_A^+(x) = \max(\alpha \mu_A(x), \min(\beta, \nu_A^+(x)) > 1 \}$$

for $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

$$P_{\alpha,\beta}(A^-) = \{ x \in A | \nu_A^-(x) = \max(\alpha \mu_A(x), \min(\beta, \nu_A^-(x)) > 1 \}$$

for $\alpha, \beta \in [-1,0]$ and $\alpha + \beta \leq -1$.

Definition: 5.4.2

Let $A$ be an bipolar intuitionistic fuzzy set of $E$, then the operator is defined by

$$Q_{\alpha,\beta}(A^+) = \{ x \in A | \mu_A^+(x) = \max(\alpha \mu_A(x), \min(\beta, \nu_A^+(x)) > 1 \}$$

for $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

$$Q_{\alpha,\beta}(A^-) = \{ x \in A | \nu_A^-(x) = \max(\alpha \mu_A(x), \min(\beta, \nu_A^-(x)) > 1 \}$$

for $\alpha, \beta \in [-1,0]$ and $\alpha + \beta \leq -1$.

Definition: 5.4.3

Let $A$ be an bipolar intuitionistic fuzzy set of $E$, then the operator is defined by

$$G_{\alpha,\beta}(A^+) = \{ x \in A | \alpha \mu_A(x) = \beta \nu_A^+(x) > 1 \}$$

for $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

$$G_{\alpha,\beta}(A^-) = \{ x \in A | \alpha \mu_A(x) = \beta \nu_A^+(x) > 1 \}$$

for $\alpha, \beta \in [-1,0]$ and $\alpha + \beta \leq -1$. 

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Theorem 5.4.4

If $A$ is an bipolar intuitionistic $M$-fuzzy group of $G$, then $P_{a,b}(A)$ is an bipolar intuitionistic $M$-fuzzy group of $G$.

Proof:

Let $x, m \in G$ then $x \in A$ and $m \in M$.

Consider $\mu_{p_{a,b}}^+(A)(mx) = \max (\alpha, \mu_{a}^+(mx))$

$$\geq \max (\alpha, \mu_{a}^+(x)) \quad \text{(by defn.2.2.1)}$$

$$= \mu_{p_{a,b}}^+(A)(x). \quad \text{(by defn.5.4.1)}$$

Therefore, $\mu_{p_{a,b}}^+(A)(mx) \geq \mu_{p_{a,b}}^+(A)(x)$ for some $m \in M$. ....(5.4.5)

Consider $\nu_{p_{a,b}}^+(A)(mx) = \min (\beta, \nu_{a}^+(mx))$

$$\leq \min (\beta, \nu_{a}^+(x)) \quad \text{(by defn.2.2.1)}$$

$$= \nu_{p_{a,b}}^+(A)(x). \quad \text{(by defn.5.4.1)}$$

Therefore, $\nu_{p_{a,b}}^+(A)(mx) \leq \nu_{p_{a,b}}^+(A)(x)$ for some $m \in M$. ....(5.4.6)

Consider $\mu_{p_{a,b}}^-(A)(mx) = \min (\alpha, \mu_{a}^-(mx))$

$$\leq \min (\alpha, \mu_{a}^-(x)) \quad \text{(by defn.2.2.1)}$$

$$= \mu_{p_{a,b}}^-(A)(x). \quad \text{(by defn.5.4.1)}$$

Therefore, $\mu_{p_{a,b}}^-(A)(mx) \leq \mu_{p_{a,b}}^-(A)(x)$ for some $m \in M$. ......(5.4.7)

Consider $\nu_{p_{a,b}}^-(A)(mx) = \max (\beta, \nu_{a}^-(mx))$

$$\geq \max (\beta, \nu_{a}^-(x)) \quad \text{(by defn.2.2.1)}$$

$$= \nu_{p_{a,b}}^-(A)(x). \quad \text{(by defn.5.4.1)}$$

Therefore, $\nu_{p_{a,b}}^-(A)(mx) \geq \nu_{p_{a,b}}^-(A)(x)$ for some $m \in M$. ......(5.4.8)

Therefore, $P_{a,b}(A)$ is an bipolar intuitionistic $M$-fuzzy group of $G$. □
Theorem: 5.4.9

If $A$ and $B$ are bipolar intuitionistic $M$ fuzzy group of $G$, then
\[ P_{\alpha,\beta}(A \cap B) = P_{\alpha,\beta}(A) \cap P_{\alpha,\beta}(B) \]
is an bipolar intuitionistic $M$ fuzzy group of $G$ and for every $\alpha, \beta \in [0,1]$ and $\alpha, \beta \in [-1,0]$ implies $\alpha + \beta \leq 1$ and $\alpha + \beta \leq -1$.

Proof:

Let $x, m \in G$ then $x, m \in A \cap B$.

Consider
\[
\mu_{P_{\alpha,\beta}}^*(A \cap B)(mx) = \max (\alpha, \mu_{A \cap B}^*(mx))
\]
\[
= \max (\alpha, \min (\mu_A^*(mx), \mu_B^*(mx))) \quad \text{(by def.3.2.1)}
\]
\[
\geq \max (\alpha, \min (\mu_A^*(x), \mu_B^*(x))) \quad \text{(by def.2.2.1)}
\]
\[
= \min (\max (\alpha, \mu_A^*(x)), \max (\alpha, \mu_B^*(x)))
\]
\[
= \min (\mu_{P_{\alpha,\beta}}^*(A)(x), \mu_{P_{\alpha,\beta}}^*(B)(x)) \quad \text{(by def.5.4.1)}
\]
\[
= \mu_{P_{\alpha,\beta}}^*(A \cap B)(x).
\]
Therefore, $\mu_{P_{\alpha,\beta}}^*(A \cap B)(mx) \geq \mu_{P_{\alpha,\beta}}^*(A) \cap \mu_{P_{\alpha,\beta}}^*(B)(x)$ for some $m \in M$ ...(5.4.10)

Consider
\[
\nu_{P_{\alpha,\beta}}^*(A \cap B)(mx) = \min (\beta, \nu_{A \cap B}^*(mx))
\]
\[
= \min (\beta, \max (\nu_A^*(mx), \nu_B^*(mx))) \quad \text{(by def.3.2.1)}
\]
\[
\leq \min (\beta, \max (\nu_A^*(x), \nu_B^*(x))) \quad \text{(by def.2.2.1)}
\]
\[
= \max (\min (\beta, \nu_A^*(x)), \min (\beta, \nu_B^*(x)))
\]
\[
= \max (\nu_{P_{\alpha,\beta}}^*(A)(x), \nu_{P_{\alpha,\beta}}^*(B)(x)) \quad \text{(by def.5.4.1)}
\]
\[
= \nu_{P_{\alpha,\beta}}^*(A \cap B)(x).
\]
Therefore, $\nu_{P_{\alpha,\beta}}^*(A \cap B)(mx) \leq \nu_{P_{\alpha,\beta}}^*(A \cap B)(x)$ for some $m \in M$ ...(5.4.11)

Consider
\[
\mu_{P_{\alpha,\beta}}^-(A \cap B)(mx) = \min (\alpha, \mu_{A \cap B}^-(mx))
\]
\[
= \min (\alpha, \max (\mu_A^-(mx), \mu_B^-(mx))) \quad \text{(by def.3.2.1)}
\]
\[
\leq \min (\alpha, \max (\mu_A^-(x), \mu_B^-(x))) \quad \text{(by def.2.2.1)}
\]
\[ = \max \left( \min (\alpha, \mu^+_A(x)), \min (\alpha, \mu^-_A(x)) \right) \]
\[ = \max (\mu^+_{p_A,\beta}(A)(x), \mu^-_{p_A,\beta}(B)(x)) \quad \text{(by def.5.4.1)} \]
\[ = \mu^+_{p_A,\beta}(A) \cap \mu^-_{p_A,\beta}(B)(x). \]

Therefore, \( \mu^+_{p_A,\beta}(A \cap B)(mx) \leq \mu^+_{p_A,\beta}(A) \cap \mu^-_{p_A,\beta}(B)(x) \) for some \( m \in M \) ... (5.4.12)

Consider \( \nu^-_{p_A,\beta}(A \cap B)(mx) = \max (\beta, \nu^-_{A \cap B}(mx)) \)
\[ = \max (\beta, \min (\nu^-_A(mx), \nu^-_B(mx))) \quad \text{(by def.3.2.1)} \]
\[ \geq \max (\beta, \min (\nu^-_A(x), \nu^-_B(x))) \quad \text{(by def.2.2.1)} \]
\[ = \min (\max (\beta, \nu^-_A(x)), \max (\beta, \nu^-_B(x))) \]
\[ = \min (\nu^-_{p_A,\beta}(A)(x), \nu^-_{p_A,\beta}(B)(x)) \quad \text{(by def.5.4.1)} \]
\[ = \nu^-_{p_A,\beta}(A) \cap \nu^-_{p_A,\beta}(B)(x). \]

Therefore, \( \nu^-_{p_A,\beta}(A \cap B)(mx) \leq \nu^-_{p_A,\beta}(A) \cap \nu^-_{p_A,\beta}(B)(x) \) for some \( m \in M \) ... (5.4.13)

Therefore, \( P_{A,\beta}(A \cap B) = P_{A,\beta}(A) \cap P_{A,\beta}(B) \) is an bipolar intuitionistic \( M \)-fuzzy group of \( G \). \( \blacksquare \)

**Theorem: 5.4.14**

If \( A \) is an bipolar intuitionistic \( M \)-fuzzy group of \( G \), then \( Q_{A,\beta}(A) \) is an bipolar intuitionistic \( M \)-fuzzy group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu^+_{Q_{A,\beta}}(A)(mx) = \min (\alpha, \mu^+_A(mx)) \)
\[ \geq \min (\alpha, \mu^+_A(x)) \quad \text{(by def.2.2.1)} \]
\[ = \mu^+_{Q_{A,\beta}}(A)(x). \quad \text{(by def.5.4.2)} \]

Therefore, \( \mu^+_{Q_{A,\beta}}(A)(mx) \geq \mu^+_{Q_{A,\beta}}(A)(x) \) for some \( m \in M \). ..... (5.4.15)
Consider $\nu^+_{Q_{\alpha,\beta}}(A)(mx) = \max(\beta, \nu^+_A(mx))$

\[ \leq \max(\beta, \nu^+_A(x)) \]  
(by def.2.2.1)

\[ = \nu^+_{Q_{\alpha,\beta}}(A)(x). \]  
(by def.5.4.2)

Therefore, $\nu^+_{Q_{\alpha,\beta}}(A)(mx) \leq \nu^+_{Q_{\alpha,\beta}}(A)(x)$ for some $m \in M$. ......(5.4.16)

Consider $\mu^+_{Q_{\alpha,\beta}}(A)(mx) = \max(\alpha, \mu^+_A(mx))$

\[ \leq \max(\alpha, \mu^+_A(x)) \]  
(by def.2.2.1)

\[ = \mu^+_{Q_{\alpha,\beta}}(A)(x). \]  
(by def.5.4.2)

Therefore, $\mu^+_{Q_{\alpha,\beta}}(A)(mx) \leq \mu^+_{Q_{\alpha,\beta}}(A)(x)$ for some $m \in M$. ......(5.4.17)

Consider $\nu^-_{Q_{\alpha,\beta}}(A)(mx) = \min(\beta, \nu^-_A(mx))$

\[ \geq \min(\beta, \nu^-_A(x)) \]  
(by def.2.2.1)

\[ = \nu^-_{Q_{\alpha,\beta}}(A)(x). \]  
(by def.5.4.2)

Therefore, $\nu^-_{Q_{\alpha,\beta}}(A)(mx) \geq \nu^-_{Q_{\alpha,\beta}}(A)(x)$ for some $m \in M$. ......(5.4.18)

Therefore, $Q_{\alpha,\beta}(A)$ is an bipolar intuitionistic $M$-fuzzy group of $G$. ■

**Theorem: 5.4.19**

If $A$ and $B$ are bipolar intuitionistic $M$-fuzzy group of $G$, then $Q_{\alpha,\beta}(A \cap B) = Q_{\alpha,\beta}(A) \cap Q_{\alpha,\beta}(B)$ is an bipolar intuitionistic $M$-fuzzy group of $G$.

**Proof:**

Let $x, m \in G$ then $x, m \in A \cap B$.

Consider $\mu^+_{Q_{\alpha,\beta}}(A \cap B)(mx) = \min(\alpha, \mu^+_A(mx))$

\[ = \min(\alpha, \min(\mu^+_A(mx), \mu^-_B(mx))) \]  
(by def.3.2.1)

\[ \geq \min(\alpha, \min(\mu^+_A(x), \mu^-_B(x))) \]  
(by def.2.2.1)
\[ = \min \left( \min (\alpha, \mu_{\alpha}^+(x)), \min (\alpha, \mu_{\beta}^+(x)) \right) \]
\[ = \min \left( \mu_{\alpha,\beta}^+(A)(x), \mu_{\alpha,\beta}^+(B)(x) \right) \quad \text{(by def. 5.4.2)} \]
\[ = \mu_{\alpha,\beta}^+(A) \cap \mu_{\alpha,\beta}^+(B)(x). \]

Therefore, \( \mu_{\alpha,\beta}^+(A \cap B)(mx) \geq \mu_{\alpha,\beta}^+(A) \cap \mu_{\alpha,\beta}^+(B)(x) \) for some \( m \in M \) ....(5.4.20)

Consider \( \nu_{\alpha,\beta}^+(A \cap B)(mx) = \max \left( \beta, \nu_{\alpha,\beta}^+(mx) \right) \]
\[ = \max (\beta, \max (\nu_{\alpha}^+(mx), \nu_{\beta}^+(mx))) \quad \text{(by def. 3.2.1)} \]
\[ \leq \max (\beta, \max (\nu_{\alpha}^+(x), \nu_{\beta}^+(x))) \quad \text{(by def. 2.2.1)} \]
\[ = \max (\beta, \nu_{\alpha,\beta}^+(x)) \quad \text{(by def. 5.4.2)} \]
\[ = \nu_{\alpha,\beta}^+(A) \cap \nu_{\alpha,\beta}^+(B)(x). \]

Therefore, \( \nu_{\alpha,\beta}^+(A \cap B)(mx) \leq \nu_{\alpha,\beta}^+(A) \cap \nu_{\alpha,\beta}^+(B)(x) \) for some \( m \in M \) ....(5.4.21)

Consider \( \mu_{\alpha,\beta}^-(A \cap B)(mx) = \max (\alpha, \mu_{\alpha,\beta}^-(mx)) \]
\[ = \max (\alpha, \max (\mu_{\alpha}^-(mx), \mu_{\beta}^-(mx))) \quad \text{(by def. 3.2.1)} \]
\[ \leq \max (\alpha, \max (\mu_{\alpha}^-(x), \mu_{\beta}^-(x))) \quad \text{(by def. 2.2.1)} \]
\[ = \max (\max (\alpha, \mu_{\alpha}^-(x)), \max (\alpha, \mu_{\beta}^-(x))) \]
\[ = \max (\mu_{\alpha,\beta}^-(A)(x), \mu_{\alpha,\beta}^-(B)(x)) \quad \text{(by def. 5.4.2)} \]
\[ = \mu_{\alpha,\beta}^-(A) \cap \mu_{\alpha,\beta}^-(B)(x). \]

Therefore, \( \mu_{\alpha,\beta}^-(A \cap B)(mx) \leq \mu_{\alpha,\beta}^-(A) \cap \mu_{\alpha,\beta}^-(B)(x) \) for some \( m \in M \) ....(5.4.22)

Consider \( \nu_{\alpha,\beta}^-(A \cap B)(mx) = \min (\beta, \nu_{\alpha,\beta}^-(mx)) \]
\[ = \min (\beta, \min (\nu_{\alpha}^-(mx), \nu_{\beta}^-(mx))) \quad \text{(by def. 3.2.1)} \]
\[ \geq \min (\beta, \min (\nu_{\alpha}^-(x), \nu_{\beta}^-(x))) \quad \text{(by def. 2.2.1)} \]
\[ = \min (\beta, \nu_{\alpha}^-(x), \min (\beta, \nu_{\beta}^-(x))) \]
\[ = \min (\nu_{\alpha,\beta}^-(A)(x), \nu_{\alpha,\beta}^-(B)(x)) \quad \text{(by def. 5.4.2)} \]
\[ = \nu_{Q_{\alpha, \beta}}^{-}(A) \cap \nu_{Q_{\alpha, \beta}}^{-}(B)(x). \]

Therefore, \( \nu_{Q_{\alpha, \beta}}^{-}(A \cap B)(mx) \geq \nu_{Q_{\alpha, \beta}}^{-}(A) \cap \nu_{Q_{\alpha, \beta}}^{-}(B)(x) \) for some \( m \in M \) \( \ldots \)(5.4.23)

Therefore, \( Q_{\alpha, \beta}(A \cap B) = Q_{\alpha, \beta}(A) \cap Q_{\alpha, \beta}(B) \) is an bipolar intuitionistic M-fuzzy group of G. \( \square \)

**Theorem: 5.4.24**

If \( A \) is an bipolar intuitionistic M-fuzzy group of G, then \( \overline{P_{\alpha, \beta}(A)} = Q_{\beta, \alpha}(A) \) is an bipolar intuitionistic M-fuzzy group of G.

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx) = \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx) \) \( \text{(by def.3.2.1)} \)

\[ = \min (\beta, \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx)) \] \( \text{(by def.5.4.1)} \)

\[ = \min (\beta, \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx)) \]

\[ \geq \min (\beta, \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(x)) \] \( \text{(by def.2.2.1)} \)

\[ = \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(x). \] \( \text{(by def.5.4.2)} \)

Therefore, \( \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx) \geq \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(x) \) for some \( m \in M \). \( \ldots \)(5.4.25)

Consider \( \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx) = \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx) \) \( \text{(by def.3.2.1)} \)

\[ = \max (\alpha, \mu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx)) \] \( \text{(by def.5.4.1)} \)

\[ = \max (\alpha, \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx)) \]

\[ \leq \max (\alpha, \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(x)) \] \( \text{(by def.2.2.1)} \)

\[ = \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(x). \] \( \text{(by def.5.4.2)} \)

Therefore, \( \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(mx) \leq \nu_{\overline{P_{\alpha, \beta}(A)}}^{+}(x) \) for some \( m \in M \). \( \ldots \)(5.4.26)
Consider \( \mu^{-}_{P_{\alpha,\beta}(A)}(mx) = \nu^{-}_{P_{\alpha,\beta}(A)}(mx) \) (by def.3.2.1)
\[ = \max (\beta, \nu^{-}_{A}(mx)) \] (by def.5.4.1)
\[ = \max (\beta, \mu^{-}_{A}(mx)) \]
\[ \leq \max (\beta, \mu^{-}_{\alpha}(x)) \] (by def.2.2.1)
\[ = \mu^{-}_{Q_{\beta,\alpha}(A)}(x). \] (by def.5.4.2)
Therefore, \( \mu^{-}_{P_{\alpha,\beta}(A)}(mx) \leq \mu^{-}_{Q_{\beta,\alpha}(A)}(x) \) for some \( m \in M. \) \text{ ....}(5.4.27)

Consider \( \nu^{-}_{P_{\alpha,\beta}(A)}(mx) = \mu^{-}_{P_{\alpha,\beta}(A)}(mx) \) (by def.3.2.1)
\[ = \min (\alpha, \mu^{-}_{A}(mx)) \] (by def.5.4.1)
\[ = \min (\alpha, \nu^{-}_{A}(mx)) \]
\[ \geq \min (\alpha, \nu^{-}_{\alpha}(x)) \] (by def.2.2.1)
\[ = \nu^{-}_{Q_{\beta,\alpha}(A)}(x). \] (by def.5.4.2)
Therefore, \( \nu^{-}_{P_{\alpha,\beta}(A)}(mx) \geq \nu^{-}_{Q_{\beta,\alpha}(A)}(x) \) for some \( m \in M. \) \text{ ....}(5.4.28)

Therefore, \( P_{\alpha,\beta}(A) = Q_{\beta,\alpha}(A) \) is an bipolar intuitionistic M - fuzzy group of \( G. \) \( \blacksquare \)

*****
5.5. Special operator $G_{\alpha,\beta}$ over on bipolar intuitionistic M-Fuzzy group.

Theorem: 5.5.1

If $A$ is an bipolar intuitionistic $M$-fuzzy group of $G$, then $G_{\alpha,\beta}(A)$ is an bipolar intuitionistic $M$-fuzzy group of $G$.

Proof:

Let $x, m \in G$ then $x, m \in A$.

Consider $\mu_{G_{\alpha,\beta}(A)}^+(mx) = \alpha \mu_A^+(mx)$

\[ \geq \alpha \mu_A^+(x) \quad \text{(by def.2.2.1)} \]

\[ = \mu_{G_{\alpha,\beta}(A)}^+(x). \quad \text{(by def.5.4.3)} \]

Therefore, $\mu_{G_{\alpha,\beta}(A)}^+(mx) \geq \mu_{G_{\alpha,\beta}(A)}^+(x)$ for some $m \in M$. \hspace{1cm} \ldots (5.5.2)

Consider $\nu_{G_{\alpha,\beta}(A)}^+(mx) = \beta \nu_A^+(mx)$

\[ \leq \beta \nu_A^+(x) \quad \text{(by def.2.2.1)} \]

\[ = \nu_{G_{\alpha,\beta}(A)}^+(x). \quad \text{(by def.5.4.3)} \]

Therefore, $\nu_{G_{\alpha,\beta}(A)}^+(mx) \leq \nu_{G_{\alpha,\beta}(A)}^+(x)$ for some $m \in M$. \hspace{1cm} \ldots (5.5.3)

Consider $\mu_{G_{\alpha,\beta}(A)}^-(mx) = \alpha \mu_A^-(mx)$

\[ \leq \alpha \mu_A^-(x) \quad \text{(by def.2.2.1)} \]

\[ = \mu_{G_{\alpha,\beta}(A)}^-(x). \quad \text{(by def.5.4.3)} \]

Therefore, $\mu_{G_{\alpha,\beta}(A)}^-(mx) \leq \mu_{G_{\alpha,\beta}(A)}^-(x)$ for some $m \in M$. \hspace{1cm} \ldots (5.5.4)

Consider $\nu_{G_{\alpha,\beta}(A)}^-(mx) = \beta \nu_A^-(mx)$

\[ \geq \beta \nu_A^-(x) \quad \text{(by def.2.2.1)} \]

\[ = \nu_{G_{\alpha,\beta}(A)}^-(x). \quad \text{(by def.5.4.3)} \]
Therefore, \( \nu_{G_{a,\beta}}(A) \) is an bipolar intuitionistic \( M \)-fuzzy group of \( G \). \( \blacksquare \)

**Theorem: 5.5.6**

If \( A \) and \( B \) are bipolar intuitionistic \( M \)-fuzzy group of \( G \), then 
\[ G_{a,\beta}(A \cap B) = G_{a,\beta}(A) \cap G_{a,\beta}(B) \]
is an bipolar intuitionistic \( M \)-fuzzy group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \cap B \).

Consider 
\[ \mu_{G_{a,\beta}}^+(A \cap B)(mx) = \alpha \mu_{A \cap B}^+(mx) \]
\[ = \alpha \left( \min \left( \mu_{A}^+(mx), \mu_{B}^+(mx) \right) \right) \quad \text{(by def.3.2.1)} \]
\[ \geq \alpha \left( \min \left( \mu_{A}^+(x), \mu_{B}^+(x) \right) \right) \quad \text{(by def.2.2.1)} \]
\[ = \min \left( \alpha \mu_{A}^+(x), \alpha \mu_{B}^+(x) \right) \]
\[ = \min \left( \mu_{G_{a,\beta}}^+(A)(x), \mu_{G_{a,\beta}}^+(B)(x) \right) \quad \text{(by def.5.4.3)} \]
\[ = \mu_{G_{a,\beta}}^+(A \cap \mu_{G_{a,\beta}}^+(B)(x). \]

Therefore, 
\[ \mu_{G_{a,\beta}}^+(A \cap B)(mx) \geq \mu_{G_{a,\beta}}^+(A \cap \mu_{G_{a,\beta}}^+(B)(x) \text{ for some } m \in M \ldots(5.5.7) \]

Consider 
\[ \nu_{G_{a,\beta}}^+(A \cap B)(mx) = \beta \nu_{A \cap B}^+(mx) \]
\[ = \beta \left( \max \left( \nu_{A}^+(mx), \nu_{B}^+(mx) \right) \right) \quad \text{(by def.3.2.1)} \]
\[ \leq \beta \left( \max \left( \nu_{A}^+(x), \nu_{B}^+(x) \right) \right) \quad \text{(by def.2.2.1)} \]
\[ = \max \left( \beta \nu_{A}^+(x), \beta \nu_{B}^+(x) \right) \]
\[ = \max \left( \nu_{G_{a,\beta}}^+(A)(x), \nu_{G_{a,\beta}}^+(B)(x) \right) \quad \text{(by def.5.4.3)} \]
\[ = \nu_{G_{a,\beta}}^+(A \cap \nu_{G_{a,\beta}}^+(B)(x). \]

Therefore, 
\[ \nu_{G_{a,\beta}}^+(A \cap B)(mx) \leq \nu_{G_{a,\beta}}^+(A \cap \nu_{G_{a,\beta}}^+(B)(x) \text{ for some } m \in M \ldots(5.5.8) \]
Consider \( \mu_{G_{a,\beta}}(A \cap B)(m) = \alpha \mu_{A(x)}(m) \)
\[
= \alpha \left( \max \left( \mu_{A}^{+}(m), \mu_{A}^{-}(m) \right) \right) \quad \text{(by def.3.2.1)}
\]
\[
\leq \alpha \left( \max \left( \mu_{A}^{+}(x), \mu_{B}^{-}(x) \right) \right) \quad \text{(by def.2.2.1)}
\]
\[
= \max \left( \alpha \mu_{A}^{+}(x), \alpha \mu_{B}^{-}(x) \right)
\]
\[
= \max \left( \mu_{G_{a,\beta}}^{+}(A)(x), \mu_{G_{a,\beta}}^{-}(B)(x) \right) \quad \text{(by def.5.4.3)}
\]
\[
= \mu_{G_{a,\beta}}^{+}(A) \cap \mu_{G_{a,\beta}}^{-}(B)(x).
\]
Therefore, \( \mu_{G_{a,\beta}}^{+}(A \cap B)(m) \leq \mu_{G_{a,\beta}}^{+}(A) \cap \mu_{G_{a,\beta}}^{-}(B)(x) \) for some \( m \in M \). (5.5.9)

Consider \( \nu_{G_{a,\beta}}^{+}(A \cap B)(m) = \beta \nu_{A(x)}(m) \)
\[
= \beta \left( \min \left( \nu_{A}^{+}(m), \nu_{A}^{-}(m) \right) \right) \quad \text{(by def.3.2.1)}
\]
\[
\geq \beta \left( \min \left( \nu_{A}^{+}(x), \nu_{B}^{-}(x) \right) \right) \quad \text{(by def.2.2.1)}
\]
\[
= \min \left( \beta \nu_{A}^{+}(x), \beta \nu_{B}^{-}(x) \right)
\]
\[
= \min \left( \nu_{G_{a,\beta}}^{+}(A)(x), \nu_{G_{a,\beta}}^{-}(B)(x) \right) \quad \text{(by def.5.4.3)}
\]
\[
= \nu_{G_{a,\beta}}^{+}(A) \cap \nu_{G_{a,\beta}}^{-}(B)(x).
\]
Therefore, \( \nu_{G_{a,\beta}}^{+}(A \cap B)(m) \geq \nu_{G_{a,\beta}}^{+}(A) \cap \nu_{G_{a,\beta}}^{-}(B)(x) \) for some \( m \in M \). (5.5.10)

Therefore, \( G_{a,\beta}(A \cap B) = G_{a,\beta}(A) \cap G_{a,\beta}(B) \) is an bipolar intuitionistic

\( M \)-fuzzy group of \( G \). \( \blacksquare \)

**Theorem:** 5.5.11

If \( A \) is an bipolar intuitionistic \( M \)-fuzzy group of \( G \), then
\[
G_{a,\beta}(I(A)) = I G_{a,\beta}(A)
\]

is an bipolar intuitionistic \( M \)-fuzzy group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{G_{a,\beta}(I(A))}(m) = \alpha \mu_{I(A)}^{+}(m) \)
\[
= \alpha \min \left( \mu_{A}^{+}(ab) \right) \quad \text{(by def.5.2.1)}
\]
\[
= \alpha \mu_{G_{a,\beta}(I(A))}^{+}(m).
\]

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\[ \geq \alpha \min (\mu_\alpha^+(b)) \quad \text{(by def.1.3.9)} \]
\[ = \min (\alpha \mu_\alpha^+(b)) \]
\[ = \mu_{IG_{a,\beta}}^+(A)(x). \quad \text{(by def.5.2.1 and 5.4.3)} \]

Therefore, \[ \mu_{G_{a,\beta}}^+(mx) \geq \mu_{IG_{a,\beta}}^+(A)(x) \] for some \( a, x, b, m \in A \) ...\( (5.5.12) \)

Consider \[ v_{G_{a,\beta}}^+(I(A))(mx) = \beta v_{I(A)}^+(mx) \]
\[ = \beta \max (v_\alpha^+(ab)) \quad \text{(by def.5.2.1)} \]
\[ \leq \beta \max (v_\alpha^+(b)) \quad \text{(by def.1.3.9)} \]
\[ = \max (\beta v_\alpha^+(b)) \]
\[ = v_{IG_{a,\beta}}^+(A)(x). \quad \text{(by def.5.2.1 and 5.4.3)} \]

Therefore, \[ v_{G_{a,\beta}}^+(I(A))(mx) \leq v_{IG_{a,\beta}}^+(A)(x) \] for some \( a, x, b, m \in A \) ...\( (5.5.13) \)

Consider \[ \mu_{G_{a,\beta}}^-(I(A))(mx) = \alpha \mu_{I(A)}(mx) \]
\[ = \alpha \max (\mu_\alpha^-(ab)) \quad \text{(by def.5.2.1)} \]
\[ \leq \alpha \max (\mu_\alpha^-(b)) \quad \text{(by def.1.3.9)} \]
\[ = \max (\alpha \mu_\alpha^-(b)) \]
\[ = \mu_{IG_{a,\beta}}^-(A)(x). \quad \text{(by def.5.2.1 and 5.4.3)} \]

Therefore, \[ \mu_{G_{a,\beta}}^-(I(A))(mx) \leq \mu_{IG_{a,\beta}}^-(A)(x) \] for some \( a, x, b, m \in A \) ...\( (5.5.14) \)

Consider \[ v_{G_{a,\beta}}^-(I(A))(mx) = \beta v_{I(A)}^-(mx) \]
\[ = \beta \min (v_\alpha^-(ab)) \quad \text{(by def.5.2.1)} \]
\[ \geq \beta \min (v_\alpha^-(b)) \quad \text{(by def.1.3.9)} \]
\[ = \min (\beta v_\alpha^-(b)) \]
\[ = v_{IG_{a,\beta}}^-(A)(x). \quad \text{(by def.5.2.1 and 5.4.3)} \]

Therefore, \[ v_{G_{a,\beta}}^-(I(A))(mx) \geq v_{IG_{a,\beta}}^-(A)(x) \] for some \( a, x, b, m \in A \) ...\( (5.5.15) \)
Therefore, $G_{\alpha,\beta}(I(A)) = I G_{\alpha,\beta}(A)$ is an bipolar intuitionistic $M$-fuzzy group of $G$. □

**Theorem: 5.5.16**

If $A$ is an bipolar intuitionistic $M$-fuzzy group of $G$, then

$G_{\alpha,\beta}(A) = G_{\beta,\alpha}(A)$ is an bipolar intuitionistic $M$-fuzzy group of $G$.

**Proof:**

Let $x, m \in G$ then $x, m \in A$.

Consider

$$\mu^+_{G_{\alpha,\beta}(A)}(mx) = \nu^+_{G_{\alpha,\beta}(A)}(mx) \quad \text{(by def.3.2.1)}$$

$$= \beta \nu^+_\Lambda(mx) \quad \text{(by def.5.4.3)}$$

$$= \beta \mu^+_\Lambda(mx) \quad \text{(by def.3.2.1)}$$

$$\geq \beta \mu^+_\Lambda(x) \quad \text{(by def.2.2.1)}$$

$$= \mu^+_{G_{\beta,\alpha}(A)}(x).$$

Therefore, $\mu^+_{G_{\alpha,\beta}(A)}(mx) \geq \mu^+_{G_{\beta,\alpha}(A)}(x)$ for some $x, m \in A$. ....(5.5.17)

Consider

$$\nu^+_{G_{\alpha,\beta}(A)}(mx) = \mu^+_{G_{\alpha,\beta}(A)}(mx) \quad \text{(by def.3.2.1)}$$

$$= \alpha \mu^+_{\Lambda}(mx) \quad \text{(by def.5.4.3)}$$

$$= \alpha \nu^+_\Lambda(mx) \quad \text{(by def.3.2.1)}$$

$$\leq \alpha \nu^+_\Lambda(x) \quad \text{(by def.2.2.1)}$$

$$= \nu^+_{G_{\beta,\alpha}(A)}(x).$$

Therefore, $\nu^+_{G_{\alpha,\beta}(A)}(mx) \leq \nu^+_{G_{\beta,\alpha}(A)}(x)$ for some $x, m \in A$. ....(5.5.18)

Consider

$$\mu^−_{G_{\alpha,\beta}(A)}(mx) = \nu^−_{G_{\alpha,\beta}(A)}(mx) \quad \text{(by def.3.2.1)}$$

$$= \beta \nu^−_{\Lambda}(mx) \quad \text{(by def.5.4.3)}$$
\[ \beta \mu_{\beta}(mx) \quad \text{(by def.3.2.1)} \]

\[ \leq \beta \mu_{\beta}(x) \quad \text{(by def.2.2.1)} \]

\[ = \mu_{\beta,\mu}(x). \]

Therefore, \( \mu_{G_{\beta,\mu}(A)}(mx) \leq \mu_{G_{\beta,\mu}(A)}(x) \) for some \( x, m \in A \). \hspace{1cm} ...(5.5.19)

Consider \( \nu_{G_{\beta,\mu}(A)}(mx) = \mu_{G_{\beta,\mu}(A)}(mx) \quad \text{(by def.3.2.1)} \)

\[ = \alpha \mu_{\beta}(mx) \quad \text{(by def.5.4.3)} \]

\[ = \alpha \nu_{\beta}(mx) \quad \text{(by def.3.2.1)} \]

\[ \geq \alpha \nu_{\beta}(x) \quad \text{(by def.2.2.1)} \]

\[ = \nu_{G_{\beta,\mu}(A)}(x). \]

Therefore, \( \nu_{G_{\beta,\mu}(A)}(mx) \geq \nu_{G_{\beta,\mu}(A)}(x) \) for some \( x, m \in A \). \hspace{1cm} ...(5.5.20)

Therefore, \( G_{\alpha,\beta}(A) = G_{\beta,\alpha}(A) \) is an bipolar intuitionistic \( M \)-fuzzy group of \( G \). \hspace{1cm} \( \blacksquare \)

*****
5.6. Special operator $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$ over bipolar intuitionistic anti M-Fuzzy group.

Theorem 5.6.1

If $A$ is an bipolar intuitionistic anti $M$-fuzzy group of $G$, then $P_{\alpha,\beta}(A)$ is an bipolar intuitionistic anti $M$-fuzzy group of $G$.

Proof:

Let $x, m \in G$ then $x \in A$ and $m \in M$.

Consider $\mu^+_{P_{\alpha,\beta}}(A)(mx) = \max(\alpha, \mu^+_{A}(mx))$

$$\leq \max(\alpha, \mu^+_{A}(x)) \quad \text{(by def.2.2.3)}$$

$$= \mu^+_{P_{\alpha,\beta}}(A)(x). \quad \text{(by def.5.4.1)}$$

Therefore, $\mu^+_{P_{\alpha,\beta}}(A)(mx) \leq \mu^+_{P_{\alpha,\beta}}(A)(x)$ for some $m \in M$. .....(5.6.2)

Consider $\nu^+_{P_{\alpha,\beta}}(A)(mx) = \min(\beta, \nu^+_{A}(mx))$

$$\geq \min(\beta, \nu^+_{A}(x)) \quad \text{(by def.2.2.3)}$$

$$= \nu^+_{P_{\alpha,\beta}}(A)(x). \quad \text{(by def.5.4.1)}$$

Therefore, $\nu^+_{P_{\alpha,\beta}}(A)(mx) \geq \nu^+_{P_{\alpha,\beta}}(A)(x)$ for some $m \in M$. .....(5.6.3)

Consider $\mu^-_{P_{\alpha,\beta}}(A)(mx) = \min(\alpha, \mu^-_{A}(mx))$

$$\geq \min(\alpha, \mu^-_{A}(x)) \quad \text{(by def.2.2.3)}$$

$$= \mu^-_{P_{\alpha,\beta}}(A)(x). \quad \text{(by def.5.4.1)}$$

Therefore, $\mu^-_{P_{\alpha,\beta}}(A)(mx) \geq \mu^-_{P_{\alpha,\beta}}(A)(x)$ for some $m \in M$. .....(5.6.4)

Consider $\nu^-_{P_{\alpha,\beta}}(A)(mx) = \max(\beta, \nu^-_{A}(mx))$

$$\leq \max(\beta, \nu^-_{A}(x)) \quad \text{(by def.2.2.3)}$$
\[ = \nu_{p_{n,\beta}}(A)(x). \]  

(by def.5.4.1)

Therefore, \( \nu_{p_{n,\beta}}(A)(mx) \leq \nu_{p_{n,\beta}}(A)(x) \) for some \( m \in M \).  

.....(5.6.5)

Therefore, \( P_{\alpha,\beta}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).  

\[ \square \]

Theorem: 5.6.6

If \( A \) and \( B \) are bipolar intuitionistic anti \( M \)-fuzzy group of \( G \), then
\[ P_{\alpha,\beta}(A \cap B) = P_{\alpha,\beta}(A) \cap P_{\alpha,\beta}(B) \] is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \) and for every \( \alpha, \beta \in [0,1] \) and \( \alpha, \beta \in [-1,0] \) implies \( \alpha + \beta \leq 1 \) and \( \alpha + \beta \leq -1 \).

Proof:

Let \( x, m \in G \) then \( x, m \in A \cap B \).

Consider  
\[ \mu_{p_{n,\beta}}(A \cap B)(mx) = \max(\alpha, \mu_{\alpha\cap\beta}(mx)) \]
\[ = \max(\alpha, \min(\mu_{n}(mx), \mu_{\beta}(mx))) \]  

(by def.3.2.1)

\[ \leq \max(\alpha, \min(\mu_{n}(x), \mu_{\beta}(x))) \]  

(by def.2.2.3)

\[ = \min(\max(\alpha, \mu_{n}(x)), \max(\alpha, \mu_{\beta}(x))) \]
\[ = \min(\mu_{p_{n,\beta}}(A)(x), \mu_{p_{n,\beta}}(B)(x)) \]  

(by def.5.4.1)

\[ = \mu_{p_{n,\beta}}(A) \cap \mu_{p_{n,\beta}}(B)(x). \]

Therefore,  
\[ \mu_{p_{n,\beta}}(A \cap B)(mx) \leq \mu_{p_{n,\beta}}(A) \cap \mu_{p_{n,\beta}}(B)(x) \) for some \( m \in M \) ...(5.6.7)

Consider  
\[ \nu_{p_{n,\beta}}(A \cap B)(mx) = \min(\beta, \nu_{\alpha\cap\beta}(mx)) \]
\[ = \min(\beta, \max(\nu_{n}(mx), \nu_{\beta}(mx))) \]  

(by def.3.2.1)

\[ \geq \min(\beta, \max(\nu_{n}(x), \nu_{\beta}(x))) \]  

(by def.2.2.3)

\[ = \max(\min(\beta, \nu_{n}(x)), \min(\beta, \nu_{\beta}(x))) \]
\[ = \max(\nu_{p_{n,\beta}}(A)(x), \nu_{p_{n,\beta}}(B)(x)) \]  

(by def.5.4.1)
\[= \nu^+_{p,a,\beta} (A) \cap \nu^+_{p,a,\beta} (B) (x).\]

Therefore, \[\nu^+_{p,a,\beta} (A \cap B) (mx) \geq \nu^+_{p,a,\beta} (A) \cap \nu^+_{p,a,\beta} (B) (x)\] for some \(m \in M\) ...(5.6.8)

Consider \[\mu^+_{p,a,\beta} (A \cap B) (mx) = \min (\alpha, \mu^+_{\wedge \cap B} (mx))\]

\[= \min (\alpha, \max (\mu^+_{A} (mx), \mu^+_{B} (mx)))\] (by def.3.2.1)

\[\geq \min (\alpha, \max (\mu^+_{A} (x), \mu^+_{B} (x)))\] (by def.2.2.3)

\[= \max (\min (\alpha, \mu^+_{A} (x)), \min (\alpha, \mu^+_{B} (x)))\]

\[= \max (\mu^+_{p,a,\beta} (A) (x), \mu^+_{p,a,\beta} (B) (x))\] (by def.5.4.1)

\[= \mu^+_{p,a,\beta} (A) \cap \mu^+_{p,a,\beta} (B) (x).\]

Therefore, \[\mu^+_{p,a,\beta} (A \cap B) (mx) \geq \mu^+_{p,a,\beta} (A) \cap \mu^+_{p,a,\beta} (B) (x)\] for some \(m \in M\) ...(5.6.9)

Consider \[\nu^-_{p,a,\beta} (A \cap B) (mx) = \max (\beta, \nu^-_{A \cap B} (mx))\]

\[= \max (\beta, \min (\nu^-_{A} (mx), \nu^-_{B} (mx)))\] (by def.3.2.1)

\[\leq \max (\beta, \min (\nu^-_{A} (x), \nu^-_{B} (x)))\] (by def.2.2.3)

\[= \min (\max (\beta, \nu^-_{A} (x)), \max (\beta, \nu^-_{B} (x)))\]

\[= \min (\nu^-_{p,a,\beta} (A) (x), \nu^-_{p,a,\beta} (B) (x))\] (by def.5.4.1)

\[= \nu^-_{p,a,\beta} (A) \cap \nu^-_{p,a,\beta} (B) (x).\]

Therefore, \[\nu^-_{p,a,\beta} (A \cap B) (mx) \leq \nu^-_{p,a,\beta} (A) \cap \nu^-_{p,a,\beta} (B) (x)\] for some \(m \in M\) ...(5.6.10)

Therefore, \[P_{a,\beta} (A \cap B) = P_{a,\beta} (A) \cap P_{a,\beta} (B)\] is an bipolar intuitionistic anti \(M\)-fuzzy group of \(G\).

\[\square\]

**Theorem: 5.6.11**

If \(A\) is an bipolar intuitionistic anti \(M\)-fuzzy group of \(G\), then \(Q_{a,\beta} (A)\) is an bipolar intuitionistic anti \(M\)-fuzzy group of \(G\).
Proof:

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu_{Q,\alpha,\beta}^+(A)(mx) = \min(\alpha, \mu_A^+(mx)) \)

\[ \leq \min(\alpha, \mu_A^+(x)) \]  
(by def.2.2.3)

\[ = \mu_{Q,\alpha,\beta}^+(A)(x). \]  
(by def.5.4.2)

Therefore, \( \mu_{Q,\alpha,\beta}^+(A)(mx) \leq \mu_{Q,\alpha,\beta}^+(A)(x) \) for some \( m \in M \).  
.....(5.6.12)

Consider \( \nu_{Q,\alpha,\beta}^+(A)(mx) = \max(\beta, \nu_A^+(mx)) \)

\[ \geq \max(\beta, \nu_A^+(x)) \]  
(by def.2.2.3)

\[ = \nu_{Q,\alpha,\beta}^+(A)(x). \]  
(by def.5.4.2)

Therefore, \( \nu_{Q,\alpha,\beta}^+(A)(mx) \geq \nu_{Q,\alpha,\beta}^+(A)(x) \) for some \( m \in M \).  
.....(5.6.13)

Consider \( \mu_{Q,\alpha,\beta}^-(A)(mx) = \max(\alpha, \mu_A^-(mx)) \)

\[ \geq \max(\alpha, \mu_A^-(x)) \]  
(by def.2.2.3)

\[ = \mu_{Q,\alpha,\beta}^-(A)(x). \]  
(by def.5.4.2)

Therefore, \( \mu_{Q,\alpha,\beta}^-(A)(mx) \geq \mu_{Q,\alpha,\beta}^-(A)(x) \) for some \( m \in M \).  
.....(5.6.14)

Consider \( \nu_{Q,\alpha,\beta}^-(A)(mx) = \min(\beta, \nu_A^-(mx)) \)

\[ \leq \min(\beta, \nu_A^-(x)) \]  
(by def.2.2.3)

\[ = \nu_{Q,\alpha,\beta}^-(A)(x). \]  
(by def.5.4.2)

Therefore, \( \nu_{Q,\alpha,\beta}^-(A)(mx) \leq \nu_{Q,\alpha,\beta}^-(A)(x) \) for some \( m \in M \).  
.....(5.6.15)

Therefore, \( Q_{\alpha,\beta}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).  
\( \blacksquare \)
Theorem: 5.6.16

If $A$ and $B$ are bipolar intuitionistic anti $M$-fuzzy group of $G$, then $Q_{\alpha,\beta}(A \cap B) = Q_{\alpha,\beta}(A) \cap Q_{\alpha,\beta}(B)$ is an bipolar intuitionistic anti $M$-fuzzy group of $G$.

Proof:

Let $x, m \in G$ then $x, m \in A \cap B$.

Consider $\mu_{Q_{\alpha,\beta}}^+(A \cap B)(mx) = \min (\alpha, \mu_{Q_{\alpha,\beta}}^+(mx))$

$\leq \min (\alpha, \min (\mu_{A}^+(mx), \mu_{B}^+(mx)))$ (by def.3.2.1)

$= \min (\alpha, \min (\mu_{A}^+(x), \mu_{B}^+(x)))$ (by def.2.2.3)

$= \min (\min (\alpha, \mu_{A}^+(x)), \min (\alpha, \mu_{B}^+(x)))$

$= \min (\mu_{Q_{\alpha,\beta}}^+(A)(x), \mu_{Q_{\alpha,\beta}}^+(B)(x))$ (by def.5.4.2)

$= \mu_{Q_{\alpha,\beta}}^+(A) \cap \mu_{Q_{\alpha,\beta}}^+(B)(x)$.

Therefore, $\mu_{Q_{\alpha,\beta}}^+(A \cap B)(mx) \leq \mu_{Q_{\alpha,\beta}}^+(A) \cap \mu_{Q_{\alpha,\beta}}^+(B)(x)$ for some $m \in M$.(5.6.17)

Consider $\nu_{Q_{\alpha,\beta}}^+(A \cap B)(mx) = \max (\beta, \nu_{Q_{\alpha,\beta}}^+(mx))$

$\leq \max (\beta, \max (\nu_{A}^+(mx), \nu_{B}^+(mx)))$ (by def.3.2.1)

$\geq \max (\beta, \max (\nu_{A}^+(x), \nu_{B}^+(x)))$ (by def.2.2.3)

$= \max (\max (\beta, \nu_{A}^+(x)), \max (\beta, \nu_{B}^+(x)))$

$= \max (\nu_{Q_{\alpha,\beta}}^+(A)(x), \nu_{Q_{\alpha,\beta}}^+(B)(x))$ (by def.5.4.2)

$= \nu_{Q_{\alpha,\beta}}^+(A) \cap \nu_{Q_{\alpha,\beta}}^+(B)(x)$.

Therefore, $\nu_{Q_{\alpha,\beta}}^+(A \cap B)(mx) \geq \nu_{Q_{\alpha,\beta}}^+(A) \cap \nu_{Q_{\alpha,\beta}}^+(B)(x)$ for some $m \in M$.(5.6.18)

Consider $\mu_{\nu_{\alpha,\beta}}^+(A \cap B)(mx) = \max (\alpha, \mu_{\nu_{\alpha,\beta}}^+(mx))$

$= \max (\alpha, \max (\mu_{A}^-(mx), \mu_{B}^-(mx)))$ (by def.3.2.1)

$\geq \max (\alpha, \max (\mu_{A}^-(x), \mu_{B}^-(x)))$ (by def.2.2.3)

$= \max (\max (\alpha, \mu_{A}^-(x)), \max (\alpha, \mu_{B}^-(x)))$
Therefore, \( \mu_{\lambda \alpha \beta_{a,\beta}}(A \cap B) \geq \mu_{\lambda \alpha \beta_{a,\beta}}(A) \cap \mu_{\lambda \alpha \beta_{a,\beta}}(B)(x) \) for some \( m \in M \). (5.6.19)

Consider \( v_{\lambda \alpha \beta_{a,\beta}}^-(A \cap B)(mx) = \min(\beta, v_{\lambda \alpha \beta_{a,\beta}}^-(mx)) \)

\[ = \min(\beta, \min(v_{\lambda \alpha \beta_{a,\beta}}^- (mx), v_{\beta \alpha \beta_{a,\beta}}^- (mx))) \]  

(by def.3.2.1)

\[ \leq \min(\beta, \min(v_{\lambda \alpha \beta_{a,\beta}}^- (x), v_{\beta \alpha \beta_{a,\beta}}^- (x))) \]  

(by def.5.4.1)

\[ = \min(\beta, v_{\lambda \alpha \beta_{a,\beta}}^-(x), \min(\beta, v_{\beta \alpha \beta_{a,\beta}}^-(x))) \]

\[ = \min(v_{\lambda \alpha \beta_{a,\beta}}^- (A)(x), v_{\beta \alpha \beta_{a,\beta}}^- (B)(x)) \]  

(by def.5.4.3)

\[ = v_{\lambda \alpha \beta_{a,\beta}}^-(A) \cap v_{\beta \alpha \beta_{a,\beta}}^- (B)(x). \]

Therefore, \( v_{\lambda \alpha \beta_{a,\beta}}^-(A \cap B)(mx) \leq v_{\lambda \alpha \beta_{a,\beta}}^-(A) \cap v_{\beta \alpha \beta_{a,\beta}}^- (B)(x) \) for some \( m \in M \). (5.6.20)

Therefore, \( Q_{\alpha \beta_{a,\beta}}(A \cap B) = Q_{\alpha \beta_{a,\beta}}(A) \cap Q_{\beta \alpha \beta_{a,\beta}}(B) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \). □

**Theorem 5.6.21**

If \( A \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \), then \( \overline{P_{\alpha \beta_{a,\beta}}(A)} = Q_{\beta \alpha \beta_{a,\beta}}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \).

Consider \( \mu^+\overline{P_{\alpha \beta_{a,\beta}}(A)}(mx) = v_{\mu^+\overline{P_{\alpha \beta_{a,\beta}}(A)}}^+(mx) \)  

(by def.3.2.1)

\[ = \min(\beta, v_{\mu^+\overline{P_{\alpha \beta_{a,\beta}}(A)}}^+(mx)) \]  

(by def.5.4.1)

\[ = \min(\beta, \mu_A^+(mx)) \]  

(by def.5.4.3)

\[ \leq \min(\beta, v_A^+(x)) \]  

(by def.2.2.3)

\[ = \mu_{Q_{\beta \alpha \beta_{a,\beta}}(A)}^+(x). \]  

(by def.5.4.2)
Therefore, $\mu_{p_{u,\beta}(A)}^+(mx) \leq \mu_{Q_{\beta,\alpha}(A)}^+(x)$ for some $m \in M$. .....(5.6.22)

Consider $\nu_{p_{u,\beta}(A)}^+(mx) = \mu_{p_{u,\beta}(A)}^+(mx)$ (by def.3.2.1)

$= \max(\alpha, \mu_{\beta}^+(mx))$ (by def.5.4.1)

$= \max(\alpha, \nu_{A}^+(mx))$

$\geq \max(\alpha, \nu_{A}^+(x))$ (by def.2.2.3)

$= \nu_{Q_{\beta,\alpha}(A)}^+(x)$. (by def.5.4.2)

Therefore, $\nu_{p_{u,\beta}(A)}^+(mx) \geq \nu_{Q_{\beta,\alpha}(A)}^+(x)$ for some $m \in M$. .....(5.6.23)

Consider $\mu_{p_{u,\beta}(A)}^-(mx) = \nu_{p_{u,\beta}(A)}^-(mx)$ (by def.3.2.1)

$= \max(\beta, \nu_{\beta}^-(mx))$ (by def.5.4.1)

$= \max(\beta, \mu_{\beta}^-(mx))$

$\geq \max(\beta, \mu_{\beta}^-(x))$ (by def.2.2.3)

$= \mu_{Q_{\beta,\alpha}(A)}^-(x)$. (by def.5.4.2)

Therefore, $\mu_{p_{u,\beta}(A)}^-(mx) \leq \mu_{Q_{\beta,\alpha}(A)}^-(x)$ for some $m \in M$. .....(5.6.24)

Consider $\nu_{p_{u,\beta}(A)}^-(mx) = \mu_{p_{u,\beta}(A)}^-(mx)$ (by def.3.2.1)

$= \min(\alpha, \mu_{\beta}^-(mx))$ (by def.5.4.1)

$= \min(\alpha, \nu_{A}^-(mx))$

$\leq \min(\alpha, \nu_{A}^-(x))$ (by def.2.2.3)

$= \nu_{Q_{\beta,\alpha}(A)}^-(x)$. (by def.5.4.2)

Therefore, $\nu_{p_{u,\beta}(A)}^-(mx) \leq \nu_{Q_{\beta,\alpha}(A)}^-(x)$ for some $m \in M$. .....(5.6.25)

Therefore, $p_{\alpha,\beta}(A) = Q_{\beta,\alpha}(A)$ is an bipolar intuitionistic anti $M$-fuzzy group of $G$. □

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5.7. Special operator $G_{\alpha,\beta}$ over on bipolar intuitionistic anti $M$-Fuzzy group.

**Theorem 5.7.1**

If $A$ is an bipolar intuitionistic anti $M$ - fuzzy group of $G$, then $G_{\alpha,\beta}(A)$ is an bipolar intuitionistic anti $M$ - fuzzy group of $G$.

**Proof:**

Let $x, m \in G$ then $x, m \in A$.

Consider

$$\mu^+_{G_{a,\beta}(A)}(mx) = \alpha \mu^+_A(mx)$$

$$\leq \alpha \mu^+_A(x)$$

(by def.2.2.3)

$$= \mu^+_G(x)$$

(by def.5.4.3)

Therefore, $\mu^+_G(x) \leq \mu^+_G(x)$ for some $m \in M$.  

.....(5.7.2)

Consider

$$\nu^+_{G_{a,\beta}(A)}(mx) = \beta \nu^+_A(mx)$$

$$\geq \beta \nu^+_A(x)$$

(by def.2.2.3)

$$= \nu^+_G(x)$$

(by def.5.4.3)

Therefore, $\nu^+_G(x) \geq \nu^+_G(x)$ for some $m \in M$.  

.....(5.7.3)

Consider

$$\mu^-_{G_{a,\beta}(A)}(mx) = \alpha \mu^-_A(mx)$$

$$\geq \alpha \mu^-_A(x)$$

(by def.2.2.3)

$$= \mu^-_G(x)$$

(by def.5.4.3)

Therefore, $\mu^-_G(x) \geq \mu^-_G(x)$ for some $m \in M$.  

.....(5.7.4)

Consider

$$\nu^-_{G_{a,\beta}(A)}(mx) = \beta \nu^-_A(mx)$$

$$\leq \beta \nu^-_A(x)$$

(by def.2.2.3)
\[ v_{G_{\alpha,\beta}}(x) = v_{G_{\alpha,\beta}}(A)(x). \]  
(by def.5.4.3)

Therefore, \( v_{G_{\alpha,\beta}}(mx) \leq v_{G_{\alpha,\beta}}(A)(x) \) for some \( m \in M \). ....(5.7.5)

Therefore, \( G_{\alpha,\beta}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).  

\[ \square \]

**Theorem: 5.7.6**

If \( A \) and \( B \) are bipolar intuitionistic anti \( M \)-fuzzy group of \( G \), then 
\[ G_{\alpha,\beta}(A \cap B) = G_{\alpha,\beta}(A) \cap G_{\alpha,\beta}(B) \]  
is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).

**Proof:**

Let \( x, m \in G \) then \( x, m \in A \cap B \).

Consider  
\[ \mu_{G_{\alpha,\beta}}^+(A \cap B)(mx) = \alpha \mu_{G_{\alpha,\beta}}^+(mx) \]
= \( \alpha \left( \min \left( \mu_{G_{\alpha,\beta}}^+(mx), \mu_{G_{\alpha,\beta}}^+(mx) \right) \right) \)  
(by def.3.2.1)
\[ \leq \alpha \left( \min \left( \mu_{G_{\alpha,\beta}}^+(x), \mu_{G_{\alpha,\beta}}^+(x) \right) \right) \]  
(by def.2.2.3)
\[ = \min \left( \alpha \mu_{G_{\alpha,\beta}}^+(x), \alpha \mu_{G_{\alpha,\beta}}^+(x) \right) \]
\[ = \min \left( \mu_{G_{\alpha,\beta}}^+(A)(x), \mu_{G_{\alpha,\beta}}^+(B)(x) \right) \]  
(by def.5.4.3)
\[ = \mu_{G_{\alpha,\beta}}^+(A) \cap \mu_{G_{\alpha,\beta}}^+(B)(x). \]

Therefore, \( \mu_{G_{\alpha,\beta}}^+(A \cap B)(mx) \leq \mu_{G_{\alpha,\beta}}^+(A) \cap \mu_{G_{\alpha,\beta}}^+(B)(x) \) for some \( m \in M \) ....(5.7.7)

Consider  
\[ v_{G_{\alpha,\beta}}^+(A \cap B)(mx) = \beta v_{G_{\alpha,\beta}}^+(mx) \]
= \( \beta \left( \max \left( v_{G_{\alpha,\beta}}^+(mx), v_{G_{\alpha,\beta}}^+(mx) \right) \right) \)  
(by def.3.2.1)
\[ \geq \beta \left( \max \left( v_{G_{\alpha,\beta}}^+(x), v_{G_{\alpha,\beta}}^+(x) \right) \right) \]  
(by def.2.2.3)
\[ = \max \left( \beta v_{G_{\alpha,\beta}}^+(x), \beta v_{G_{\alpha,\beta}}^+(x) \right) \]
\[ = \max \left( v_{G_{\alpha,\beta}}^+(A)(x), v_{G_{\alpha,\beta}}^+(B)(x) \right) \]  
(by def.5.4.3)
\[ = \nu_{G_{a,\beta}}^+(A) \cap \nu_{G_{a,\beta}}^-(B)(x). \]

Therefore, \[ \nu_{G_{a,\beta}}^+(A \cap B)(mx) \geq \nu_{G_{a,\beta}}^+(A) \cap \nu_{G_{a,\beta}}^-(B)(x) \] for some \( m \in M \) \( \ldots \) (5.7.8)

Consider \( \mu_{G_{a,\beta}}^-(A \cap B)(mx) = \alpha \mu_{\lambda \cap \mu}^-(mx) \)
\[ = \alpha \max (\mu_{\lambda}^-(mx), \mu_{\beta}^-(mx)) \] (by def. 3.2.1)
\[ \geq \alpha \max (\mu_{\lambda}^-(x), \mu_{\beta}^-(x)) \] (by def. 2.2.3)
\[ = \max (\alpha \mu_{\lambda}^-(x), \alpha \mu_{\beta}^-(x)) \]
\[ \geq \max (\mu_{G_{a,\beta}}^-(A)(x), \mu_{G_{a,\beta}}^-(B)(x)) \] (by def. 5.4.3)
\[ = \mu_{G_{a,\beta}}^-(A) \cap \mu_{G_{a,\beta}}^-(B)(x). \]

Therefore, \[ \mu_{G_{a,\beta}}^-(A \cap B)(mx) \geq \mu_{G_{a,\beta}}^- (A) \cap \mu_{G_{a,\beta}}^- (B)(x) \] for some \( m \in M \) \( \ldots \) (5.7.9)

Consider \[ \psi_{G_{a,\beta}}^-(A \cap B)(mx) = \beta \psi_{\lambda \cap \mu}^- (mx) \]
\[ = \beta \min (\psi_{\lambda}^-(mx), \psi_{\beta}^-(mx)) \] (by def. 3.2.1)
\[ \leq \beta \min (\psi_{\lambda}^-(x), \psi_{\beta}^-(x)) \] (by def. 2.2.3)
\[ = \min (\beta \psi_{\lambda}^-(x), \beta \psi_{\beta}^-(x)) \]
\[ = \min (\psi_{G_{a,\beta}}^- (A)(x), \psi_{G_{a,\beta}}^- (B)(x)) \] (by def. 5.4.3)
\[ = \psi_{G_{a,\beta}}^- (A) \cap \psi_{G_{a,\beta}}^- (B)(x). \]

Therefore, \[ \psi_{G_{a,\beta}}^- (A \cap B)(mx) \leq \psi_{G_{a,\beta}}^- (A) \cap \psi_{G_{a,\beta}}^- (B)(x) \] for some \( m \in M \) \( \ldots \) (5.7.10)

Therefore, \( G_{a,\beta} (A \cap B) = G_{a,\beta} (A) \cap G_{a,\beta} (B) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).

\[ \square \]

**Theorem 5.7.11**

If \( A \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \), then \( G_{a,\beta} (I(A)) = I G_{a,\beta} (A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G \).

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Proof:

Let \( x,m \in G \) then \( x,m \in A \).

Consider \( \mu_{G,A,G,A}^+ (mx) = \alpha \mu_{I(A)}^+ (mx) \)

\[
= \alpha \min ( \mu_A^+ (ab) ) \quad \text{(by def.5.2.1)}
\]

\[
\leq \alpha \min ( \mu_A^+ (b) ) \quad \text{(by def.1.3.9)}
\]

\[
= \min ( \alpha \mu_A^+ (b) )
\]

\[
= \mu_{IG_a,G_a}^+ (x). \quad \text{(by def.5.2.1 and 5.4.3)}
\]

Therefore, \( \mu_{G,A,G,A}^+ (mx) \leq \mu_{IG_a,G_a}^+ (x) \) for some \( a,x,b,m \in A \) ....\( (5.7.12) \)

Consider \( \nu_{G,A,G,A}^+ (mx) = \beta \nu_{I(A)}^+ (mx) \)

\[
= \beta \max ( \nu_A^+ (ab) ) \quad \text{(by def.5.2.1)}
\]

\[
\geq \beta \max ( \nu_A^+ (b) ) \quad \text{(by def.1.2.9)}
\]

\[
= \max ( \beta \nu_A^+ (b) )
\]

\[
= \nu_{IG_a,G_a}^+ (x). \quad \text{(by def.5.2.1 and 5.4.3)}
\]

Therefore, \( \nu_{G,A,G,A}^+ (mx) \geq \nu_{IG_a,G_a}^+ (x) \) for some \( a,x,b,m \in A \) ....\( (5.7.13) \)

Consider \( \mu_{G,A,G,A}^- (mx) = \alpha \mu_{I(A)}^- (mx) \)

\[
= \alpha \min ( \mu_A^- (ab) ) \quad \text{(by def.5.2.1)}
\]

\[
\geq \alpha \min ( \mu_A^- (b) ) \quad \text{(by def.1.3.9)}
\]

\[
= \min ( \alpha \mu_A^- (b) )
\]

\[
= \mu_{IG_a,G_a}^- (x). \quad \text{(by def.5.2.1 and 5.4.3)}
\]

Therefore, \( \mu_{G,A,G,A}^- (mx) \geq \mu_{IG_a,G_a}^- (x) \) for some \( a,x,b,m \in A \) ....\( (5.7.14) \)

Consider \( \nu_{G,A,G,A}^- (mx) = \beta \nu_{I(A)}^- (mx) \)

\[
= \beta \max ( \nu_A^- (ab) ) \quad \text{(by def.5.2.1)}
\]

\[
\leq \beta \max ( \nu_A^- (b) ) \quad \text{(by def.1.3.9)}
\]
\[ \max (\beta v_{\alpha}(b)) = v_{I_{G_{\alpha,\beta}}(A)}(x). \]  
\[ \text{(by def.5.2.1 and 5.4.3)} \]

Therefore, \( v^{-}_{G_{\alpha,\beta} I(A)}(mx) \leq v^{-}_{I_{G_{\alpha,\beta}}(A)}(x) \) for some \( a, x, b, m \in A. \)  
\[ (5.7.15) \]

Therefore, \( G_{\alpha,\beta}(I(A)) = I_{G_{\alpha,\beta}}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G. \)  
\[ \square \]

**Theorem: 5.7.16**

If \( A \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G, \) then \( G_{\alpha,\beta}(A) = G_{\beta,\alpha}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G. \)

**Proof:**

Let \( x, m \in G \) then \( x, m \in A. \)

Consider \( \mu^{+}_{G_{\alpha,\beta}(A)}(mx) = v^{+}_{G_{\alpha,\beta}(A)}(mx) \) \[ \text{(by def.3.2.1)} \]

\[ = \beta v^{+}_{\alpha}(mx) \] \[ \text{(by def.5.4.3)} \]

\[ = \beta \mu^{+}_{\alpha}(mx) \] \[ \text{(by def.3.2.1)} \]

\[ \leq \beta \mu^{+}_{\alpha}(x) \] \[ \text{(by def.2.2.3)} \]

\[ = \mu^{+}_{G_{\alpha,\beta}(A)}(x). \]

Therefore, \( \mu^{+}_{G_{\alpha,\beta}(A)}(mx) \leq \mu^{+}_{G_{\beta,\alpha}(A)}(x) \) for some \( x, m \in A. \)  
\[ (5.7.17) \]

Consider \( v^{+}_{G_{\alpha,\beta}(A)}(mx) = \mu^{+}_{G_{\alpha,\beta}(A)}(mx) \) \[ \text{(by def.3.2.1)} \]

\[ = \alpha \mu^{+}_{\alpha}(mx) \] \[ \text{(by def.5.4.3)} \]

\[ = \alpha v^{+}_{\alpha}(mx) \] \[ \text{(by def.3.2.1)} \]

\[ \geq \alpha v^{+}_{\alpha}(x) \] \[ \text{(by def.2.2.3)} \]

\[ = v^{+}_{G_{\beta,\alpha}(A)}(x). \]

Therefore, \( v^{+}_{G_{\alpha,\beta}(A)}(mx) \geq v^{+}_{G_{\beta,\alpha}(A)}(x) \) for some \( x, m \in A. \)  
\[ (5.7.18) \]
Consider \( \mu_{G_{\alpha,\beta}(A)}^-(mx) = \nu_{G_{\alpha,\beta}(A)}^-(mx) \) (by def.3.2.1)
\[ = \beta \nu_{A}^-(mx) \] (by def.5.4.3)
\[ = \beta \mu_{A}^-(mx) \] (by def.3.2.1)
\[ \geq \beta \mu_{A}^-(x) \] (by def.2.2.3)
\[ = \mu_{G_{\beta,\alpha}(A)}^-(x). \]
Therefore, \( \mu_{G_{\alpha,\beta}(A)}^-(mx) \geq \mu_{G_{\beta,\alpha}(A)}^-(x) \) for some \( x, m \in A. \) \ ...(5.7.19)

Consider \( \nu_{G_{\alpha,\beta}(A)}^-(mx) = \mu_{G_{\alpha,\beta}(A)}^-(mx) \) (by def.3.2.1)
\[ = \alpha \mu_{A}^-(mx)) \] (by def.5.4.3)
\[ = \alpha \nu_{A}^-(mx)) \] (by def.3.2.1)
\[ \leq \alpha \nu_{A}^-(x)) \] (by def.2.2.3)
\[ = \nu_{G_{\beta,\alpha}(A)}^-(x). \]
Therefore, \( \nu_{G_{\alpha,\beta}(A)}^-(mx) \leq \nu_{G_{\beta,\alpha}(A)}^-(x) \) for some \( x, m \in A. \) \ ...(5.7.20)

Therefore, \( G_{\alpha,\beta}(A) = G_{\beta,\alpha}(A) \) is an bipolar intuitionistic anti \( M \)-fuzzy group of \( G. \)

\[ \blacksquare \]

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