Chapter 3

Modeling Count Data using Inflated Probability Distributions

This chapter presents a brief introduction to the concept of zero-inflated and mixed distribution in modeling of count data. Further it also presents zero inflated and multi point inflated versions of some mixed distributions is provided. Maximum likelihood estimation method is used for estimating the parameters of the distribution. A comparison study is performed to show the advantage of the proposed distributions over the standard count distributions with respect to various measures such as chi-square test of goodness-of-fit, $p$ - value, log-likelihood, AIC.

3.1 Modeling Count Data using Mixed and Inflated models

Count data in many applications shows inflation at some particular points in its support. Poisson distribution has a long history for modeling count data. But due to the equi-dispersion property of this distribution, in certain applications it may not provide adequate fitting due to the over dispersion nature of count data. Hence, negative binomial distribution was utilized by many researchers in various applications for modeling over dispersed count data. The studies show that negative binomial (NB) distribution provides a better fit compared to Poisson distribution for modeling the over dispersed count data. NB distribution is a mixture of Poisson and Gamma distribution. For the NB distribution the probability mass function, mean and variance are given as

$$p(y, r, p) = \binom{r+x-1}{x} p^r (1-p)^x$$  \hspace{1cm} (3.1)

where $x = 0,1,2,\ldots, r > p$ and $0 < p < 1$

$$\text{Mean} = E(X) = \frac{r(1-p)}{p}$$  \hspace{1cm} (3.2)

$$\text{Variance} = V(X) = \frac{r(1-p)}{p^2}$$  \hspace{1cm} (3.3)
However one major cause of over dispersion is the existence of excess number of zero counts in the data. Therefore in certain circumstances Poisson and negative binomial distributions cannot model the count data adequately, since the proportion of zeros in the observed data may far exceed under the assumptions of the traditional count distribution models such as Poisson and negative binomial distribution. As a result, for incorporating these excess zero counts in these distributions, researchers in this fields tried to modify these distributions. Thus, inflated models and mixed distribution models were established in the literature. Mixture distributions can be simply implemented to a data set in which two or more populations are mixed together. Mixing some life time distributions with Poisson and negative binomial distribution provide a better fit to the count data (Sankaran, 1970; Sichel, 1975; Zamani and Ismail, 2010.). Details and formulations of the mixed Poisson distributions are not included in this chapter. In this chapter we concentrate only on the mixed negative binomial distributions and their zero inflated and multi point inflated versions, since mixed negative binomial provides better fit compared to mixed Poisson distributions. Zamani and Ismail (2010) showed that mixing negative binomial distribution with Lindley distribution provide better fit to the count data with excess number of zero counts and for the distribution having thick tail. Exponential distribution is used as a mixing component of NB distribution for modeling the over dispersed count data. We also considered negative binomial –Lindley (NB-L) mixture for modeling count data. Ghitani et al., (2008b) showed that Lindly distribution performs better compared to exponential distribution in many ways, since Lindley distribution has decreasing mean residual life function and increasing hazard rate function. Further, mixing negative binomial with Lindley provides better fit compared to negative binomial with exponential. According to Zamani and Ismail (2010), the specification and characteristics of the NB-L distribution are as follows.
\[ p(x, r, \theta) = \left( \frac{r + x - 1}{x} \right) \sum_{j=0}^x \binom{x}{j} (-1)^j \frac{\theta^2}{\theta + 1} \frac{(\theta + r + j + 1)}{(\theta + r + j)^2} \quad \text{for } x = 1, 2, 3, \ldots; \ r, \theta > 0 \quad (3.4) \]

The factorial moment of NB-Lindley distribution is

\[ \mu_{[k]}(X) = \frac{\Gamma(r + k)}{\Gamma r} \sum_{j=0}^k \binom{k}{j} (-1)^j \frac{\theta^2}{\theta + 1} \frac{(\theta - k + j + 1)}{(\theta - k + j)^2}, k = 0, 1, 2, \ldots \quad (3.5) \]

For the NB-Lindley distribution, it can be obtained that

\[ E(X) = r \left( \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right) \]

\[ V(X) = \left[ (r + r^2) \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} - \frac{(r + 2r^2)\theta^3}{(\theta + 1)(\theta - 1)^2} + r^2 \right] - r^2 \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right]^2 \]

Lindley distribution is also a mixture of exponential and gamma distribution, and was introduced by Lindley (1958). The functional form of the Lindley distribution can be written as

\[ f(x, \theta) = \frac{\theta^2}{\theta + 1} (1 + x)e^{-\theta x}, \quad x > 0, \theta > 0 \quad (3.8) \]

For the Lindley distribution the mean and variance can be obtained as

\[ E(X) = \frac{\theta + 2}{\theta(\theta + 1)} \quad (3.9) \]
\[ V(X) = \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta + 1)^2} \quad (3.10) \]

We proposed a zero inflated and multipoint inflated structure of the negative binomial–Lindley distribution to model the claim count distribution, since it provides a clear description of over dispersion caused by excess zero counts. We also introduced the zero
inflated versions of negative binomial–two parameter Lindley (NB-TPL) distribution and negative binomial–Sushila (NB-S) distributions.

NB-TPL distribution was proposed by Denthet et al., (2016), which is a combination of NB and two parameter Lindley (TPL) distribution proposed by Shanker et al., (2013). Shankar et al., (2013) showed that TPL is an extension of one parameter–Lindley distribution and provide better fit to the waiting and survival time data in terms of goodness of fit tests.

The specification of the model is obtained as follows

\[ f(x, a, b) = \frac{b^2}{b + a} (1 + ax)e^{-bx}, x > 0, \theta > 0, \alpha > -\theta \] (3.11)

This distribution reduces to the one parameter –Lindley distribution at \( \alpha = 1 \) and reduces to the exponential distribution at \( \alpha = 0 \). The mean and variance of the distribution can be attained as follows

\[ E(X) = \frac{b + 2a}{b(b + a)} \] (3.12)

\[ V(X) = \frac{b^2 + 4ba + 2a^2}{b^2(b + a)^2} \] (3.13)

Denthet et al., (2016) showed that NB-TPL distribution provides a better fit to the zero inflated count data compared to NB-L distribution based on the K-S test, \( p \)-values and AIC.

The formulations and characteristics of the NB-TPL distribution are obtained as follows

\[ p(x, r, a, b) = \binom{r + x - 1}{x} \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \times \left( \frac{a + b + r + j}{(b + a)(b + r + j)^2} \right) \] (3.14)

where \( x = 0, 1, 2, \ldots; b > 0 \) and \( a + b > 0 \)

The factorial moment of the NB-TPLD can be obtained as

\[ \mu_{(k)}(X) = \frac{\Gamma(r + k)}{\Gamma(r)} \left( \frac{b^2}{b + a} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left( \frac{b + a - k + j}{(b - k + j)^2} \right) \right) \] (3.15)

where \( k = 0, 1, 2, \ldots; b > 0 \) and \( a + b > 0 \)
From the factorial moments of the distribution the mean and variance of NB-TPLD can be obtained as

\[ E(X) = r(m_1 - m_0) \]

\[ \text{Var}(X) = (r^2 + r)m_2 - (2r^2 + r + r^2m_1)m_1 + (r^2 + 2r^2m_1)m_0 - r^2m_0^2 \]  

where \( \frac{b^2}{b+a} \frac{b+a-s}{(b-a)^2} \)

Another negative binomial mixture distribution, we considered in this chapter is negative binomial-Sushila (NB-S) distribution (Yamrubboon et al., 2017) which is obtained by mixing the negative binomial distribution with Sushila distribution, which can be used as an alternative to modeling the count data with large number of zeros. Sushila distribution is proposed by Shanker et.al (2013) of which Lindley distribution is a particular case and they showed that Sushila distribution provides a better fit compared to Lindley distribution. It is a two parameter continuous distribution with density function

\[ f(x, \alpha, \theta) = \frac{\theta^2}{\alpha(\theta+1)} \left(1 + \frac{x}{\alpha}\right) e^{-\frac{x}{\alpha}}, \quad x > 0, \theta > 0, \alpha > 0 \]

For this distribution the first four raw moments and central moments are given below

\[ \mu_1 = \frac{\alpha(\theta+2)}{\theta(\theta+1)} \], \[ \mu_2 = \frac{2\alpha^2(\theta+3)}{\theta^2(\theta+1)} \], \[ \mu_3 = \frac{6\alpha^3(\theta+4)}{\theta^3(\theta+1)} \], \[ \mu_4 = \frac{24\alpha^4(\theta+5)}{\theta^4(\theta+1)} \]

\[ \mu_2^* = \frac{\alpha^2(\theta^2 + 4\theta + 2)}{\theta^2(\theta+1)^2} \], \[ \mu_3^* = \frac{2\alpha^3(\theta^3 + 6\theta^2 + 6\theta + 2)}{\theta^3(\theta+1)^3} \], \[ \mu_4^* = \frac{3\alpha^4(3\theta^4 + 24\theta^3 + 44\theta^2 + 32\theta + 8)}{\theta^4(\theta+1)^4} \]

Yamrubboon et al., (2017) showed that mixture of this Sushila distribution with NB distribution can be used as a better alternative model for the zero inflated and over dispersed count data analysis. They obtained the PMF of the distribution as follows :
\[
f(x,r,\alpha,\theta) = \frac{\theta^2}{(\theta+1)} \left( \frac{x+r-1}{x} \right) \sum_{j=0}^{r} \binom{x}{j} (-1)^j \left( \frac{\theta + \alpha(r+j) + 1}{(\theta + \alpha(r+j))^2} \right) \]

where \( x = 0, 1, 2, \ldots \); \( r, \alpha, \theta > 0 \). The mean, variance and factorial moments of the NB-S distribution are given respectively by

\[
E(X) = r \left[ \frac{\theta^2(\theta - \alpha + 1)}{(\theta+1)(\theta - \alpha)^2} - 1 \right]
\]

\[
V(X) = \frac{r^2 \delta_1 + r \delta_1 - 2r \delta_2 - r^2 \delta_2^2}{\delta_1} + r
\]

where \( \delta_1 = \frac{(\theta+1)}{\theta^2} \), \( \delta_2 = \frac{(\theta - \alpha + 1)}{(\theta - \alpha)^2} \), \( \delta_3 = \frac{(\theta - 2\alpha + 1)}{(\theta - 2\alpha)^2} \) and

\[
\mu_k[X] = E[X(X-1)...(X-k+1)] = \left( \frac{\Gamma(r+k)}{\Gamma r} \right) \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{\theta^2(\theta - (k-j)\alpha + 1)}{(\theta+1)(\theta - (k-j)\alpha)^2}
\]

In this chapter, we proposed zero inflated versions of the above mentioned NB-L, NB-TPL and NB-S distributions named as zero inflated negative binomial-Lindley (ZINB-L) distribution, zero inflated negative binomial-two parameter Lindley (ZINB-TPL) distribution and zero inflated negative binomial-Sushila (ZINB-S) distribution. The reason behind for developing the zero inflated versions of the mixed negative binomial distributions is that, zero inflated models accounts for over dispersion caused by excess zero counts. We also proposed a multipoint inflated negative binomial–Lindley (MPINB-L) distribution for modeling the count data where inflation occurs at any number of points in its support.

3.1.1 Structure of Zero-Inflated Models

The probability mass function of the zero-inflated count model can be written in the following form

\[
p(X = x) = \begin{cases} 
\omega + (1-\omega)g(0, \Theta) & \text{if } x = 0 \\
(1-\omega)g(x, \Theta) & \text{if } x = 1, 2, \ldots
\end{cases}
\]
where $X$ is the count random variable, $g(x, \Theta)$ is the PMF of $X$ with parameter space $\Theta$ and $
abla$ represents the zero inflation parameter which is the proportion of excess zero counts and $0 < \nabla < 1$. The zero inflated distributions has been formulated to deal with the over dispersed count data with excess zero counts. Lambert (1992) suggested zero inflated Poisson distribution for handling the purely zero inflated data, and provided the specification and characteristics of the distribution as follows

$$P(X = x / \lambda, \omega) = \begin{cases} \omega + (1-\omega)e^{-\lambda} ; & x = 0 \\ (1-\omega)\frac{e^{-\lambda} \lambda^x}{x!} ; & x > 0 \end{cases}$$ (3.24)

where $\lambda > 0$ and $0 < \omega < 1$.

The mean and variance of the distribution can be written as

$$Mean = E(X) = (1-\omega)\lambda$$

$$Variance = V(X) = \lambda(1-\omega)(1+\omega\lambda)$$ (3.26)

If the data has over dispersion, zero inflated negative binomial distribution is more appropriate model (Neelon.et.al; 2010). It is obtained as a mixture of the degenerate distribution at zero and a NB distribution $(p = e^{-\lambda})$ as follows

$$p(X = x / \lambda, r, \omega) = \begin{cases} \omega + (1-\omega)e^{-\lambda r} , & if \ x = 0 \\ (1-\omega)\left\{ \frac{x+r-1}{x} \right\} e^{-\lambda r} (1-e^{-\lambda})^x , & if \ x > 0 \end{cases}$$ (3.27)

The mean and variance of the distribution are

$$E(X) = (1-\omega)\lambda$$ (3.28)

$$V(X) = \lambda(1-\omega)(1+\lambda(\omega+r))$$ (3.29)

ZINB distribution is an good alternative to ZIP distribution for modeling over dispersed data. Further zero inflated generalized Poisson (ZIGP) distribution and zero inflated mixed distributions etc are proposed in the literature.
3.2 Zero-Inflated Mixed Models

Both mixed models and zero inflated models were adopted by many researchers for modeling the count data with large number of zeros and thick tail. But some recent studies shows that zero inflated mixed models provides better fit to the count data with excess number of zeros. Some of the previous works related to zero inflated mixed models are zero inflated negative binomial-generalized exponential (ZINB-GE) distribution (Aryuyuen et al., 2014) and zero inflated negative binomial- crack (ZINB-CR) distribution (Saengthong et al., 2015) etc. The specifications of the models and their characteristics are obtained as follows.

ZINB-GE distribution is a new mixture distribution used for zero inflated count modeling by combining both Bernoulli and negative binomial–generalized exponential distribution. The PMF of the distribution can be written as

\[
p(X = x / r, \alpha, \beta, \omega) = \begin{cases} 
\omega + (1 - \omega)M_{(r)}, & \text{if } x = 0 \\
(1 - \omega) \left(\frac{x + r - 1}{x}\right) \sum_{j=0}^{x} \left(\begin{array}{c}
x \\
j 
\end{array}\right) (-1)^{j} M_{(r+j)}, & \text{if } x > 0 
\end{cases}
\] (3.30)

where \(0 < \omega < 1, r, \alpha, \beta > 0\) and \(M_{(\omega)}\) is specified as follows

\[
M_{(\omega)} = \frac{\Gamma(\alpha + 1) \Gamma(1 + u / \beta)}{\Gamma(\alpha + u / \beta + 1)} \text{ for } r, \alpha, \beta > 0
\]

The mean and variance of the distribution can be obtained as follows

\[
E(X) = (1 - \omega)(r \delta_{(1)} - r) \quad (3.31)
\]

\[
V(X) = r[(r + 1) \delta_{(2)} - (2r + 1) \delta_{(1)} + r](1 - \omega) - [r(\delta_{(1)} - 1)(1 - \omega)]^{2} 
\] (3.32)

where \(\delta_{(u)} = \frac{\Gamma(\alpha + 1) \Gamma(1 - u / \beta)}{\Gamma(\alpha - u / \beta + 1)}\)

The functional form of the distribution and characteristics of the ZINB-CR distribution can be obtained as
\[ p(X = x \mid r, \lambda, \theta, \gamma, \omega) = \begin{cases} \omega + (1 - \omega) \frac{\exp\left(\lambda(1 - \sqrt{1 + 2\theta r})\right)\left(1 - \gamma(1 - \sqrt{1 + 2\theta r})\right)}{\sqrt{1 + 2\theta r}}, & \text{if } x = 0 \\ (1 - \omega) \left(1 + x - 1\right) \sum_{j=0}^{X} \binom{X}{j} (-1)^{j} \frac{\exp\left(\lambda(1 - \sqrt{1 + 2\theta(r + j)})\right)}{\sqrt{1 + 2\theta(r + j)}} \times \left(1 - \gamma(1 - \sqrt{1 + 2\theta(r + j)})\right), & \text{if } x > 0 \end{cases} \]

(3.33)

where \( 0 < \omega < 1, r, \alpha, \beta > 0 \) and \( 0 \leq \gamma \leq 1 \)

The mean and variance of the distribution are given by

\[ E(X) = (1 - \omega) r \left(1 - \gamma(1 - \delta)\right) \frac{\exp\left(\lambda(1 - \delta)\right)}{\delta} - 1 \]  

(3.34)

\[ V(X) = (1 - \omega) r \left(\frac{(r^2 + r)(1 - \gamma(1 - \varsigma))\exp\left(\lambda(1 - \varsigma)\right)}{\varsigma}\right) - r \left(\frac{(1 - \gamma(1 - \delta))\exp\left(\lambda(1 - \delta)\right)}{\delta}\right) - \omega(1 - \omega) \left(\frac{2r^2(1 - \gamma(1 - \delta))\times \exp\left(\lambda(1 - \delta)\right)}{\delta} - r^2\right) \]

\[-\left(\frac{(1 - \omega) r (1 - \gamma(1 - \delta))\times \exp\left(\lambda(1 - \delta)\right)}{\delta}\right)^2 \]  

(3.35)

where \( \delta = \sqrt{1 - 2\theta} \) and \( \varsigma = \sqrt{1 - 4\theta} \)

In the following sections, we proposed some zero inflated version of mixed models and they are named as zero inflated negative binomial-Lindley (ZINB-L) distribution, zero inflated negative binomial-two parameter Lindley (ZINB-TPL) distribution and zero inflated negative binomial-Sushila (ZINB-S) distribution. Further the performances of these models were compared with already existing count models for modeling over-dispersed count data.

### 3.2.1 Zero Inflated Negative Binomial-Lindley Distribution

Zero inflated mixed negative binomial distribution has been proposed with the aim of obtaining a more flexible model for modeling the count data with high density at zero. The
fundamental statistical properties of this distribution are derived and for making inferences about the parameters of this distribution maximum likelihood estimation method is used. The performance of this distribution is illustrated with the help of real count data set available in literature.

**Theorem 3.1:** If \( X \sim ZINB - L(\omega, r, \theta) \) distribution, then the PMF of \( X \) can be written as

\[
p(x; r, \omega, \theta) = \begin{cases} 
\omega + (1 - \omega) \frac{\theta^2}{(\theta + 1)^2} \frac{\theta + r + 1}{(\theta + 1)^2} & ; x = 0 \\
(1 - \omega) \left( r + x + 1 \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{(\theta + 1)^2} \frac{\theta + r + j + 1}{(\theta + r + j)^2} & ; x > 0; r, \theta > 0 
\end{cases}
\]  

(3.36)

where \( x = 0, 1, 2, \ldots; \theta > 0, r > 0, 0 < \omega < 1 \).

**Proof:**

If \( X|\lambda \sim ZINB(r, p = e^{-\lambda}, \omega) \) and \( \lambda \sim Lindley(\theta) \) then the PMF of zero inflated negative binomial-Lindley distribution can be obtained as follows,

\[
p(x; r, \omega, \theta) = \int_0^\infty p(x; \omega, r, p = e^{-\lambda}) f(\lambda; \theta) d\lambda
\]

where \( p(x; \omega, r, p = e^{-\lambda}) \) is the PMF of the zero inflated negative binomial distribution given in equation (3.27) and \( f(\lambda; \theta) \) is the density function of the one parameter Lindley distribution given as follows

\[
f(\lambda; \theta) = \frac{\theta^2}{(\theta + 1)} (1 + \lambda) e^{-\theta \lambda}, \; \lambda > 0, \theta > 0
\]  

(3.37)

when \( x = 0 \), the PMF of \( ZINB - L(\omega, r, \theta) \) can be obtained as

\[
p(x; r, \omega, \theta) = \int_0^\infty \left[ \omega + (1 - \omega) e^{-\lambda x} \right] f(\lambda; \theta) d\lambda
\]
\[ p(x; r, \omega, \theta) = \omega + (1 - \omega)M_\lambda(-r) \]

The PMF of the \textit{ZINB} \textit{–} \textit{L}(\omega, r, \theta) at \: x = 0 \: \text{is obtained by substituting the MGF of Lindley distribution with } z = -r. \: \text{Therefore the PMF of ZINB distribution at } \: x = 0 \: \text{is obtained as follows}

\[ p(x; r, \omega, \theta) = \omega + (1 - \omega) \frac{\theta^2}{\theta + 1} \frac{\theta + r + 1}{(\theta + r)^2} \]

where the MGF of the Lindley distribution is

\[ M_\lambda(z) = \frac{\theta^2}{\theta + 1} \frac{\theta - z + 1}{(\theta + z)^2} \]  

Similarly, we can obtain the PMF of \textit{ZINB} \textit{–} \textit{L}(\omega, r, \theta) at \: x \neq 0 \: \text{is obtained by}

\[ p(x; r, \omega, \theta) = \omega \left(1 - \omega\right) \sum_{j=0}^{x-r} \binom{x-r-1}{j} (-1)^j \left(e^{-\lambda}\right)^j f(\lambda; \theta) d\lambda \]

Using binomial expansion, we get

\[ e^{-\lambda x} \left(1 - e^{-\lambda}\right)^x = e^{-\lambda x} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left(e^{-\lambda}\right)^j \]

\[ = \sum_{j=0}^{x} \binom{x}{j} (-1)^j e^{-\lambda (r+j)} \]

Therefore

\[ p(x; r, \omega, \theta) = (1 - \omega) \left(\sum_{j=0}^{x-r} \binom{x-r-1}{j} (-1)^j e^{-\lambda (r+j)}\right) f(\lambda; \theta) d\lambda \]
\[ p(x; r, \omega, \theta) = (1 - \omega) \left( \sum_{j=0}^{x+r-1} \binom{x}{j} (-1)^j M(x + r - j) \right) \]

And using equation (3.38) obtained the PMF of ZINB–L \((\omega, r, \theta)\) at \(x \neq 0\) is obtained as

\[ p(x; r, \omega, \theta) = (1 - \omega) \left( \sum_{j=0}^{x+r-1} \binom{x}{j} (-1)^j \frac{\theta^2 (\theta + \alpha(r + j) + 1)}{(\theta + 1)(\theta + \alpha(r + j))^2} \right) \]

When \(\omega = 0\), ZINB-L distribution reduces to NB-L distribution. Following Figure 3.1 shows the comparison of ZINB-L distribution with NB-L distribution with various values of the parameters. Figure 3.1 shows the comparison of ZINB-L distribution with NB-L distribution with some parameter values and figure 3.2 shows the PMF of the ZINB–L distribution with different values of parameters \(\theta\) and \(r\).

**Figure 3.1:** PMF of NB-Lindley and ZINB-Lindley distribution for various values of parameters
Characteristics of the ZINB-Lindley distribution.

**Theorem 3.2:** If $X \sim ZINB - L(\omega, r, \theta)$ then the factorial moments of order $k$ of $X$ is

$$
\mu_k(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} \sum_{j=0}^{k} \binom{k}{j} (-1)^{j} \left( \frac{\theta^2}{\theta + 1} \right)^{\frac{\theta^2}{\theta + 1} + \frac{\theta - k + j + 1}{\theta - k + j}}
$$

(3.39)

**Proof:**

According to Gomez et al., (2008), the factorial moment of order $k$ of the mixed NB distribution where $p = e^{-\lambda}$ can be written as

$$
\mu_k(X) = (X(X - 1)) \ldots (X - k + 1) = \frac{\Gamma(r + k)}{\Gamma r} E_{\lambda} \left( \frac{1 - e^{-\lambda}}{e^{-\lambda}} \right)^k
$$

Then, the factorial moments of the zero inflated mixed negative binomial distribution can be written as

$$
\mu_k(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} E_{\lambda} \left( \frac{1 - e^{-\lambda}}{e^{-\lambda}} \right)^k
$$

$$
= (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} E_{\lambda} \left( e^\lambda - 1 \right)^k
$$

$$
= (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} \sum_{j=0}^{k} \binom{k}{j} (-1)^{j} E \left( e^{\lambda(k-j)} \right)
$$

$$
\mu_k(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} \sum_{j=0}^{k} \binom{k}{j} (-1)^{j} M_{\lambda}(k - j)
$$
where \( M_A(k-j) \) is the MGF of the Lindley distribution at \((k-j)\). Then by inserting the MGF of the Lindley distribution given in equation (3.38), the factorial moment of ZINB-L distribution can be obtained.

**Mean and Variance**

Putting \( k = 1, 2 \) in equation (3.39) obtained the first two moments and variance of the proposed distribution are obtained as follows:

\[
E(X) = (1 - \omega) r \frac{\theta^2}{\theta + 1} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right] \tag{3.40}
\]

\[
E(X)^2 = (1 - \omega) \left\{ (r + r^2) \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} - (r + 2r^2) \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} + r^2 \right\} \tag{3.41}
\]

\[
V(X) = (1 - \omega) \left[ (r + r^2) \left( \frac{\theta^2}{\theta + 1} \right) \frac{(\theta - 1)}{(\theta - 2)^2} - \frac{(r + 2r^2)\theta^3}{(\theta + 1)(\theta - 1)^2} + r \right]
- \left[ r(1 - \omega) \frac{\theta^2}{\theta + 1} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right] \right]^2 \tag{3.42}
\]

**Random Variate Generation of ZINB-Lindley Distribution**

To generate a random variable \( X \) from the \( ZINB - L(\omega, r, \theta) \) one can use the following algorithm:

1) Generate \( U \) from the uniform distribution, \( U (0,1) \).

2) Set \( \lambda = -1 - \frac{1}{\theta} - W_{-1}\left( \frac{(\theta + 1)(U - 1)}{e^{(\theta + 1)}} \right) \), where \( W_{-1} \) is the negative branch of the Lambert \( W \) function and \( \lambda \sim Lindley(\theta) \).

3) Generate \( Y \) from the \( NB(r, p = e^{-\lambda}) \) distribution.

4) Generate \( U^* \) from the uniform distribution, \( U (0,1) \).

5) If \( U^* > \pi \) then set \( X = Y \); otherwise, \( X = 0 \).
Parameter Estimation of ZINB-L Distribution

In this study, parameters of the ZINB-L distribution are obtained by the maximum likelihood estimation method. For defining the likelihood function, considered an indicator function as follows

\[
I(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \in \{1, 2, 3, \ldots\}
\end{cases}
\] (3.43)

with the above function, the likelihood function of the ZINB-L distribution can be written as

\[
L(\omega, r, \theta) = \prod_{i=1}^{n} \{I(x) p(x = 0) + (1 - I(x)) p(x \neq 0)\}
\]

That is

\[
L(\omega, r, \theta) = \prod_{i=1}^{n} \left[ \omega + (1 - \omega) \frac{\theta^2}{(\theta + 1)} \frac{\theta + r + 1}{(\theta + r)^2} \right] + (1 - \theta) \left[ \frac{\theta^2}{(\theta + 1)} \frac{(\theta + r + j + 1)}{(\theta + r + j)^2} \right]
\]

(3.44)

By taking

\[
A = \omega + (1 - \omega) \frac{\theta^2}{(\theta + 1)} \frac{\theta + r + 1}{(\theta + r)^2}
\]

\[
B = (1 - \theta) \left[ \frac{\theta^2}{(\theta + 1)} \frac{(\theta + r + j + 1)}{(\theta + r + j)^2} \right]
\]

Then the log-likelihood function can be written as follows

\[
l(\omega, r, \theta) = \log L(\omega, r, \theta) = \sum_{i=1}^{n} \log \{IA + (1 - I)B\}
\] (3.45)

The first order partial derivatives of the likelihood function \(l(\omega, r, \theta)\) of the ZINB-L distribution with respect to the parameters \(\omega, r\) and \(\theta\) are derived in the following differential equations:
\[
\frac{\partial}{\partial \theta} l(\omega, r, \theta) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \times \frac{\partial}{\partial \theta} \{IA + (1-I)B\} \tag{3.46}
\]

where

\[
\frac{\partial}{\partial \theta} IA = \frac{\partial}{\partial \theta} \left\{ \omega + (1-\omega) \frac{\theta^2}{\theta + 1} \frac{\theta + r + 1}{(\theta + r)^2} \right\} = l(1-\omega) \frac{\partial}{\partial \theta} \left[ \frac{\theta^2}{\theta + 1} \frac{\theta + r + 1}{(\theta + r)^2} \right]
\]

\[
= l(1-\omega) \left\{ (\theta + 1)(\theta + r)^2 \times \left[ 2\theta(\theta + r + 1) + \theta^2(-1) \right] - \theta^2(\theta + r + 1) \right\} \frac{(\theta + 1)^2 + (\theta + r)^2}{(\theta + 1)^3 (\theta + r)^3}
\]

\[
= l(1-\omega) \left[ \frac{\theta(\theta + 1)(\theta + r)^2}{(\theta + 1)^3 (\theta + r)^3} \right] \left\{ (\theta + r + 2)(\theta + r + 1) - \theta^2(\theta + r + 1) \right\}
\]

\[
= l(1-\omega) \left[ \frac{\theta(\theta + 1)(\theta + r)^2}{(\theta + 1)^3 (\theta + r)^3} \right] \left[ (\theta + r + 1)(3\theta + r + 2) - \theta^2(\theta + r + 1) \right]
\] \tag{3.47}

and

\[
\frac{\partial}{\partial \theta} (1-I)B = (1-I) \frac{\partial}{\partial \theta} \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{\theta + 1} \frac{\theta + r + j + 1}{(\theta + r + j)^2}
\]

\[
= (1-I)(1-\omega) \left\{ \binom{x + r - 1}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{\theta + 1} \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right\}
\]

\[
= (1-I)(1-\omega) \left\{ \binom{x + r - 1}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left[ \frac{(\theta + 1)(\theta + r + j)^2}{(\theta + 1)^2 (\theta + r + j)^2} \frac{(\theta + r + j + 1)^2}{(\theta + r + j + 1)^2} \right] - \theta^2(\theta + r + j + 1)^2 \frac{(\theta + r + j + 1)^2}{(\theta + r + j)^2} \right\}
\]
\[
= (1-I)(1-\omega) \left\{ \sum_{j=0}^{x+r-1} \binom{x}{j} (-1)^j \frac{\theta(\theta+1)(\theta+r+j)^2(3\theta+2r+2j+2)}{(\theta+1)^2(\theta+r+j)^3} - \theta^2(\theta+r+j+1)(\theta+r+j)^2(3\theta+2r+2j+2) \right\}
\]

Therefore, \( \frac{\partial}{\partial \theta} (1-I)B \) can be written as

\[
\frac{\partial}{\partial \theta} (1-I)B = (1-I)(1-\omega) \left\{ \sum_{j=0}^{x+r-1} \binom{x}{j} (-1)^j \frac{\theta(\theta+1)(\theta+r+j)^2(3\theta+2r+2j+2)}{(\theta+1)^2(\theta+r+j)^3} - \theta^2(\theta+r+j+1)(\theta+r+j)^2(3\theta+2r+2j+2) \right\}
\]

(3.48)

Substituting equation (3.47) and (3.48) in (3.46), \( \frac{\partial}{\partial \theta} l(\pi,r,\theta) \) can be written as

\[
\frac{\partial}{\partial \theta} l(\omega,r,\theta) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \left\{ \frac{(\theta+1)(\theta+r)(3\theta+2r+2)}{(\theta+1)^2(\theta+r)^3} \right\} \left\{ \sum_{j=0}^{x+r-1} \binom{x}{j} (-1)^j \right\}
\]

(3.49)
Similarly,

\[
\frac{\partial}{\partial r} l(\omega, r, \theta) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \frac{\partial}{\partial r} (IA + (1-I)B)
\]  

(3.50)

\[
\frac{\partial}{\partial r} IA = \frac{\partial}{\partial A} I \left[ \omega + (1-\omega) \frac{\theta^2}{\theta+1} \frac{\theta + r + 1}{(\theta + r)^2} \right] = I(1-\omega) \frac{\theta^2}{\theta+1} \frac{\theta + r + 1}{(\theta + r)^2}
\]

\[
= I(1-\omega) \frac{\theta^2}{\theta+1} \left[ \frac{(\theta + r)^2 - (\theta + r + 1) \times 2(\theta + r)}{(\theta + r)^3} \right]
\]

\[
= I(1-\omega) \frac{\theta^2}{\theta+1} \left[ \frac{(\theta + r) - 2(\theta + r + 1)}{(\theta + r)^3} \right]
\]

\[
= I(1-\omega) \frac{\theta^2}{\theta+1} \left[ \frac{\theta + r - 2\theta - 2r - 2}{(\theta + r)^3} \right]
\]

\[
\frac{\partial}{\partial r} IA = I(1-\omega) \frac{\theta^2}{\theta+1} \left( \frac{-(\theta + r + 2)}{(\theta + r)^3} \right)
\]  

(3.51)

\[
\frac{\partial}{\partial \theta} (1-I)B = \frac{\partial}{\partial r} \left[ (1-I)(1-\omega) \left( \frac{x+r-1}{x} \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{\theta^2}{\theta+1} \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right]
\]

Since \( \left( \frac{x+r-1}{x} \right) = \frac{(x+r-1)!}{(r-1)!x!} = \frac{\Gamma(x+r)}{\Gamma(x+1)} \)

\[
\frac{\partial}{\partial \theta} (1-I)B \text{ can be written as}
\]

\[
\frac{\partial}{\partial \theta} (1-I)B = (1-I) \frac{(1-\omega)\theta^2}{\theta+1} \frac{\partial}{\partial r} \left( \frac{\Gamma(x+r)}{\Gamma(x+1)} \frac{\sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right)
\]

\[
= (1-I) \frac{(1-\omega)\theta^2}{(\theta+1)\Gamma(x+1)} \frac{\partial}{\partial r} \left( \frac{\Gamma(x+r)}{\Gamma(x+1)} \frac{\sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right)
\]
\[
(1 - I) \frac{(1 - \omega)\theta^2}{(\theta + 1)\Gamma(x + 1)} \left[ \frac{\Gamma(x + r)}{\Gamma r} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left( \frac{(\theta + r + j)^2 - (\theta + r + j + 1)}{(\theta + r + j)^3} \right) \right] \\
+ \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta + r + j + 1}{(\theta + r + j)^2} \left( \frac{\Gamma r \Gamma'_{(x + r)} - \Gamma_{(x + r)} \Gamma'(r)}{(\Gamma r)^2} \right)
\]

Substituting (3.51) and (3.52) in (3.50), \( \frac{\partial}{\partial r} l(\omega, r, \theta) \) can be obtained as

\[
\frac{\partial}{\partial r} l(\omega, r, \theta) = \sum_{i=1}^{n} \frac{1}{IA + (1 - I)B} \left[ \frac{1}{\theta + 1} \frac{(1 - \omega)\theta^2}{(\theta + 1)\Gamma(x + 1)} \left[ -\frac{(\theta + r + 2)}{(\theta + r)^3} \right] + (1 - I) \frac{(1 - \omega)\theta^2}{(\theta + 1)\Gamma(x + 1)} \right] \\
\times \left[ \frac{\Gamma(x + r)}{\Gamma r} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta + r + j + 2}{(\theta + r + j)^3} \right] \\
\times \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta + r + j + 1}{(\theta + r + j)^2} \left( \frac{\Gamma r \Gamma'_{(x + r)} - \Gamma_{(x + r)} \Gamma'(r)}{(\Gamma r)^2} \right)
\]

similarly

\[
\frac{\partial}{\partial \pi} l(\omega, r, \theta) = \sum_{i=1}^{n} \frac{1}{IA + (1 - I)B} \times \frac{\partial}{\partial \omega} \left( IA + (1 - I)B \right)
\]

where

\[
\frac{\partial}{\partial \pi} IA = \frac{\partial}{\partial \pi} (I\omega) + I (1 - \omega) \left[ \frac{\theta^2}{\theta + 1} \frac{\theta + r + 1}{(\theta + r)^2} \right]
\]

\[
\frac{\partial}{\partial \omega} IA = I \left[ 1 - \frac{\theta^2}{\theta + 1} \frac{\theta + r + 1}{(\theta + r)^2} \right]
\]

and

\[
\frac{\partial}{\partial \omega} (1 - I)B = \frac{\partial}{\partial \pi} (1 - I) \left[ (1 - \omega) \left( \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{\theta + 1} \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right) \right]
\]

\[
= (1 - I) \left( \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{\theta + 1} \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right) \times \frac{\partial}{\partial \omega} (1 - \omega)
\]
\[
\frac{\partial}{\partial \omega} (1-I)B = -(1-I) \left( x + r + 1 \right) \frac{\theta^2}{\theta + 1} \sum_{j=0}^{r} \left( \frac{x}{j} \right) (-1)^j \frac{\theta + r + j + 1}{(\theta + r + j)^2}
\] (3.56)

Substituting equation (3.55) and (3.56) in (3.54), \(\frac{\partial}{\partial \pi} l(\omega, r, \theta)\) can be obtained as follows

\[
\frac{\partial}{\partial \pi} l(\omega, r, \theta) = \frac{1}{IA + (1-I)B} \left[ \left( \frac{1-\theta^2}{\theta+1} - \frac{\theta+r+1}{(\theta+r)^2} \right) - (1-I) \left( \frac{x+r-1}{x} \right) \frac{\theta^2}{\theta+1} \sum_{j=0}^{r} \left( \frac{x}{j} \right) (-1)^j \frac{\theta + r + j + 1}{(\theta + r + j)^2} \right]
\] (3.57)

The above three partial derivative equations cannot be solved analytically by equating these equations to zero. Therefore we adopted Newton-Raphson technique, which is a simple iterative numerical method to estimate the maximum likelihood estimators. Solving the resulting normal equations simultaneously using R software, we obtained the MLE solutions of \(\hat{\omega}, \hat{r}\) and \(\hat{\theta}\).

**Application of ZINB-L Distribution**

Here we considered a real data set which shows over dispersion and excess zero counts. The data set was taken from Zamani and Ismail (2010). It consists of the number of accidents per each policy. The data set is depicted in Table 3.1. The data set has 82.9 % values of zeros and dispersion index is 1.348, signifying that the data set is over dispersed and proportion of zero count is much higher than usual. We fitted the traditional standard distributions for modeling count data such as Poisson distribution, negative binomial distribution, NB-L distribution and ZINB-L distribution. We computed the measures for chi-square test of goodness- of-fit, \(p\)-value and log-likelihood for the given data.

The results are provided in Table 3.1 shows that Poisson and negative binomial distribution provides poor fit compared to NB-L and ZINB-L distributions. Also we obtained that ZINB-L distribution provides relatively better performance over NB-L distribution for the given over dispersion and excess zero count data.
Table 3.1: Observed and expected frequencies of the number of accidents under the policy using Poisson, NB, NB-L and ZINB-L distributions

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Number of Drivers</th>
<th>Fitted Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Poisson</td>
</tr>
<tr>
<td>0</td>
<td>7840</td>
<td>7638.3</td>
</tr>
<tr>
<td>1</td>
<td>1317</td>
<td>1634.6</td>
</tr>
<tr>
<td>2</td>
<td>239</td>
<td>174.9</td>
</tr>
<tr>
<td>3</td>
<td>42</td>
<td>12.5</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>0.7</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>8+</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Parameter estimates:
\[ \hat{\lambda} = 0.214, \hat{\hat{\lambda}} = 0.70, \hat{\hat{\hat{\lambda}}} = 4.63, \hat{\hat{\hat{\hat{\lambda}}}} = 1.433 \]

Chi-squares:
- Poisson: 138.34, P-value: <0.01
- NB: 7.277, P-value: 0.026
- NB-L: 6.366, P-value: 0.095
- ZINB-L: 3.22, P-value: 0.359

Log-likelihood:
- Poisson: -5490.78
- NB: -5348.00
- NB-L: -5344.70
- ZINB-L: -5342.5

3.2.2 Zero Inflated Negative Binomial-Two Parameter Lindley (ZINB-TPL) Distribution

The proposed ZINB-L distribution is also obtained with the intension of generating a more flexible alternative for modeling the count data with over dispersion and high density of frequency at zero. The vital statistical measures of this distribution are derived. Maximum likelihood estimation is employed for estimating the parameters of the proposed distribution. Further, we performed curve analysis of the distribution to access the behavior of the parameters of the distribution. Finally, we presented an application of ZINB-TPLD for modeling the claim count data with an excess number of zeros and evaluated the goodness of fit of ZINB-TPL distribution with other suitable distributions.
**Theorem 3.3:** If \( X \sim ZINB - TPL (\omega, r, a, b) \), then the probability mass function of \( X \) is

\[
p(x; \omega, r, a, b) = \begin{cases} 
\omega + (1 - \omega) \frac{b^2}{b + a} \frac{(a + b + r)}{(b + r)^2} ; & x = 0 \\
(1 - \omega) \frac{r + x - 1}{x} \sum_{j=0}^{r} \left( \frac{x}{x} \right)^{-1} \left( 1 - \frac{x}{x} \right)^{-1} \frac{b^2}{b + a} \frac{(b + a + r + j)}{(b + r + j)^2} ; & x = 1, 2, 3, \ldots 
\end{cases} 
\]  
(3.58)

where \( x = 0, 1, 2, \ldots ; b > 0, a + b > 0, 0 \leq \omega \leq 1 \).

**Proof:**

If \( X/\lambda \sim ZINB(r, p = e^{-\lambda}, \omega) \) and \( \lambda \sim TPL(a, b) \) then the PMF of zero inflated negative binomial-Lindley distribution can be obtained as follows,

\[
p(x; \omega, r, a, b) = \int_0^\infty p(x; \omega, r, p = e^{-\lambda}) f(\lambda; a, b) d\lambda
\]

where \( p(x; \omega, r, p = e^{-\lambda}) \) is the PMF of the zero inflated negative binomial distribution given in equation (3.27) and \( f(\lambda; a, b) \) is the density function of the one parameter Lindley distribution given as follows

\[
f(\lambda; a, b) = \frac{b^2}{b + a} (1 + a\lambda) e^{-b\lambda}, \lambda > 0, a + b > 0
\]  
(3.59)

Then the PMF of \( ZINB - TPL (\omega, r, a, b) \) at \( x = 0 \), can be obtained as

\[
p(x; \omega, r, a, b) = \int_0^\infty (\omega + (1 - \omega) e^{-\lambda}) f(\lambda; a, b) d\lambda
\]

\[
= \int_0^\infty \omega f(\lambda; a, b) d\lambda + \int_0^\infty (1 - \omega) e^{-\lambda} f(\lambda; a, b) d\lambda
\]

\[
= \omega + (1 - \omega) \int_0^\infty e^{-\lambda} f(\lambda; a, b) d\lambda
\]

\[
p(x; \omega, r, a, b) = \omega + (1 - \omega) M_2(-r)
\]
Then the PMF of the $ZINB-TPL(\omega, r, a, b)$ at $x = 0$ is obtained by substituting the MGF of Lindley distribution with $z = r$. Therefore the PMF of ZINB-TPL distribution at $x = 0$ is obtained as follows

$$p(x; \omega, r, a, b) = \omega + (1 - \omega) \frac{b^2}{b + a} \frac{a + b + r}{(b + r)^2}$$

where the MGF of the Two parameter Lindley distribution is given below

$$M_\lambda(z) = \frac{b^2}{b + a} \frac{a + b - z}{(b - z)^2} \quad (3.60)$$

Similarly we can obtain the PMF of $ZINB-TPL(\omega, r, \theta)$ at $x \neq 0$ as

$$p(x; \omega, r, a, b) = \int_0^\infty \left(1 - \omega \right) \left( \frac{x + r - 1}{x} \right) e^{-a} \left( 1 - e^{-\lambda} \right)^x f(\lambda; a, b) d\lambda$$

$$= (1 - \omega) \left( \frac{x + r - 1}{x} \right) \int_0^\infty e^{-a} \left( 1 - e^{-\lambda} \right)^x f(\lambda; a, b) d\lambda$$

Using binomial expansion, we get

$$p(x; \omega, r, a, b) = \omega \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^x \binom{x}{j} (-1)^j e^{-\lambda} e^{-(r+j)} f(\lambda; a, b) d\lambda$$

$$= (1 - \omega) \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^x \binom{x}{j} (-1)^j M_\lambda(-(r+j))$$

Using MGF of the two parameter Lindley distribution in equation (3.60) obtained the PMF of $ZINB-TPL(\omega, r, \theta)$ at $x \neq 0$ as follows

$$p(x; \omega, r, a, b) = (1 - \omega) \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^x \binom{x}{j} (-1)^j \frac{a + b + r + j}{(b + a)(b + r + j)}$$
Figure 3.3: PMF of the $NB-TPL(r, a, b)$ and $ZINB-TPL(\omega, r, a, b)$ distribution for different values of parameters.

When $\omega = 0$, ZINB-TPL distribution reduces to NB-TPL distribution. Some possible shapes of the NBTP-L distribution and ZINB-TPL distribution are given in Figure 3.3. The PMF of the ZINB-TPLD for different values of parameters $(\omega, r, a, b)$ are depicted in Figure 3.4.
Properties of the ZINB-TPL Distribution

**Theorem 3.4:** If \( X \sim ZINB-TPLD(\omega, r, a, b) \), the factorial moments of order \( k \) of \( X \) is

\[
\mu_{[k]}(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma(r)} \frac{b^2}{b + a} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \frac{b + a - k + j}{(b - k + j)^2}
\]

(3.61)

where \( k = 0, 1, 2, \ldots; b > 0, a + b > 0, 0 \leq \omega \leq 1 \)

**Proof:**

The factorial moment of the zero inflated mixed negative binomial distribution can be written in the form

\[
\mu_{[k]}(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma(r)} \frac{b^2}{b + a} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j M_\lambda(k - j)
\]

Substituting the MGF of the two parameter Lindley distribution given in equation (3.60) in \( M_\lambda(k - j) \) at \( z = k - j \), we obtain the factorial moments of ZINB-TPL distribution.
Moments, Skewness and Kurtosis

The moments of \( X \) about the origin can be attained simply by putting \( k = 1,2,3,4 \) in equation (3.61) and are given below.

When \( k = 1 \) we obtain \( E(X) \) as

\[
E(X) = (1 - \omega) \frac{\Gamma(r + 1)}{\Gamma r} \frac{b^2}{b + a} \sum_{j=0}^{1} \frac{1}{(-1)^j} \left\{ \frac{(b + a - 1 + j)}{(b - 1 + j)^2} \right\}
\]

\[
= (1 - \omega) r \frac{b^2}{b + a} \left[ \frac{b + a - 1}{(b - 1)^2} - \frac{b + a}{b^2} \right]
\]

\[
= (1 - \omega) r \left[ \frac{b^2}{b + a} \frac{b + a - 1}{(b - 1)^2} - \frac{b^2}{b + a} \frac{b + a}{b^2} \right]
\]

\[
= (1 - \omega) r \left( \frac{b^2}{(b + a)(b - 1)^2} - 1 \right)
\]

\[
\mu_1 = E(X) = (1 - \omega) r \left[ d_1 - d_0 \right] \tag{3.62}
\]

where \( d_s = \frac{b^2}{b + a} \frac{b + a - g}{(b - g)^2} \)

When \( k = 2 \) we obtain \( E(X(X - 1)) \) as

\[
E(X(X - 1)) = (1 - \omega) \frac{\Gamma(r + 2)}{\Gamma r} \frac{b^2}{b + a} \sum_{j=0}^{2} \frac{2}{(-1)^j} \left\{ \frac{(b + a - 2 + j)}{(b - 2 + j)^2} \right\}
\]

\[
= (1 - \omega) r \frac{b^2}{b + a} \left[ \frac{(b + a - 2)}{(b - 2)^2} - 2 \frac{(b + a - 2 + 1)}{(b - 1)^2} + \frac{(b + a)}{b^2} \right]
\]

\[
= (1 - \omega) r \left[ \frac{b^2}{b + a} \left( \frac{(b + a - 2)}{(b - 2)^2} \right) - 2 \frac{b^2}{b + a} \left( \frac{b + a - 1}{(b - 1)^2} \right) + \frac{b^2}{b^2} \left( \frac{b + a}{b + a} \right) \right]
\]

\[
= (1 - \omega) r \left[ \frac{b^2}{b + a} \left( \frac{(b + a - 2)}{(b - 2)^2} \right) - 2 \frac{b^2}{b + a} \left( \frac{b + a - 1}{(b - 1)^2} \right) + 1 \right]
\]

\[
E(X^2) \text{ can be written as}
\]
\[ E(X^2) = (1 - \omega) r \left[ \frac{b^2}{b+a} \left( \frac{(b+a-2)}{(b-2)^2} - 2 \left( \frac{b+a-1}{(b-1)^2} \right) + 1 + \frac{b^2}{(b+a)} \left( \frac{b+a-1}{(b-1)^2} - 1 \right) \right) \right] \]

\[ = (1 - \omega) r \frac{b^2}{b+a} \left[ \frac{(b+a-2)}{(b-2)^2} - 2 \left( \frac{b+a-1}{(b-1)^2} \right) + \frac{b+a-1}{(b-1)^2} \right] \]

\[ \mu'_2 = E(X^2) = (1 - \omega) r \frac{b^2}{b+a} \left[ \frac{(b+a-2)}{(b-2)^2} - \frac{(b+a-1)}{(b-1)^2} \right] = (1 - \omega) r [d_2 - d_1] \quad (3.63) \]

When \( k = 3 \), we will get \( E(X(X-1)(X-2)) \) as

\[ E[X(X-1)(X-2)] = (1 - \omega) \frac{\Gamma(r+2)}{\Gamma r} b^2 \sum_{j=0}^{\frac{3}{2}} \binom{2}{j} (-1)^j \left[ \frac{b+a-3+j}{(b-3+j)^2} \right] \]

\[ = (1 - \omega) r \frac{b^2}{b+a} \left[ \frac{b+a-3}{(b-3)^2} - 3 \left( \frac{b+a-2}{(b-2)^2} \right) + 3 \left( \frac{b+a-1}{(b-1)^2} \right) - \frac{b+a}{b^2} \right] \]

\[ = (1 - \omega) r \frac{b^2}{b+a} \left[ \frac{b+a-3}{(b-3)^2} - 3 \left( \frac{b+a-2}{(b-2)^2} \right) + 3 \left( \frac{b+a-1}{(b-1)^2} \right) - \frac{b+a-0}{(b-0)^2} \right] \]

\[ = (1 - \omega) r [d_3 - 3d_2 + 3d_1 - d_0] \]

From the above equations \( E(X^3) = E(X(X-1)(X-2)) + 3E(X^2) - 2E(X) \) can be written as

\[ E(X^3) = (1 - \omega) r [d_3 - 3d_2 + 3d_1 - d_0] + 3(1 - \omega) r [d_2 - d_1] - 2(1 - \omega) r (d_1 - d_0) \]

\[ = (1 - \omega) r [d_3 - 3d_2 + 3d_1 - d_0 + 3d_2 - 3d_1 - 2d_0 + 2d_1] \]

So

\[ \mu'_3 = E(X^3) = (1 - \omega) r [d_3 - 2d_1 + d_0] \quad (3.64) \]

When substituting \( k = 4 \) in the factorial moment expression in equation (3.60), we obtained

\[ E[X(X-1)(X-2)(X-3)] \]

\[ = (1 - \omega) r \frac{b^2}{b+a} \sum_{j=0}^{4} \binom{4}{j} (-1)^j \left[ \frac{b+a-4+j}{(b-4+j)^2} \right] \]

\[ = (1 - \omega) r \frac{b^2}{b+a} \left[ \frac{b+a-4}{(b-4)^2} - 4 \left( \frac{b+a-3}{(b-3)^2} \right) + 6 \left( \frac{b+a-2}{(b-2)^2} \right) - 4 \left( \frac{b+a-1}{(b-1)^2} \right) + \frac{b+a}{b^2} \right] \]
Then, 

\[ E(X^4) = E[(X-1)(X-2)(X-3)] = 6E(X^2) - 11E(X^3) + 6E(X) \]

\[ = (1-\omega)r[d_4 - 4d_3 + 6d_2 - 4d_1 + d_0] \]

\[ \mu_4' = E(X^4) = (1-\omega)r[d_4 - 2d_3 - 5d_2 + d_1 + d_0] \]  \hspace{1cm} (3.65)

The variance of the distribution can be written as

\[ V(X) = E(X^2) - [E(X)]^2 \]

\[ = (1-\omega)r \left\{ \frac{b^2}{b+a} \left[ \frac{(b+a-2)}{(b-2)^2} - \frac{b+a-1}{(b-1)^2} \right] \right\} - (1-\omega)^2 r^2 \left\{ \left[ \frac{b^2}{b+a} \right] \frac{b+a-1}{(b-1)^2} - 1 \right\} \]

\[ = (1-\omega)r \left[ \frac{b^2}{b+a} \left[ \frac{(b+a-2)}{(b-2)^2} - \frac{b+a-1}{(b-1)^2} \right] \right] - (1-\omega)^2 r^2 \left[ \frac{b^2}{b+a} \left( \frac{b+a-1}{(b-1)^2} - 1 \right) \right] \]

\[ \mu_2 = V(X) = (1-\omega)r[(d_2 - d_1) - (1-\omega)r(d_1 - d_0)^2] \]  \hspace{1cm} (3.66)

Similarly,

\[ \mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 \]

\[ = (1-\omega)r\left\{(d_3 - 2d_1 + d_0) - (1-\omega)r(d_1 - d_0)\left[3(d_2 - d_1) + (1-\omega)r(d_1 - d_0)^2\right]\right\} \]  \hspace{1cm} (3.67)

\[ \mu_4 = \mu_4' - 4\mu_1'\mu_2' + 6(\mu_1')^2 - 3(\mu_1')^4 \]

\[ = (1-\omega)r\left\{(d_4 - 2d_3 - 5d_2 + d_1 + d_0) - (1-\omega)r(d_1 - d_0)\left[4(d_3 - 2d_1 + d_0) + 6(1-\omega)r(d_2 - d_1)(d_1 - d_0)\right]\right\} \]

\[ - 3(1-\omega)^2 r^2 (d_1 - d_0)^3 \]  \hspace{1cm} (3.68)

The coefficients of skewness \( \gamma_1 = \frac{\mu_3}{\mu_2^{1/2}} \) and kurtosis \( \gamma_2 = \frac{\mu_4}{\mu_2^2} \) measures are obtained as follows
\begin{align*}
\gamma_1 &= \left\{ \left( d_3 - 2d_1 + d_0 \right) - (1 - \omega)r(d_1 - d_0) \right\} \left[ \frac{3(d_3 - d_1) + 2(1 - \omega)r(d_1 - d_0)^2}{(1 - \omega)r} \right]^{1/2} \\
\gamma_2 &= \left\{ \frac{4(d_3 - 2d_1 + d_0) + 6(1 - \omega)r(d_2 - d_1)(d_1 - d_0)}{(1 - \omega)r(d_2 - d_1)(d_1 - d_0)^2} \right\}^{1/2}
\end{align*}

(3.69) \quad (3.70)

**Parameter Estimation of ZINB-TPL**

We considered maximum likelihood estimation (MLE) method for the estimation the parameters of the ZINB-TPL distribution. Let \( x_1, x_2, \ldots, x_n \) be \( n \) independent random observations from the proposed ZINB-TPL distribution. Then the likelihood function of the vector of parameters \((\omega, r, a, b)\) is of the form

\[
L(\omega, r, a, b) = \prod_{i=1}^{n} I \left( \omega + (1 - \omega) \frac{b^2}{b + a} \frac{(a + b + r)}{(b + r)^2} + (1 - I) \right) \left[ \left( 1 - \omega \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \right] \left( \frac{b^2}{b + a} \frac{(b + a + r + f)}{(b + r + f)^2} \right)
\]

(3.71)

where

\[
I(x) = \begin{cases} 1 ; & \text{if } x = 0 \\ 0 ; & \text{if } x \in \{1,2,\ldots\} \end{cases}
\]

Take

\[
A = \omega + (1 - \omega) \frac{b^2}{b + a} \frac{(a + b + r)}{(b + r)^2} \quad \text{and} \quad B = \left( 1 - \omega \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{b^2}{b + a} \frac{(b + a + r + f)}{(b + r + f)^2}
\]

The log likelihood function can be written as

\[
l(\omega, r, a, b) = \log L(\omega, r, a, b) = \sum_{i=1}^{n} \log(I A + (1 - I)B)
\]

(3.72)
The normal equations are obtained by differentiating the log-likelihood function of the ZINB-TPL distribution with respect to the parameters \( \omega, r, a \) and \( b \). The normal equations are obtained as follows

\[
\frac{\partial}{\partial a} I(\omega, r, a, b) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \times \frac{\partial}{\partial a} \{IA + (1-I)B\} 
\]

(3.73)

where

\[
\frac{\partial}{\partial a} IA = \frac{\partial}{\partial a} I \left\{ \omega + (1-\omega) \frac{(a+b+r)b^2}{(b+a)(b+r)^2} \right\}
\]

\[
= I (1-\omega) \frac{\partial}{\partial a} \left\{ \frac{(a+b+r)b^2}{(b+a)(b+r)^2} \right\}
\]

\[
= I (1-\omega) \frac{b^2}{(b+r)^2} \left[ \frac{(b+a)-(a+b+r)}{(b+a)^2} \right]
\]

\[
= I (1-\omega) \frac{b^2}{(b+r)^2} \left[ \frac{b+a-a-b-r}{(b+a)^2} \right]
\]

\[
= I (1-\omega) \frac{b^2}{(b+r)^2} \left[ \frac{-r}{(b+a)^2} \right]
\]

\[
\frac{\partial}{\partial a} IA = I(1-\omega) \frac{-rb^2}{(b+r)^2(b+a)^2}
\]

(3.74)

and

\[
\frac{\partial}{\partial a} (1-I)B = \frac{\partial}{\partial a} (1-I) \left[ (1-\omega) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(a+b+r+j)b^2}{(b+a)(b+r+j)^2} \right]
\]

\[
= (1-I)(1-\omega) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\partial}{\partial a} \left[ \frac{(a+b+r+j)b^2}{(b+a)(b+r+j)^2} \right]
\]
\[
(1 - I)(1 - \omega) \left( x + r - 1 \right) x \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \left\{ \frac{\frac{\partial}{\partial a} \left[ (a + b + r + j) b^2 \right] (b + a) (b + r + j)^2}{(b + a) (b + r + j)^2} \right\}
\]

\[
= (1 - I)(1 - \omega) \left( x + r - 1 \right) x \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \left\{ \frac{b^2 \left[ (b + a) (b + r + j)^2 \right]}{(b + a)^2 (b + r + j)^2} \right\}
\]

\[
= (1 - I)(1 - \omega) \left( x + r - 1 \right) x \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \left\{ \frac{\left[ (b + a) (b + r + j)^2 \right]}{(b + a)^2 (b + r + j)^2} \right\}
\]

\[
= (1 - I)(1 - \omega) \left( x + r - 1 \right) x \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \left\{ \frac{b^2 (b + r + j)^2 \left[ (b + a - a - b - r - j) \right]}{(b + a)^2 (b + r + j)^2} \right\}
\]

\[
\frac{\partial}{\partial a} (1 - I)B = (1 - I)(1 - \omega) \left( x + r - 1 \right) x \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \left\{ \frac{b^2 (r + j)}{(b + a)^2 (b + r + j)^2} \right\}
\] (3.75)

substitute equations (3.74) and (3.75) in (3.73) we get

\[
\frac{\partial}{\partial a} l(\omega, r, a, b) = - \sum_{i=1}^{n} \frac{1}{IA + (1 - I)B} \left[ \frac{I(1 - \omega)}{(b + r)^2 (b + a)^2} + (1 - I)(1 - \omega) \right]
\times \left[ \left( r + x - 1 \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \left\{ \frac{b^2 (r + j)}{(b + a)^2 (b + r + j)^2} \right\} \right]
\] (3.76)

Similarly

\[
\frac{\partial}{\partial b} l(\omega, r, a, b) = \sum_{i=1}^{n} \frac{1}{IA + (1 - I)B} \times \frac{\partial}{\partial b} \left[ IA + (1 - I)B \right]
\] (3.77)

where
\[
\frac{\partial}{\partial b} I_A = \frac{\partial}{\partial b} I \bigg[ \omega + (1 - \omega) \left( \frac{a + b + r}{b + a} \right)^2 (b + r)^2 \bigg] = I(1 - \omega) \frac{\partial}{\partial b} \left[ \frac{a + b + r}{b + a} \right] (b + r)^2
\]

\[
= I(1 - \omega) \left\{ \left[ (b + a)(b + r)^2 \right] \frac{\partial}{\partial b} \left[ \frac{a + b + r}{b + a} \right] - (a + b + r)b^2 \frac{\partial}{\partial b} \left[ \frac{b + a}{b + r} \right] \right\}
\]

\[
= I(1 - \omega) \frac{(b + a)(b + r)^2 (a + b + r)2b + (a + b + r)^2 b^2}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + a)(a + b + r)^2 b(b + r)(b + r - b)}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)2rb(a + b + r) - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)2rb(a + b + r) - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)2rb(a + b + r) - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]

\[
= I(1 - \omega) \frac{(b + r)(b + a)(b + r)^2 r - (b + r)^2 b^2 r}{(b + a)^2 (b + r)^2}
\]
\[ I(1 - \omega) \frac{br}{(b + a)^2(b + r)} \left[ (b + r)a + 2ab + 2a^2 \right] \]

\[ I(1 - \omega) \frac{br(ab + ar + 2ab + 2a^2)}{(b + a)^2(b + r)^3} \]

\[ \frac{\partial}{\partial b} I A = I(1 - \omega) \frac{br(2a^2 + 3ab + ar)}{(b + a)^2(b + r)^3} \]  

(3.78)

and

\[ \frac{\partial}{\partial b} (1 - I) B = (1 - I)(1 - \omega) \left( x + r - 1 \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(a + b + r + j)b^2}{(b + a)(b + r + j)^2} \]

\[ \frac{b + a)(b + r + j)^2 [b^2 + (a + b + r + j) \times 2b] - (a + b + r + j)b^2 [2(b + a)(b + r + j)] + (b + r + j)^2}{(b + a)^2(b + r + j)^4} \]

\[ \left( x + r - 1 \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left\{ \frac{(b + a)(b + r + j)^2 [b^2 + 2b(a + b + r + j)] - (b + r + j)b^2 [2(b + a)(b + r + j)]}{(b + a)^2(b + r + j)^4} \right\} \]

\[ (1 - I)(1 - \omega) \left( x + r - 1 \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left\{ \frac{(b + a)(b + r + j)^2 [b^2 + 2b(a + b + r + j)] - (b + r + j)b^2 [2(b + a)(b + r + j)]}{(b + a)^2(b + r + j)^4} \right\} \]
\[
(1 - I)(1 - \omega) \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left\{ \frac{(b + a)(b + r + j)b^2 + 2b(b + a)(b + r + j)}{(a + b + r + j) - 2b^2(a + b + r + j)(b + a)} \right\}
= \frac{1}{(b + a)^2[b + r + j]^3} \left\{ (a + b + r + j)(b + a) \right\}
\times \left\{ \frac{2b(b + r + j) - 2b^2}{b + a - a - b - r - j} \right\}
= \frac{b^2(r + j)}{(b + a)^2[b + r + j]^3}
= \frac{2b(a + b + r + j)(b + a)(r + j)}{(b + r + j)b^2(r + j)}
= \frac{b(r + j)^2}{(b + a)^2[b + r + j]^3}
\]

\[
\frac{\partial}{\partial b} (1 - I)B = (1 - I)(1 - \omega) \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left\{ \frac{2(a + b + r + j)(b + a)}{b(r + j) + b(b + r + j)} \right\}
\] 

Substitute (3.78) and (3.79) in (3.77), we obtained \( \frac{\partial}{\partial b} l(\omega, r, a, b) \) as
\[
\frac{\partial}{\partial b} l(\omega,r,a,b) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \left[ I(1-\omega) \frac{rb(2a^2 + 3ab + ar)}{(b+r)(b+a)^2} + (1-I)(1-\omega) \left( \frac{r + x - 1}{x} \right) \sum_{j=0}^{x} \left( \begin{array}{c} x \\ j \end{array} \right) (-1)^j \right] \\
\times \left[ b(r+j) \left( \frac{2(a+b+r+j)(b+a)}{(b+a)^2(b+r+j)} \right) \right]
\]

\[
\frac{\partial}{\partial r} l(\omega,r,a,b) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \times \frac{\partial}{\partial r} \left[ IA + (1-I)B \right]
\]

where

\[
\frac{\partial}{\partial r} IA = I(1-\omega) \frac{\partial}{\partial r} \left[ \frac{(a+b+r)b^2}{(b+a)(b+r)^2} \right]
\]

\[
= I(1-\omega) \left[ \frac{(b+a)(b+r)^2 \frac{\partial}{\partial r} [(a+b+r)b^2] - (a+b+r)b^2 \frac{\partial}{\partial r} [(b+a)(b+r)^2]}{(b+a)^2(b+r)^4} \right]
\]

\[
= I(1-\omega) \left[ \frac{(b+a)(b+r)^2 b^2 - (a+b+r)b^2 (b+a) 2(b+r)}{(b+a)^2(b+r)^4} \right]
\]

\[
= I(1-\omega) \left[ \frac{b^2(b+r) [(b+a)(b+r) - 2(a+b)(b+a)]}{(b+a)^2(b+r)^4} \right]
\]

\[
= I(1-\omega) \left[ \frac{(b+a)b^2(b+r)[b+r - 2a - 2b - 2r]}{(b+a)^2(b+r)^4} \right]
\]

\[
= I(1-\omega) \left[ \frac{(b+a)b^2(b+r)[-2a - b - r]}{(b+a)^2(b+r)^4} \right]
\]

\[
= I(1-\omega) \left[ \frac{b^2(-2a - (b+r))}{(b+a)(b+r)^3} \right]
\]
\[
\frac{\partial}{\partial r} IA = I(1 - \omega) \left[ -b^2 \frac{(2a + b + r)}{(b + a)(b + r)^2} \right]
\]

(3.82)

and

\[
\frac{\partial}{\partial r} (1 - I)B = (1 - I)(1 - \omega) \frac{\partial}{\partial r} \left[ \sum_{j=0}^{\infty} \left( \frac{x}{j} \right)^{(-1)^j} \frac{(a + b + r + j)b^2}{(b + a)(b + r + j)^2} \right]
\]

\[
= (1 - I)(1 - \omega) \left[ \sum_{j=0}^{\infty} \frac{x}{j} \right]^{(-1)^j} \frac{(b + r + j)^2}{(b + a)} \left[ \frac{b^2}{(b + r + j)^2} \left( -2(a + b + r + j) \frac{(b + r + j)}{(b + r + j)^2} \right) \right]
\]

\[
= (1 - I)(1 - \omega) \left[ \sum_{j=0}^{\infty} \frac{x}{j} \right]^{(-1)^j} \frac{b^2}{(b + a)} \left[ \frac{(b + r + j)}{(b + r + j)^2} \right] \left[ \frac{(b + r + j)}{b + a} \right]
\]

\[
= (1 - I)(1 - \omega) \left[ \sum_{j=0}^{\infty} \frac{x}{j} \right]^{(-1)^j} \frac{b^2}{(b + a)} \left[ \frac{(b + r + j)(1 - 2a)}{(b + r + j)^2} \right]
\]

\[
= (1 - I)(1 - \omega) \left[ \sum_{j=0}^{\infty} \frac{x}{j} \right]^{(-1)^j} \frac{b^2}{(b + a)} \left[ \frac{(2a + b + r + j)}{(b + r + j)^2} \right]
\]

\[
\frac{\partial}{\partial r} (1 - I)B = -(1 - I)(1 - \omega) \left[ \sum_{j=0}^{\infty} \frac{x}{j} \right]^{(-1)^j} \frac{b^2}{(b + a)} \left[ \frac{(2a + b + r + j)}{(b + r + j)^2} \right]
\]

(3.83)

Substitute (3.82) and (3.83) in (3.81), we obtained \( \frac{\partial}{\partial r} l(\omega, r, a, b) \) as
\[ \frac{\partial}{\partial r} l(\omega, r, a, b) = -\sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \left[ I(1-\omega) \frac{b^2(2a+b+r)}{(b+r)^3(b+a)} + (1-I)(1-\omega) \right] \times \left( r + x - 1 \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{b^2(2a+b+r)}{(b+a)(b+r+j)} \]  

(3.84)

\[ \frac{\partial}{\partial \omega} \log L = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \frac{\partial}{\partial \omega} \{IA + (1-I)B\} \]  

(3.85)

where

\[ \frac{\partial}{\partial \omega} IA = I \left\{ \omega + (1-\omega) \frac{(a+b+r)b^2}{(b+a)(b+r)^2} \right\} \]

\[ = I \left\{ 1 + \frac{(a+b+r)b^2}{(b+a)(b+r)^2} \times -1 \right\} \]

\[ = \frac{\partial}{\partial \omega} IA = I \left\{ 1 - \frac{(a+b+r)b^2}{(b+a)(b+r)^2} \right\} \]  

(3.86)

and

\[ \frac{\partial}{\partial \omega} (1-I)B = (1-I) \frac{\partial}{\partial \omega} \left[ (1-\omega) \left( \frac{x+r-1}{x} \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{(a+b+r+j)b^2}{(b+a)(b+r+j)^2} \right] \]

\[ = -\left( 1-I \right) \left( \frac{x+r-1}{x} \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{(a+b+r+j)b^2}{(b+a)(b+r+j)^2} \]  

(3.87)

Substituting (3.86) and (3.87) in (3.85), we obtained \( \frac{\partial}{\partial \omega} l(\omega, r, a, b) \) as

\[ \frac{\partial}{\partial \omega} l(\omega, r, a, b) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \left[ I \left\{ 1 - \frac{b^2(a+b+r)}{(b+r)^2(b+a)} \right\} \right. \]

\[ \left. - (1-I) \left( \frac{x+r-1}{x} \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \right\} \times \frac{b^2(a+b+r+j)}{(b+a)(b+r+j)^2} \]  

(3.88)
For estimating the parameters of the ZINB-TPL distribution, we have to solve the above equations by equating to zero. Since these equations do not have closed form, analytical estimation of parameters is difficult. Hence Newton-Raphson technique can be utilized for approximating the maximum likelihood estimators using R software and obtained the ML estimators $\hat{\omega}, \hat{r}, \hat{a}, \hat{b}$ of ZINB-TPL distribution for any given data.

**Application of ZINB-TPL Distribution using Real Data Set**

To demonstrate the application of ZINB-TPLD, we considered a data set with excess number of zeros and over dispersion. The data set represents the number of hospital stays by US residents aged 66 and over, and is given in Table 3.2 (Flynn and Francis (2009)). It is observed that around 80% of the observations show zero values and index of dispersion is 1.88, hence it is considered as an over dispersed and zero inflated data set. While comparing this ZINB-TPL distribution with already existing count distributions used for modeling over dispersed and zero inflated count data such as NB, ZINB, NB-L, NB-TPL distributions, we obtained that ZINB-TPL distribution provides better performance compared to the existing count distributions for modeling over dispersed count data. It is observed from Table 3.2 that the ZINB-TPL distribution model is better than NB, ZINB, NB-L, NB-TPL distributions based on the values of $\chi^2$, $p$-value and AIC for the given over-dispersed data.
Table 3.2: Observed and Expected Number of hospital stays using NB, ZINB,NB-L,NB-TPL and ZINB-TPL distributions

<table>
<thead>
<tr>
<th>Number of hospital stays</th>
<th>Observed value</th>
<th>Expected value by fitting distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NB</td>
<td>ZINB</td>
</tr>
<tr>
<td>0</td>
<td>3541</td>
<td>3262.17</td>
</tr>
<tr>
<td>1</td>
<td>599</td>
<td>673.15</td>
</tr>
<tr>
<td>2</td>
<td>176</td>
<td>256.66</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>112.82</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>52.88</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>25.71</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>12.80</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>6.48</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>3.32</td>
</tr>
</tbody>
</table>

Parameter estimates

<table>
<thead>
<tr>
<th>Parameter estimates</th>
<th>( \hat{r} = 0.371 )</th>
<th>( \hat{r} = 3.969 )</th>
<th>( \hat{r} = 2.098 )</th>
<th>( \hat{r} = 1.478 )</th>
<th>( \hat{r} = 1.573 )</th>
<th>( \hat{b} = 5.41 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{p} = 0.556 )</td>
<td>( \hat{p} = 0.842 )</td>
<td>( \hat{p} = 6.552 )</td>
<td>( \hat{p} = 6.852 )</td>
<td>( \hat{p} = 8.555 )</td>
<td>( \hat{b} = 8.555 )</td>
<td></td>
</tr>
<tr>
<td>( \hat{\omega} = 0.604 )</td>
<td>( \hat{\omega} = 6.391 )</td>
<td>( \hat{\omega} = 1.083 )</td>
<td>( \hat{\omega} = 7.517 )</td>
<td>( \hat{\omega} = 0.144 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \chi^2 )</th>
<th>84.22</th>
<th>58.31</th>
<th>14.27</th>
<th>5.41</th>
<th>3.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p ) – value</td>
<td>&lt;0.001</td>
<td>&lt;0.001</td>
<td>0.147</td>
<td>0.530</td>
<td>0.907</td>
</tr>
<tr>
<td>AIC</td>
<td>6023.25</td>
<td>6078</td>
<td>6017.77</td>
<td>6007.70</td>
<td>6006.45</td>
</tr>
</tbody>
</table>

3.2.3 Zero Inflated Negative Binomial-Sushila (ZINB-S) Distribution

We proposed ZINB-S distribution as a compound of zero-inflated negative binomial (ZINB) distribution and Sushila distribution and it is named as zero-inflated negative binomial–Sushila (ZINB-S) distribution. It can be used as an alternative and effective way of modeling over dispersed count data. The probability mass function (PMF) and some vital characteristics of ZINB-S distribution are derived. MLE method is employed for estimating the model parameters. Further the example is given to show that ZINB-S provides a better fit compare to traditional models for over dispersed count data.
Theorem 3.5: If $X \sim ZINB - S(\omega, r, \alpha, \theta)$ distribution, then the probability mass function of $X$ can be obtained as

$$g(x; r, \omega, \alpha, \theta) = \begin{cases} \omega + (1 - \omega) \frac{\theta^2 (\theta + r \alpha + 1)}{(\theta + 1)(\theta + r \alpha)^2}, & \text{when } x = 0 \\ (1 - \omega) \frac{\theta^2}{(\theta + 1)} + \left( x + r - 1 \right) \sum_{j=0}^{x} \left[ \begin{array}{c} x \\ j \end{array} \right] (-1)^j \left( \frac{x}{(\theta + \alpha(r + j) + 1)(\theta + \alpha(r + j))^2} \right), & \text{when } x > 0 \end{cases}$$

(3.89)

where $x = 0, 1, 2, \ldots ; r, \alpha, \theta > 0, \ 0 < \omega < 1$.

Proof: If $X/\lambda \sim ZINB(r, p = e^{-\lambda}, \omega)$ and $\lambda \sim Sushila(\alpha, \theta)$ then the PMF of zero inflated negative binomial-Lindley distribution can be obtained as follows,

$$p(x; r, \omega, \alpha, \theta) = \int_{0}^{\infty} p(x; r, \omega, p = e^{-\lambda}) f(\lambda; \alpha, \theta) d\lambda$$

where $p(x; r, \omega, p = e^{-\lambda})$ is the PMF of the zero inflated negative binomial distribution given in equation (3.27) and $f(\lambda; \alpha, \theta)$ is the density function of the Sushila distribution given as follows

$$f(\lambda; \alpha, \theta) = \frac{\theta^2}{\alpha^2(\alpha + 1)} \frac{\lambda^\alpha}{\theta^\alpha} e^{-\lambda} \theta^\alpha, \ \lambda > 0, \theta > 0, \alpha > 0$$

Then the PMF of $ZINB - S(\omega, r, \alpha, \theta)$ at $x = 0$, can be obtained as

$$p(x; r, \omega, \alpha, \theta) = \int_{0}^{\infty} p(X = 0|\lambda) f(\lambda; \alpha, \theta) d\lambda$$

$$= \int_{0}^{\infty} \left[ \omega + (1 - \omega) e^{-\lambda r} \right] f(\lambda; \alpha, \theta) d\lambda$$

$$= \int_{0}^{\infty} \omega f(\lambda; \alpha, \theta) d\lambda + \int_{0}^{\infty} (1 - \omega) e^{-\lambda r} f(\lambda; \alpha, \theta) d\lambda$$
\[ p(x; r, \omega, \alpha, \theta) = \omega + (1 - \omega)\int_{0}^{\infty} e^{-x} f(\lambda; \alpha, \theta) d\lambda \]

Then by substituting the MGF of Sushila distribution with \( z = -r \), we obtained the PMF of the \( ZINB - S(\omega, r, \alpha, \theta) \) at \( x = 0 \). Therefore, the PMF of ZINB-S distribution at \( x = 0 \) can be written as follows

\[ p(x; r, \omega, \alpha, \theta) = \omega + (1 - \omega) \frac{\theta^2 (\theta + r \alpha + 1)}{\theta^2 (\theta + r \alpha)^2} \]

where the MGF of the Sushila distribution is given below

\[ M_{\lambda}(z) = \frac{\theta^2 (\theta - \alpha z + 1)}{(\theta + 1)(\theta - \alpha z)^2} \quad (3.90) \]

Similarly, we can obtain the PMF of \( ZINB - S(\omega, r, \alpha, \theta) \) at \( x \neq 0 \) as

\[ p(x; r, \omega, \alpha, \theta) = (1 - \omega) \left( \sum_{j=0}^{\infty} \binom{x + r - 1}{j} \frac{(-1)^j e^{-\lambda}}{j!} \int_{0}^{\infty} e^{-x} f(\lambda; \alpha, \theta) d\lambda \right) \]

Using binomial expansion, we get

\[ p(x; r, \omega, \alpha, \theta) = (1 - \omega) \left( \frac{x + r - 1}{x} \sum_{j=0}^{\infty} \binom{x}{j} (-1)^j \int_{0}^{\infty} e^{-x} f(\lambda; \alpha, \theta) d\lambda \right) \]

Using MGF of the Sushila distribution in equation (3.90) obtained the PMF of \( ZINB - S(\omega, r, \alpha, \theta) \) at \( x \neq 0 \) is obtained as follows

---

Note: The material presented in Subsection 3.2.3 has been published under the title ‘Zero – Inflated Negative binomial-Sushila Distribution and its Applications’ in International Journal of Pure and Applied Mathematics, 117 (Special issue 13), pp. 117-126, 2017.
When the inflation parameter take the value zero \((\omega = 0)\), ZINB-S distribution reduces to NB-S distribution and when \(\omega = 0\) and \(\alpha = 1\), ZINB –S distribution reduces to Negative Binomial –Lindley (NB-L) distribution. Hence the ZINB-S can be expressed as a comprehensive form of NB-S distribution and NB-L distribution. The PMF curve pattern of ZINB-S distribution for different values of parameters \(\theta\) and \(\omega\) are depicted in figure 3.5.

![Figure 3.5: PMF of ZINB-S distribution for different values of parameters \(\theta\) and \(\omega\)](image)

**Characteristics of ZINB-S distribution**

Some basic characteristics of the distribution are established in this section.

**Theorem 6:** \(X \sim ZINB-S(\omega, r, \alpha, \theta)\), we obtained the factorial moments of order \(k\) of \(X\) as

\[
\mu_k = (1 - \omega) \left( \frac{\Gamma(r + k)}{\Gamma r} \right) \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{\theta^2(\theta - (k - j)\alpha + 1)}{(\theta + 1)(\theta - (k - j)\alpha)^2} \tag{3.91}
\]

**Proof:**

The factorial moments of the zero inflated mixed negative binomial distribution can be written as

\[
\mu_k(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} E_{\lambda} \left( \frac{1 - e^{-\lambda}}{e^{-\lambda}} \right)^k
\]
\[ E(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} E_{\lambda} \left( e^{\lambda} - 1 \right)^k \]

\[ = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} \sum_{j=0}^{k} \binom{k}{j} (-1)^j E(e^{\lambda(k-j)}) \]

\[ \mu_k(X) = (1 - \omega) \frac{\Gamma(r + k)}{\Gamma r} \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_{\lambda}(k-j) \tag{3.92} \]

where \( M_{\lambda}(k-j) \) is the MGF of the Sushila distribution at \((k-j)\). Then by substituting the MGF of Sushila distribution given in equation (3.90) in equation (3.92), we obtained the factorial moment of ZINB-S distribution. Using this expression of factorial moments we can easily obtain the mean and variance of the ZINB-S distribution.

Substituting \( k = 1 \), in equation (3.92) we obtained the mean of the ZINB-S distribution as

\[ E(X) = (1 - \omega) \left( \frac{\Gamma r + 1}{\Gamma r} \right) \frac{\theta^2}{(\theta + 1)} \sum_{j=0}^{1} \binom{1}{j} (-1)^j \frac{(\theta - (1-j)\alpha + 1)}{(\theta - (1-j)\alpha)^2} \]

\[ = (1 - \omega) \left[ \frac{\theta^2(\theta - \alpha + 1)}{(\theta + 1)(\theta - \alpha)^2} - 1 \right] \tag{3.93} \]

For \( k = 2 \), we get

\[ E(X(X-1)) = (1 - \omega) \left( \frac{\Gamma(r + 2)}{\Gamma r} \right) \frac{\theta^2}{(\theta + 1)} \sum_{j=0}^{2} \binom{2}{j} (-1)^j \frac{(\theta - (2-j)\alpha + 1)}{(\theta - (2-j)\alpha)^2} \]

\[ = (1 - \omega) r(r+1) \frac{\theta^2}{(\theta + 1)} \left( \frac{(\theta - 2\alpha + 1)}{(\theta - 2\alpha)^2} - \frac{2(\theta - \alpha + 1)}{(\theta - \alpha)^2} + \frac{(\theta + 1)}{(\theta)} \right) \]

\[ = (1 - \omega) r(r+1) \left( \frac{\theta^2}{(\theta + 1)} \frac{(\theta - 2\alpha + 1)}{(\theta - 2\alpha)^2} - \frac{\theta^2}{(\theta + 1)} \frac{2(\theta - \alpha + 1)}{(\theta - \alpha)^2} + 1 \right) \]

Then, \( E(X^2) \) can be written as

\[ E(X^2) = E(X(X-1)) + E(X) \]

\[ = (1 - \omega) r(r+1) \left( \frac{\theta^2}{(\theta + 1)} \frac{(\theta - 2\alpha + 1)}{(\theta - 2\alpha)^2} - \frac{(r+1)\theta^2}{(\theta + 1)} \frac{2(\theta - \alpha + 1)}{(\theta - \alpha)^2} + 1 \right) \]
\[ E(X^2) = (1 - \omega) r \frac{\theta^2}{(\theta + 1)} \left( \frac{(r+1)(\theta - 2\alpha + 1)}{(\theta - 2\alpha)^2} - \frac{2(r+1)(\theta - \alpha + 1)}{(\theta - \alpha)^2} - \frac{(\theta - \alpha + 1)}{(\theta - \alpha)^2} \right) \] (3.94)

From equations (3.93) and (3.94), variance of ZINB-S distribution can be obtained as follows

\[ V(X) = E(X^2) - (E(X))^2 \]

\[ V(X) = r(1 - \omega) \left\{ \frac{\theta^2}{(\theta + 1)} \left( \frac{(r+1)(\theta - 2\alpha + 1)}{(\theta - 2\alpha)^2} - \frac{2(r+1)(\theta - \alpha + 1)}{(\theta - \alpha)^2} - \frac{(\theta - \alpha + 1)}{(\theta - \alpha)^2} \right) \right\} 
- r(1 - \omega) \left[ \frac{\theta^2(\theta - \alpha + 1)}{(\theta + 1)(\theta - \alpha)^2} - 1 \right]^2 \]

\[ V(X) = r(1 - \omega) \left\{ ((r+1)\delta_2 - 2(r+1)\delta_1 - \delta_1) - r(1 - \omega)\left[ \delta_1 - 1 \right]^2 \right\} \] (3.95)

where \( \delta_1 = \frac{\theta^2}{(\theta + 1)} \left( \frac{\theta - \alpha + 1}{(\theta - \alpha)^2} \right) \), \( \delta_2 = \frac{\theta^2}{(\theta + 1)} \left( \frac{\theta - 2\alpha + 1}{(\theta - 2\alpha)^2} \right) \), \( \delta_3 = \frac{\theta^2}{(\theta + 1)} \left( \frac{\theta - 3\alpha + 1}{(\theta - 3\alpha)^2} \right) \)

**Parameter Estimation of ZINB-S Distribution**

The maximum likelihood estimation method for estimating the model parameters is used. We used the indicator function defined in equation (3.43) for obtaining the likelihood function of the distribution. The likelihood function can written as follows

\[ L(\omega, r, \alpha, \theta) = \prod_{i=1}^{n} \left( \omega + (1 - \omega) \frac{\theta^2(\theta + r\alpha + 1)}{(\theta + 1)(\theta + r\alpha)^2} \right) \]

\[ + (1 - \omega) \left[ (1 - \omega) \left( \frac{\theta^2}{(\theta + 1)} \left( \frac{x + r - 1}{x} \right) \right) \right] \]

\[ \times \sum_{j=0}^{x} \left( \frac{x}{j} \right)(-1)^j \left( \frac{\theta + \alpha(r + j) + 1}{(\theta + \alpha(r + j))^2} \right) \] (3.96)

Take \( A = \omega + (1 - \omega) \frac{\theta^2(\theta + r\alpha + 1)}{(\theta + 1)(\theta + r\alpha)^2} \) and

\[ B = \left( 1 - \omega \right) \left( \frac{\theta^2}{(\theta + 1)} \right) \left( \frac{x + r - 1}{x} \right) \sum_{j=0}^{x} \left( \frac{x}{j} \right)(-1)^j \left( \frac{\theta + \alpha(r + j) + 1}{(\theta + \alpha(r + j))^2} \right) \]
Then the log likelihood function is in the form

\[
I(\omega, r, \alpha, \theta) = \log L(\omega, r, \alpha, \theta) = \sum_{i=1}^{n} \log (I A + (1 - I)B)
\]

(3.97)

The partial derivatives of the log likelihood function are obtained by differentiating equation (3.97) with respect to the parameters \(\omega, r, \alpha, \theta\) as follows

\[
\frac{\partial}{\partial \theta} I(\omega, r, \alpha, \theta) = \sum_{i=1}^{n} \frac{1}{IA + (1 - I)B} \times \frac{\partial}{\partial \theta} [IA + (1 - I)B]
\]

(3.98)

where

\[
\frac{\partial}{\partial \theta} I(A) = I \frac{\partial}{\partial \theta} \left\{ \omega + (1 - \omega) \frac{\theta^2(\theta + r \alpha + 1)}{(\theta + 1)(\theta + r \alpha)^2} \right\}
\]

\[
= I(1 - \omega) \frac{\partial}{\partial \theta} \left\{ \frac{\theta^2(\theta + r \alpha + 1)}{(\theta + 1)(\theta + r \alpha)^2} \right\}
\]

\[
= I(1 - \omega) \left[ \frac{(\theta + 1)(\theta + r \alpha)[(3\theta + 2r \alpha + 2)]}{(\theta + 1)^2(\theta + r \alpha)^3} - \frac{\theta^2(\theta + r \alpha + 1)[3\theta + r \alpha + 2]}{(\theta + 1)^2(\theta + r \alpha)^3} \right]
\]

(3.99)
\[
\frac{\partial}{\partial \theta} (1-I)B = (1-I) \left( \frac{\partial}{\partial \theta} (1-\omega) \left( \frac{x+r-1}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{\theta+1} \frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2} \right) \right)
\]
\[
= (1-I)(1-\omega) \left( \frac{x+r-1}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{\theta^2}{\theta+1} \frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2} \right) \tag{3.100}
\]

and
\[
\frac{\partial}{\partial \theta} \left[ \frac{\theta^2}{(\theta+1)^2} \frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2} \right] = \left[ \frac{(\theta+1)(\theta+\alpha(r+j))^2 \left( 2\theta(\theta+\alpha(r+j)+1)+\theta^2 \right) }{(\theta+1)^2(\theta+\alpha(r+j))^2} - \theta^2 \frac{\theta+\alpha(r+j)+1}{(\theta+1)(\theta+\alpha(r+j))^2} \left( \frac{3\theta+2\alpha(r+j)+2}{(\theta+1)^2} \right) \right] \tag{3.101}
\]

substituting equation (3.101) in (3.100) we obtain \( \frac{\partial}{\partial \theta} (1-I)B \) as
\[
\frac{\partial}{\partial \theta} (1-I)B = (1-I)(1-\omega) \left( \frac{x+r-1}{x} \right) \sum_{j=0}^{x} \binom{x}{j} (-1)^j \left[ \frac{\theta(\theta+1)(\theta+\alpha(r+j))^2 [(3\theta+2\alpha(r+j)+2)]}{(\theta+1)^3(\theta+\alpha(r+j))^3} - \theta^2 \frac{\theta+\alpha(r+j)+1}{(\theta+1)^2(\theta+\alpha(r+j))^2} \left[ \frac{3\theta+\alpha(r+j)+2}{(\theta+1)^2} \right] \right] \tag{3.102}
\]

Substituting (3.99) and (3.102) in (3.98) we obtain \( \frac{\partial}{\partial \theta} l(\omega, r, \alpha, \theta) \) as
Similarly,

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \left[ I(1-\omega) \theta \left[ \frac{(\theta + r\alpha)(\theta + 1)(3\theta + 2r\alpha + 2)}{(\theta + 1)^2 (\theta + r)^3} \right] + (1-I)(1-\omega) \right]$$

\(\times \left( x + r - 1 \right) \sum_{j=0}^{x} \left[ -\theta(\theta + 1) \left( \frac{\theta + \alpha(r + j)}{(\theta + 1)^2 (\theta + (\theta + r + j))^3} \right) \right] \) \hspace{1cm} (3.103)

$$\frac{\partial l}{\partial r} = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \frac{\partial}{\partial r} \left( IA + (1-I)B \right)$$

(3.104)

where

$$\frac{\partial IA}{\partial r} = I \frac{\partial}{\partial r} \left[ \omega(1-\omega) \theta^2 (\theta + r\alpha + 1) \right]$$

$$= I \left( (1-\omega) \theta^2 \left[ \frac{(\theta + r\alpha)^2 - 2\alpha(\theta + r\alpha + 1)(\theta + r\alpha)}{(\theta + r\alpha)^3} \right] \right)$$

$$= I \left( (1-\omega) \theta^2 \alpha \left[ \frac{\theta + r\alpha - 2(\theta + r\alpha + 1)}{(\theta + r\alpha)^3} \right] \right)$$

$$\frac{\partial IA}{\partial r} = I \left( (1-\omega) \theta^2 \alpha \left[ \frac{-(\theta + r\alpha + 2)}{(\theta + r\alpha)^3} \right] \right)$$

(3.105)

and

$$\frac{\partial(1-I)B}{\partial r} = (1-I) \frac{\partial}{\partial r} \left[ (1-\omega) \theta^2 \left( \frac{x + r - 1}{x} \sum_{j=0}^{x} \left( \frac{x}{j} \right) (-1)^j \frac{\theta + \alpha(r + j) + 1}{(\theta + \alpha(r + j))^3} \right) \right]$$
Consider,

\[
\frac{\partial}{\partial r} \left\{ \frac{\Gamma(x+r)}{\Gamma(x+1)} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j)+1)}{(\theta + \alpha(r+j))^4} \right\} = \frac{\Gamma(x+r)}{\Gamma r} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j))^2(\alpha - 2\alpha(\theta + \alpha(r+j)+1)(\theta + \alpha(r+j))}{(\theta + \alpha(r+j))^4}
\]

\[
+ \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j)+1)}{(\theta + \alpha(r+j))^3} \frac{\Gamma r (\Gamma(x+r) - \Gamma(x+r)(r)\Gamma'(r))}{(\Gamma r)^2}
\]

\[
\frac{\partial}{\partial r} \left\{ \frac{\alpha \Gamma(x+r)}{\Gamma r} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j))(\theta + \alpha(r+j)+1)}{(\theta + \alpha(r+j))^3} \right\} = \frac{\alpha \Gamma(x+r)}{\Gamma r} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j)+1)}{(\theta + \alpha(r+j))^2} \frac{\Gamma r (\Gamma(x+r) - \Gamma(x+r)(r)\Gamma'(r))}{(\Gamma r)^2}
\]

Substituting equation (3.107) in equation (3.106), we obtain \(\frac{\partial(1-1)B}{\partial r}\) as

\[
\frac{\partial(1-1)B}{\partial r} = (1-1)(1-\omega) \frac{\partial^2}{\partial (\theta+1) \partial r} \left\{ \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j)+1)}{(\theta + \alpha(r+j))^3} \right\}
\]

\[
+ \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(\theta + \alpha(r+j)+1)}{(\theta + \alpha(r+j))^2} \frac{\Gamma r (\Gamma(x+r) - \Gamma(x+r)(r)\Gamma'(r))}{(\Gamma r)^2}
\]

\[
(3.108)
\]
Substitute equation (3.105) and (3.108) in equation (3.104) we obtained

\[
\frac{\partial}{\partial r} l(\omega, r, \alpha, \theta) = \sum_{j=1}^{n} \frac{1}{IA + (1 - l)B} \left[ I \left[ \frac{(1 - \omega)\theta^2}{\theta + 1} \times \frac{-(\theta + r \alpha + 2)}{(\theta + r \alpha)^2} \right] + (1 - l)(1 - \omega) \frac{\theta^2}{\theta + 1} \right] \\
\quad \quad + \frac{\sum_{j=0}^{x} \left( \begin{array}{c} x \\ j \end{array} \right) (-1)^{j+1} \left( \frac{\Gamma(x + r)}{(\theta + \alpha (r + j + 1))} \right) \times \left( \frac{\Gamma(r) \Gamma'(x + r)}{(\Gamma)^2} \right)}{\Gamma r} \right]
\]

(3.109)

and

\[
\frac{\partial}{\partial \omega} l(\omega, r, \alpha, \theta) = \sum_{j=1}^{n} \frac{1}{IA + (1 - l)B} \frac{\partial}{\partial \omega} \left[ (IA + (1 - l)B) \right]
\]

(3.110)

\[
\frac{\partial (IA)}{\partial \omega} = I \frac{\partial}{\partial \omega} \left[ \omega + (1 - \omega) \frac{\theta^2(\theta + r \alpha + 1)}{(\theta + 1)(\theta + r \alpha)^2} \right]
\]

(3.111)

\[
\frac{\partial B}{\partial \omega} = \frac{\partial}{\partial \omega} \left[ (1 - \omega) \frac{\theta^2}{(\theta + 1)} \left( \frac{x + r - 1}{x} \sum_{j=0}^{x} \left( \begin{array}{c} x \\ j \end{array} \right) (-1)^{j} \left( \frac{\theta + \alpha (r + j + 1)}{(\theta + \alpha (r + j))^2} \right) \right) \right]
\]

(3.112)
Substituting equation (3.11) and (3.12) in (3.100) we obtained

$$
\frac{\partial}{\partial \omega} l(\omega, r, \alpha, \theta) = \sum_{j=1}^{\tilde{\gamma}} \frac{1}{IA + (1 - I)B} \left[ I \left[ 1 - \frac{\theta^2 (\theta + r \alpha + 1)}{(\theta + 1)(\theta + r \alpha)^2} \right] \right] 
$$

\begin{equation}
\left[ \left( \frac{\theta^2}{(\theta + 1)} \right) \left( \frac{x + r - 1}{x} \right) \right] 
- (1 - I) \left[ \frac{\theta^2}{(\theta + 1)} \left( \frac{x}{x} \right) \sum_{j=0}^{\tilde{\gamma}} x(-1)^{j} \left( \frac{(\theta + \alpha (r + j) + 1)}{(\theta + \alpha (r + j))^{2}} \right) \right] \tag{3.113}
\end{equation}

and finally,

$$
\frac{\partial}{\partial \alpha} l(\omega, r, \alpha, \theta) = \sum \log(IA + (1 - I)B) \tag{3.114}
$$

where

$$
\frac{\partial A}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left[ \omega + (1 - \omega) \frac{\theta^2 (\theta + r \alpha + 1)}{(\theta + 1)(\theta + r \alpha)^2} \right] \tag{3.115}
$$

and

$$
\frac{\partial}{\partial \alpha} \left( \frac{(\theta + r \alpha + 1)}{(\theta + r \alpha)^2} \right) = \frac{r (\theta + r \alpha)^3 - 2 (\theta + r \alpha + 1) (\theta + r \alpha)}{(\theta + r \alpha)^4} \tag{3.116}
$$

substituting (3.116) in (3.115) we obtain \( \frac{\partial A}{\partial \alpha} \) as

$$
\frac{\partial A}{\partial \alpha} = -I \left[ \frac{(1 - \omega) \theta^2}{\theta + 1} \times \frac{r (\theta + r \alpha + 2)}{(\theta + r \alpha)^3} \right] \tag{3.117}
$$

$$
\frac{\partial B}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left( (1 - \omega) \frac{\theta^2}{(\theta + 1)} \sum_{j=0}^{\tilde{\gamma}} \frac{x(-1)^{j} (\theta + \alpha (r + j) + 1)}{(\theta + \alpha (r + j))^{2}} \right) \tag{3.118}
$$
\[
\frac{\partial}{\partial \alpha} \left[ \frac{(\theta + \alpha (r + j) + 1)}{(\theta + \alpha (r + j))^2} \right] = \frac{(\theta + \alpha (r + j))^2 (r + j) - (\theta + \alpha (r + j) + 1)}{(\theta + \alpha (r + j))^2} \times 2(\theta + \alpha (r + j))(r + j)
\]

\[
\frac{\partial}{\partial \alpha} \left[ \frac{(\theta + \alpha (r + j) + 1)}{(\theta + \alpha (r + j))^2} \right] = \frac{(r + j)(\theta + \alpha (r + j) + 2)}{(\theta + \alpha (r + j))^3}
\]

(3.119)

Substituting equation (3.119) in (3.118) we obtained \( \frac{\partial B}{\partial \alpha} \) as

\[
\frac{\partial B}{\partial \alpha} = (1 - 1) \left( 1 - \omega \right) \frac{\theta^2}{(\theta + 1)} \left( x + r - 1 \right) \sum_{j=0}^{x} \left( \begin{array}{c} x \\ j \end{array} \right) (-1)^j \left[ \frac{(r + j)(\theta + \alpha (r + j) + 2)}{(\theta + \alpha (r + j))^3} \right]
\]

(3.120)

Inserting equations (3.120) and (3.117) in equation (3.114) we obtained

\[
\frac{\partial l(\omega, r, \alpha, \theta)}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{LA + (1 - L)B} \left[ -1 \left( 1 - \omega \right) \theta^2 \frac{x (\theta + r \alpha + 2)}{(\theta + r \alpha)^3} + (1 - L) \left( 1 - \omega \right) \frac{\theta^2}{(\theta + 1)} \right] \times \left( x + r - 1 \right) \sum_{j=0}^{x} \left( \begin{array}{c} x \\ j \end{array} \right) (-1)^j \left[ \frac{(r + j)(\theta + \alpha (r + j) + 2)}{(\theta + \alpha (r + j))^3} \right]
\]

(3.121)

We obtained the maximum likelihood estimators \( \hat{\omega}, \hat{r}, \hat{\alpha}, \hat{\theta} \) of the ZINB-S distribution using R software via Newton-Raphson method, since obtaining the MLEs by equating the normal equations given in (3.103), (3.109), (3.113) and (3.121) to zero is little difficult due to the non-existence of closed form solutions. The parameter estimates are provided in table 3.3.

**Application of ZINB-S Distribution**

For demonstrating the performance of the ZINB-S distribution for modeling the over dispersed count data with excess number of zero counts, we considered a real data set from the paper of Zamani and Ismail (2010), which provides information on 9,461 automobile insurance policies. The data shows the number of accidents per each policy. The proposed ZINB-S distribution is compared with Poisson, NB and NB-S distributions. The observed and expected frequencies of the above mentioned distributions are given in Table 3.3. The
performance of these distributions are evaluated by means of chi-square statistic, log-
likelihood and \( p \)-values. Table 3.3 shows that NB-S and ZINB-S distribution provide better
performance compared to traditional distributions. Moreover, ZINB-S distribution performs
better than the NB-S distribution.

Table 3.3. Observed and expected frequencies of the number of accidents under the policy
using Poisson, NB, NB-S and ZINB-S distributions

<table>
<thead>
<tr>
<th>Number of accidents</th>
<th>Number of policies</th>
<th>Expected value by fitting distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Poisson</td>
<td>NB</td>
</tr>
<tr>
<td>0</td>
<td>7638.3</td>
<td>7843.3</td>
</tr>
<tr>
<td>1</td>
<td>1634.6</td>
<td>1290.2</td>
</tr>
<tr>
<td>2</td>
<td>174.9</td>
<td>257.7</td>
</tr>
<tr>
<td>3</td>
<td>12.5</td>
<td>54.5</td>
</tr>
<tr>
<td>4</td>
<td>0.7</td>
<td>11.8</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>2.6</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>0.6</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>8+</td>
<td>0</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Parameter estimates

\( \hat{\lambda} = 0.214 \)
\( \hat{\rho} = 0.70 \)
\( \hat{\lambda} = 2.0180 \)
\( \hat{\rho} = 1.4210 \)
\( \hat{\alpha} = 0.0100 \)
\( \hat{\theta} = 0.765 \)
\( \hat{\alpha} = 0.8552 \)
\( \hat{\omega} = 0.1881 \)
\( \hat{\omega} = 0.4450 \)

Chi-squares

<table>
<thead>
<tr>
<th></th>
<th>Poisson</th>
<th>NB</th>
<th>NB-S</th>
<th>ZINB-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson</td>
<td>293.8</td>
<td>8.65</td>
<td>5.8976</td>
<td>0.7078</td>
</tr>
<tr>
<td>NB</td>
<td>&lt;0.01</td>
<td>0.07</td>
<td>0.32</td>
<td>0.8714</td>
</tr>
<tr>
<td>NB-S</td>
<td>-5490.78</td>
<td>-5348.00</td>
<td>-5344.00</td>
<td>-5330.87</td>
</tr>
</tbody>
</table>

3.3 Multipoint Inflated Mixed Models

Many zero inflated, mixed and zero inflated mixed distributions are available in the
literature. Researchers tried to continue alternatives for over dispersed and inflated count
data. Most of the studies discussed only about the zero inflated count modeling. But, in
certain applications the count data may be inflated at a non-zero point or more than two
points. Thus some nonzero inflated models and double point inflated models are also
proposed in the area of count modeling.
3.3.1 Structure of Single and Multipoint Inflated Models

In some applications, the count data shows inflation of observations necessarily at the point zero but also any point in its support. In this situation, the single point inflated models can be used for modeling this type of data. The probability mass function of the single point inflated count model can be written in the following form

\[
p(x / \omega_i, \theta) = \begin{cases} 
\omega_i + (1 - \omega_i)p(i / \theta) & \text{when } x = i \\
(1 - \omega_i)p(x / \theta) & \text{when } x = 0,1,2,\ldots, x \neq i 
\end{cases} 
\]

where \( i \) may be any non-negative integer and \( p(x / \theta) \) represents the PMF of the underlying distribution with mixing parameter \( \omega_i \) which ranges between

\[
-\frac{p(i / \theta)}{1-p(i / \theta)} < \omega_i < 1
\]

Sometimes the inflation occur more than a single point. In this circumstance, the probability distribution of the inflated model can be written in the following form.

\[
p(x | \omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_k}, \theta) = \begin{cases} 
\omega_o + \left(1 - \sum_{j=0}^{k} \omega_j\right)p(0 | \theta); & \text{when } x = 0 \\
\omega_i + \left(1 - \sum_{j=1}^{k} \omega_j\right)p(1 | \theta); & \text{when } x = 1 \\
\omega_2 + \left(1 - \sum_{j=2}^{k} \omega_j\right)p(2 | \theta); & \text{when } x = 2 \\
& \vdots \\
\omega_k + \left(1 - \sum_{j=k}^{k} \omega_j\right)p(k | \theta); & \text{when } x = k \\
\left(1 - \sum_{j=0}^{k} \omega_j\right)p(x | \theta); & \text{when } x = k + 1, k + 2, \ldots
\end{cases}
\]

where \( i \) may be any non-negative integer and \( p(x / \theta) \) represents the PMF of the underlying probability distribution.
3.3.2 Multipoint Inflated-Negative Binomial-Lindley Distribution

In this section, we provided two different inflated models: the first model is proposed for modeling the count data where inflation occurs at single point in its support and in the second model inflation occurs at two or more points in its support.

Single Point Inflated Negative Binomial-Lindley Distribution

We proposed a single point inflated negative binomial mixture distribution named as single point inflated negative binomial-Lindley (SPINB-L) distribution for modeling the count data where inflation occurs at a single point in its support.

The PMF is defined as follows

\[
p(x / \omega_i, \theta) = \begin{cases} \omega_i + (1 - \omega_i) p(x | r, \theta) & \text{when } x = i \\ (1 - \omega_i) p(x | r, \theta) & \text{when } x = 0, 1, 2, \ldots; x \neq i \end{cases}
\]

(3.124)

where

\[
p(x | r, \theta) = \sum_{r=0}^{\infty} \frac{x^r}{r!} \left(1 + \frac{\theta}{\theta + 1}\right)^{r+1} \frac{\theta^2}{(\theta + 1)^2} \; ; \; x = 0, 1, 2, \ldots; r, \theta > 0
\]

Mean and Variance of SPINB-L Distribution

The mean and variance of the SPI-NBL distribution are given by

\[
E(X) = i \omega_i + (1 - \omega_i) \sum_{r=0}^{\infty} \frac{x^r}{r!} \left(1 + \frac{\theta}{\theta + 1}\right)^{r+1} \frac{\theta^2}{(\theta + 1)^2} \left(\frac{\theta}{\theta + 1}\right)^2 - 1
\]

(3.125)

\[
E(X^2) = i^2 \omega_i + (1 - \omega_i) \sum_{r=0}^{\infty} \frac{x^r}{r!} \left(1 + \frac{\theta}{\theta + 1}\right)^{r+1} \frac{\theta^2}{(\theta + 1)^2} \left(\frac{\theta}{\theta + 1}\right)^3 - 1
\]
\[
E(X^2) = i^2 \omega_i + (1 - \omega_i) \left[ (r + r^2) \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} - \left( r + 2r^2 \right) \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} + r^2 \right]
\] (3.126)

\[
V(X) = i^2 \omega_i + (1 - \omega_i) \left[ (r + r^2) \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} - \left( r + 2r^2 \right) \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} + r^2 \right]
- \left\{ i \omega_i + (1 - \omega_i) r \frac{\theta^2}{\theta + 1} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right] \right\}^2
\]

\[
= i^2 \omega_i + (1 - \omega_i) r \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} + (1 - \omega_i) r^2 \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} - (1 - \omega_i) r \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} + (1 - \omega_i) 2 r^2 \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} + (1 - \omega_i) r^2 - i^2 \omega_i^2
\]

\[
- 2 i \omega_i (1 - \omega_i) r \frac{\theta^2}{(\theta + 1)} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right] - (1 - \omega_i) r^2 \frac{\theta^4}{(\theta + 1)^2} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right]^2
\]

\[
= i^2 \omega_i - i^2 \omega_i^2 + (1 - \omega_i) r \frac{\theta^2}{(\theta + 1)} \left[ \frac{\theta - 1}{(\theta - 2)^2} + \frac{r.(\theta - 1)}{(\theta - 2)^2} + \frac{\theta}{(\theta - 1)^2} + \frac{2 r \theta}{(\theta - 1)^2} - 2 i \omega_i \left\{ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right\} \right]
+ (1 - \omega_i) r^2 - (1 - \omega_i)^2 r^2 \frac{\theta^4}{(\theta + 1)^2} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right]^2
\]

\[
= i^2 \omega_i - i^2 \omega_i^2 + (1 - \omega_i) r \frac{\theta^2}{(\theta + 1)} \left[ \frac{\theta - 1}{(\theta - 2)^2} (1 + r) + \frac{\theta}{(\theta - 1)^2} (1 + 2r) - 2 i \omega_i \left\{ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right\} \right]
+ (1 - \omega_i) r^2 \left[ 1 - (1 - \omega_i) r^2 \frac{\theta^4}{(\theta + 1)^2} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right]^2 \right]
\]

\[
= i^2 \omega_i - i^2 \omega_i^2 + (1 - \omega_i) r^2 + r \frac{\theta^2}{\theta + 1} \left[ \frac{(1 - \omega_i)(\theta - 1)(1 + r)}{(\theta - 2)^2} + (1 - \omega_i) \frac{\theta}{(\theta - 1)^2} (1 + 2r) \right]
+ (1 - \omega_i) r \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right\] \]

\[
= i^2 \omega_i - i^2 \omega_i^2 + (1 - \omega_i) r^2 + r \frac{\theta^2}{\theta + 1} \left[ \frac{(\theta + 1)(1 + r)}{(\theta - 2)^2} + \frac{\theta}{(\theta - 1)^2} (1 + 2r) \right]
+ (1 - \omega_i) r \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right\] \]

\[
= \left[ \frac{(1 - \omega_i)(\theta - 1)(1 + r)}{(\theta - 2)^2} + (1 - \omega_i) \frac{\theta}{(\theta - 1)^2} (1 + 2r) \right]
+ (1 - \omega_i) r \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right\] \]

\[
= \left[ \frac{(\theta + 1)(1 + r)}{(\theta - 2)^2} + \frac{\theta}{(\theta - 1)^2} (1 + 2r) \right]
+ (1 - \omega_i) r \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right\] \]
Parameter Estimation of SPINB-L Distribution

Let \( x_1, x_2, \ldots, x_n \) be \( n \) independent random observations from the proposed SPI-NBL distribution, then the likelihood function of the vector of parameters \( \Theta = (\omega, r, \theta) \) is of the form

\[
L(\Theta) = \prod_{h=1}^{n} \left[ (\omega_h + (1 - \omega_h) A) I_k \right] \left[ (1 - \omega_h) C \right] \text{ for } k = 0, 1, 2, \ldots; k \neq i
\]

where

\[
I_k(x) = \begin{cases} 
1; & \text{if } x = k \\
0; & \text{otherwise}
\end{cases}
\]

And

\[
A = \sum_{i=0}^{x} \binom{i}{l} (-1)^l \frac{\theta^2 \theta + r + l + 1}{\theta + 1 (\theta + r + l)^2}
\]

\[
C = \sum_{x=0}^{\infty} \binom{x}{l} (-1)^l \frac{\theta^2 \theta + r + l + 1}{\theta + 1 (\theta + r + l)^2}
\]

The log likelihood function can be written as

\[
l(\Theta) = \log L(\Theta) = \sum_{h=1}^{n} I_{\{i\}} \log [(\omega_h + (1 - \omega_h) A] + \sum_{h=1}^{n} I_{\{k\}} \log [(1 - \omega_h) C]
\]

To obtain the parameters using maximum likelihood estimation method, we differentiate the log likelihood function of the SPINB-L with respect to the parameters in the parameter space \( \Theta = (\omega, r, \theta) \). The normal equations are given below.

\[
\frac{\partial}{\partial \theta} l(\Theta) = \sum_{h=1}^{n} I_{\{i\}} \frac{1}{\omega_h + (1 - \omega_h) A} \left( 1 - \omega_h \right) \frac{\partial}{\partial \theta} (A) + \sum_{h=1}^{n} I_{\{k\}} \frac{1}{(1 - \omega_h) C} \left( 1 - \omega_h \right) \frac{\partial}{\partial \theta} (C)
\]
Consider

\[
\frac{\partial}{\partial \theta}(A) = \frac{\partial}{\partial \theta} \left[ \sum_{i=0}^{r+i-1} \binom{i}{l} (-1)^l \frac{\theta^2}{(\theta+1)(\theta+r+l)^2} \right]
\]

\[
= \binom{r+i-1}{i} \sum_{l=0}^{r} \binom{i}{l} (-1)^l \left[ \frac{(\theta+1)(\theta+r+l)^2}{(\theta+1)(\theta+r+l+1)+\theta^2} \right]
\]

\[
- \left( \frac{-\theta^2(\theta+r+l+1)(\theta+1)(\theta+r+l)+(\theta+r+l)^2}{(\theta+1)(\theta+r+l)^2} \right)
\]

\[
= \binom{r+i-1}{i} \sum_{l=0}^{r} \binom{i}{l} (-1)^l \left[ \theta(\theta+1)(\theta+r+l)(3\theta+2r+2l+2) \right]
\]

\[
- \left( \frac{-\theta(\theta+r+l+1)(3\theta+r+l+2)}{(\theta+1)^2(\theta+r+l)^3} \right)
\]

(3.131)

Consider

\[
\frac{\partial}{\partial \theta}(C) = \binom{r+x-1}{x} \sum_{l=0}^{r} \binom{x}{l} (-1)^l \left[ \theta(\theta+1)(\theta+r+l)(3\theta+2r+2l+2) \right]
\]

\[
- \left( \frac{-\theta(\theta+r+l+1)(3\theta+r+l+2)}{(\theta+1)^2(\theta+r+l)^3} \right)
\]

(3.132)

Substituting equations (3.131) and (3.132) in (3.130) we obtained

\[
\frac{\partial}{\partial \theta} l(\Theta) = \sum_{h=1}^{n} \frac{l_{(i)}}{\omega_i + (1-\omega_i)A} \left( 1 - \omega_i \right) \binom{r+i-1}{i} \sum_{l=0}^{r} \binom{i}{l} (-1)^l \left[ \frac{\theta(\theta+1)(\theta+r+l)(3\theta+2r+2l+2)}{(\theta+1)^2(\theta+r+l)^3} \right]
\]

\[
+ \sum_{h=1}^{n} \frac{l_{(k)}}{C} \binom{r+x-1}{x} \sum_{l=0}^{r} \binom{x}{l} (-1)^l \left[ \frac{\theta(\theta+1)(\theta+r+l)(3\theta+2r+2l+2)}{(\theta+1)^2(\theta+r+l)^3} \right]
\]

(3.133)

similarly we derived \( \frac{\partial}{\partial r} l(\Theta) \) as

\[
\frac{\partial}{\partial r} l(\Theta) = \sum_{h=1}^{n} \frac{l_{(i)}}{\omega_i + (1-\omega_i)A} \left( 1 - \omega_i \right) \frac{\partial}{\partial r} (A) + \sum_{h=1}^{n} \frac{l_{(k)}}{C} \frac{\partial}{\partial r} (C)
\]

(3.134)
Consider,

$$\frac{\partial}{\partial r} \left( A \right) = \frac{\partial}{\partial r} \left[ \binom{r+i-1}{i} \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{\theta^2}{\theta+1} \frac{\theta+r+l+1}{(\theta+r+l)^2} \right]$$

$$= \frac{\theta^2}{\theta+1} \left\{ \binom{r+i-1}{i} \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{\partial}{\partial r} \left[ \frac{\theta+r+l+1}{(\theta+r+l)^2} \right] \right\}$$

$$+ \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{\theta+r+l+1}{\theta+r+l} \frac{\partial}{\partial r} \left[ \frac{\Gamma(i+1) \Gamma(r+l)}{(\theta+r+l)^2} \right]$$

$$= \frac{\theta^2}{\theta+1} \left\{ \frac{\Gamma(r+i)}{\Gamma(i+1) \Gamma(r)} \sum_{l=0}^{i} \binom{i}{l} (-1)^l \left[ \frac{(\theta+r+l)^2-(\theta+r+l+1) \times 2(\theta+r+1)}{(\theta+r+l)^4} \right] \right\}$$

$$+ \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{\theta+r+l+1}{\theta+r+l} \frac{1}{\Gamma(i+1)} \left\{ \frac{\Gamma r \Gamma'(r+i)-\Gamma(r+i) \Gamma'(r)}{(\Gamma r)^2} \right\}$$

$$\frac{\partial}{\partial r} \left( A \right) = \frac{\theta^2}{\theta+1} \left\{ \frac{\Gamma(r+i) \Gamma(l)}{\Gamma(i+1) \Gamma(r)} \sum_{l=0}^{i} \binom{i}{l} (-1)^l \left[ \frac{(\theta+r+l)-2(\theta+r+1)}{(\theta+r+l)^3} \right] \right\}$$

$$+ \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{\theta+r+l+1}{\theta+r+l} \frac{1}{\Gamma(i+1)} \left\{ \frac{\Gamma r \Gamma'(r+i)-\Gamma(r+i) \Gamma'(r)}{(\Gamma r)^2} \right\}$$

(3.135)

$$\frac{\partial}{\partial r} \left( C \right) = \frac{\theta^2}{\theta+1} \left\{ \frac{\Gamma(r+x) \Gamma(x)}{\Gamma(x+1) \Gamma(r)} \sum_{l=0}^{x} \binom{x}{l} (-1)^l \left[ \frac{(\theta+r+l)-2(\theta+r+1)}{(\theta+r+l)^3} \right] + \right\}$$

$$+ \sum_{l=0}^{x} \binom{x}{l} (-1)^l \frac{\theta+r+l+1}{\theta+r+l} \frac{1}{\Gamma(x+1)} \left\{ \frac{\Gamma r \Gamma'(r+x)-\Gamma(r+x) \Gamma'(r)}{(\Gamma r)^2} \right\}$$

(3.136)

Substituting (3.135) and (3.136) in (3.134) we get
These equations can be solved numerically by using R software.

**Two point Inflated Negative Binomial-Lindley distribution**

When excess number of counts occurs at two support points, we can consider a two point inflated negative binomial Lindley distribution. The PMF of TPINB-L distribution is defined as follows

\[
p(x | \theta, r, \omega_i, \omega_j) = \begin{cases} 
   \omega_i + (1 - \omega_i - \omega_j)p(i/r, \theta); & \text{when } x = i \\
   \omega_j + (1 - \omega_i - \omega_j)p(i/r, \theta); & \text{when } x = j \\
   (1 - \omega_i - \omega_j)p(x/r, \theta); & \text{when } x = 0, 1, 2, \ldots; \ x \neq i, j
\end{cases}
\]  

(3.139)
where

\[
p(x | r, \theta) = \left(\frac{r+x-1}{x}\right)^{\infty} \sum_{l=0}^{\infty} \left(\frac{x}{l}\right)^{(-1)^l} \frac{\theta^2}{\theta+1} \frac{(\theta+r+l+1)}{(\theta+r+l)^2}, \quad x = 0, 1, 2, 3, \ldots; r, \theta > 0
\]

Mean and Variance of TPI-NBL Distribution

\[
E(X) = i \{ \omega_i + (1 - \omega_i - \omega_j) A \} + j \{ \omega_j + (1 - \omega_i - \omega_j) B \} + \sum_{x=0; x \neq i, j}^{\infty} x \{ (1 - \omega_i - \omega_j) C \}
\]

where

\[
A = \left(\frac{r+i-1}{i}\right)^{\infty} \sum_{l=0}^{\infty} \left(\frac{x}{l}\right)^{(-1)^l} \frac{\theta^2}{\theta+1} \frac{\theta+r+l+1}{(\theta+r+l)^2}
\]

\[
B = \left(\frac{r+j-1}{j}\right)^{\infty} \sum_{l=0}^{\infty} \left(\frac{x}{l}\right)^{(-1)^l} \frac{\theta^2}{\theta+1} \frac{\theta+r+l+1}{(\theta+r+l)^2}
\]

\[
C = \left(\frac{r+x-1}{x}\right)^{\infty} \sum_{l=0}^{\infty} \left(\frac{x}{l}\right)^{(-1)^l} \frac{\theta^2}{\theta+1} \frac{\theta+r+l+1}{(\theta+r+l)^2}
\]

Solving this, we get

\[
E(X) = i \omega_i + j \omega_j + (1 - \omega_i - \omega_j) r \frac{\theta^2}{\theta+1} \left[ \frac{\theta^3}{(\theta+1)(\theta-1)^2} - 1 \right]
\]

and

\[
E(X^2) = \sum_{x=0}^{\infty} x^2 p(x)
\]

\[
= \sum_{x=0}^{\infty} x^2 \{ \omega_i + (1 - \omega_i - \omega_j) A \} + \sum_{x=0}^{\infty} x^2 \{ \omega_j + (1 - \omega_i - \omega_j) B \} + \sum_{x=0}^{\infty} x^2 \{ (1 - \omega_i - \omega_j) C \}
\]

\[
E(X^2) = i^2 \{ \omega_i + (1 - \omega_i - \omega_j) A \} + j^2 \{ \omega_j + (1 - \omega_i - \omega_j) B \} + \sum_{x=0; x \neq i, j}^{\infty} x^2 \{ (1 - \omega_i - \omega_j) C \}
\]

Solving this, we get

\[
E(X^2) = i^2 \omega_i + j^2 \omega_j + (1 - \omega_i - \omega_j) \left[ \frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2} - \frac{\theta^3}{(\theta+1)(\theta-1)^2} + r^2 \right]
\]
\[
V(X) = i^2 \omega_i + j^2 \omega_j + (1 - \omega_i - \omega_j) \left[ (r + r^2) \frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2} - (r + 2r^2) \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} + 2r^2 \right] \]

(3.142)

\[
- \left\{ i \omega_i + j \omega_j + (1 - \omega_i - \omega_j) r \frac{\theta^2}{\theta + 1} \left[ \frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1 \right] \right\}
\]

**Parameter Estimation of the TPINB-L Distribution**

Let \( x_1, x_2, \ldots, x_n \) be \( n \) independent random observations from the proposed TPINB-L distribution. Then the likelihood function of the vector of parameters \( \Theta = (\omega_i, \omega_j, r, \theta) \) is of the form

\[
L(\Theta) = \prod_{k=1}^{n} \left[ \omega_i + (1 - \omega_i - \omega_j) A^{(i)} [\omega_j + (1 - \omega_i - \omega_j) B^{(j)} x (1 - \omega_i - \omega_j) C^{(i,j)}] \right]_{k=0} \text{, } k = 0, 1, 2, \ldots; k \neq i, j
\]

(3.143)

where \( I_k(x) = \begin{cases} 
1; & \text{if } x = k \\
0; & \text{otherwise} 
\end{cases} \)

where

\[
A = \left( r + i - 1 \right) \left( \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{\theta^2}{\theta + 1 (\theta + r + l)^2} \right)
\]

\[
B = \left( r + j - 1 \right) \left( \sum_{l=0}^{j} \binom{j}{l} (-1)^l \frac{\theta^2}{\theta + 1 (\theta + r + l)^2} \right)
\]

\[
C = \left( r + x - 1 \right) \left( \sum_{l=0}^{x} \binom{x}{l} (-1)^l \frac{\theta^2}{\theta + 1 (\theta + r + l)^2} \right)
\]
Then, the log likelihood function can be written as

\[
I(\Theta) = \log L(\Theta) = \sum_{i=1}^{n} \log \left( \left[ \omega_i + (1 - \omega_i - \omega_j)A \right] \left[ \omega_j + (1 - \omega_i - \omega_j)B \right] \left[ (1 - \omega_i - \omega_j)C \right] \right)
\]

\[
= \sum_{h=1}^{n} I_{(i)} \log \left[ \omega_i + (1 - \omega_i - \omega_j)A \right] + \sum_{h=1}^{n} I_{(j)} \log \left[ \omega_j + (1 - \omega_i - \omega_j)B \right] + \sum_{h=1}^{n} I_{(k)} \log \left[ (1 - \omega_i - \omega_j)C \right]
\]

(3.144)

To obtain the MLE, the partial derivatives w.r.t. \( \theta, r, \omega_i, \omega_j \) can be derived as follows

\[
\frac{\partial}{\partial r} I(\Theta) = \sum_{h=1}^{n} I_{(i)} \left[ \frac{1}{\omega_i + (1 - \omega_i - \omega_j)A} (1 - \omega_i - \omega_j) \frac{\partial}{\partial r} (A) \right] + \sum_{h=1}^{n} I_{(j)} \left[ \frac{1}{\omega_j + (1 - \omega_i - \omega_j)B} (1 - \omega_i - \omega_j) \frac{\partial}{\partial r} (B) \right] + \sum_{h=1}^{n} I_{(k)} \frac{1}{C} \frac{\partial}{\partial r} (C)
\]

(3.145)

where

\[
A_i = \frac{\partial}{\partial r} (A) = \frac{\theta^2}{\theta + 1} \left\{ \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{(\theta + r + l)}{(\theta + r + l)^3} + \sum_{l=0}^{i} \binom{i}{l} (-1)^l \frac{(\theta + r + l + 1)^2}{(\theta + r + l)^2} \right\}
\]
\[ B_i = \frac{\partial}{\partial r} (B) = \frac{\theta^2}{(\theta+1)} \left( r + j - 1 \right) \sum_{l=0}^{j} \binom{j}{l} (-1)^l \left[ \frac{(\theta + r + l)}{(\theta + r + l)^3} \right] + \sum_{l=0}^{j} \frac{1}{\Gamma(j+1)} \left( \Gamma'(r + l + 1) \right) \left( (\theta + r + l + 1) \right) \left( (\theta + r + l) \right)^2 \] 

\[ C_i = \frac{\partial}{\partial r} (C) = \frac{\theta^2}{(\theta+1)} \left( r + x - 1 \right) \sum_{l=0}^{x} \binom{x}{l} (-1)^l \left[ \frac{(\theta + r + l)}{(\theta + r + l)^3} \right] + \sum_{l=0}^{x} \frac{1}{\Gamma(x+1)} \left( \Gamma'(x + 1) \Gamma r \right) \left( (\theta + r + x + 1) \right) \left( (\theta + r + x) \right)^2 \] 

Similarly

\[ \frac{\partial}{\partial \theta} I(\Theta) = \sum_{h=1}^{n} I_{(i)} \left[ \frac{1}{\omega_i + (1 - \omega_i - \omega_j)A} \right] \left( 1 - \omega_i - \omega_j \right) \frac{\partial}{\partial \theta} (A) + \sum_{h=1}^{n} I_{(j)} \left[ \frac{1}{\omega_j + (1 - \omega_i - \omega_j)B} \right] \left( 1 - \omega_i - \omega_j \right) \frac{\partial}{\partial \theta} (B) + \sum_{h=1}^{n} I_{(k)} \left[ \frac{1}{C} \right] \frac{\partial}{\partial \theta} (C) \]

\[ = \sum_{h=1}^{n} I_{(i)} \left[ \frac{1}{\omega_i + (1 - \omega_i - \omega_j)A} \right] \left( 1 - \omega_i - \omega_j \right) \times A_2 + \sum_{h=1}^{n} I_{(j)} \left[ \frac{1}{\omega_j + (1 - \omega_i - \omega_j)B} \right] \left( 1 - \omega_i - \omega_j \right) \times B_2 + \sum_{h=1}^{n} I_{(k)} \left[ \frac{1}{C} \right] \times C_2 \] 

(3.146)
where

\[ A_2 = \frac{\partial}{\partial \theta} (A) = \binom{r + i - 1}{i} \sum_{l=0}^{i} \binom{i}{l} (-1)^l \begin{pmatrix} \theta \left( \theta + r + l \right) \left( \theta + 2r + 2l + 2 \right) \\ \theta \left( \theta + r + l + 1 \right) \left( \theta + r + l + 2 \right) \end{pmatrix} \left( \theta + 1 \right)^2 \left( \theta + r + l \right)^3 \] 

\[ B_2 = \frac{\partial}{\partial \theta} (B) = \binom{r + j - 1}{j} \sum_{l=0}^{j} \binom{j}{l} (-1)^l \begin{pmatrix} \theta \left( \theta + r + l \right) \left( \theta + 2r + 2l + 2 \right) \\ \theta \left( \theta + r + l + 1 \right) \left( \theta + r + l + 2 \right) \end{pmatrix} \left( \theta + 1 \right)^2 \left( \theta + r + l \right)^2 \] 

\[ C_2 = \frac{\partial}{\partial \theta} (C) = \binom{r + x - 1}{x} \sum_{l=0}^{x} \binom{x}{l} (-1)^l \begin{pmatrix} \theta \left( \theta + r + l \right) \left( \theta + 2r + 2l + 2 \right) \\ \theta \left( \theta + r + l + 1 \right) \left( \theta + r + l + 2 \right) \end{pmatrix} \left( \theta + 1 \right)^2 \left( \theta + r + l \right)^2 \] 

\[ \frac{\partial}{\partial \omega_i} l(\Theta) = \sum_{h=1}^{n} I_{(i)} \frac{1}{\omega_i + (1 - \omega_i - \omega_j)A} \left( 1 - A \right) - \sum_{h=1}^{n} I_{(j)} \frac{1}{\omega_j + (1 - \omega_i - \omega_j)B} \left( 1 - B \right) - \sum_{h=1}^{n} I_{(k)} \frac{1}{(1 - \omega_i - \omega_j)} \] 

\[ \frac{\partial l(\Theta)}{\partial \omega_j} = -\sum_{h=1}^{n} I_{(i)} \frac{1}{\omega_i + (1 - \omega_i - \omega_j)A} \times A + \sum_{h=1}^{n} I_{(j)} \frac{1}{\omega_j + (1 - \omega_i - \omega_j)B} \times (1 - B) - \sum_{h=1}^{n} I_{(k)} \frac{1}{(1 - \omega_i - \omega_j)} \] 

These equations can be solved using R software for approximating the maximum likelihood estimates.

**Application of SPINB-L and TPINB-L Distributions**

In this section, we also considered data set from Zamani and Ismail (2010), which shows excess number of zero counts and over dispersion. The data set shows the number of accidents per each policy. Here we model the data using single point inflated negative binomial-Lindley distribution and two points inflated negative binomial-Lindley distribution.

For SPINB-L distribution, we considered the data inflated only at the point 0. For TPINB-L, we considered the data inflated at both points 0 and 1 and the data were fitted to the
Poisson, NB, NB-L, SPINB-L and TPINB-L distribution. All the parameter estimates in this study are obtained using R software. The results are presented in Table 3.4.

Table 3.4 displays the observed frequencies and fitted frequencies of the number of accidents. The predicted values of the SPINB-L (at the point 0) and TPINB-L distribution (at the points 0 and 1) are more accurate than conventional Poisson and negative binomial distribution. The chi-square values, p values and log-likelihood corresponding to each distribution show the superiority of the proposed SPINB-L and TPINB-L distribution. Further among these two proposed models TPINB-L distribution performs better compared to SPINB-L distribution, since it predicts zero counts and one counts more accurately than SPINB-L distribution.

Table 3.4: Observed and Expected frequencies of the number of accidents per policy

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Number of Drivers</th>
<th>Fitting Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Poisson</td>
<td>NB</td>
</tr>
<tr>
<td>0</td>
<td>7840</td>
<td>7638.3</td>
</tr>
<tr>
<td>1</td>
<td>1317</td>
<td>1634.6</td>
</tr>
<tr>
<td>2</td>
<td>239</td>
<td>174.9</td>
</tr>
<tr>
<td>3</td>
<td>42</td>
<td>12.5</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>0.7</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>8+</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Parameter estimates

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\lambda} = 0.214$</th>
<th>$\hat{\phi} = 0.70$</th>
<th>$\hat{\gamma} = 4.63$</th>
<th>$\hat{\theta} = 23.55$</th>
<th>$\hat{\omega} = 0.443$</th>
<th>$\hat{\alpha} = 165$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\phi}$</td>
<td>0.765</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.765</td>
<td>0.765</td>
<td>0.765</td>
<td>0.765</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>$\hat{\omega}$</td>
<td>0.443</td>
<td>0.443</td>
<td>0.443</td>
<td>0.443</td>
<td>0.443</td>
<td>0.443</td>
</tr>
<tr>
<td>Chi-squares</td>
<td>138.34</td>
<td>7.277</td>
<td>6.366</td>
<td>3.22</td>
<td>1.1932</td>
<td></td>
</tr>
<tr>
<td>p value</td>
<td>&lt;0.01</td>
<td>0.026</td>
<td>0.095</td>
<td>0.359</td>
<td>0.94553</td>
<td></td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>-5490.78</td>
<td>-5348.00</td>
<td>-5344.70</td>
<td>-5342.5</td>
<td>-5342.0</td>
<td></td>
</tr>
</tbody>
</table>