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SOME BIORTHOGONAL POLYNOMIALS SUGGESTED
BY THE LAGUERRE POLYNOMIALS

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Joseph D. E. Konhauser discussed two polynomial sets
\{Y_\alpha(x; k)\} and \{Z_\alpha(x; k)\}, which are biorthogonal with respect
to the weight function \(x^\alpha e^{-x}\) over the interval \((0, \infty)\), where
\(\alpha > -1\) and \(k\) is a positive integer. The present paper
attempts at exploring certain novel approaches to these
biorthogonal polynomials in simple derivations of their
several interesting properties. Many of the results obtained
here are believed to be new; others were proven in the
literature by employing markedly different techniques.

1. Introduction. Konhauser ([10]; see also [9]) has considered
two classes of polynomials \(Y_\alpha(x; k)\) and \(Z_\alpha(x; k)\), where \(Y_\alpha(x; k)\) is a
polynomial in \(x\), while \(Z_\alpha(x; k)\) is a polynomial in \(x^\alpha\), \(\alpha > -1\) and
\(k \in \{1, 2, 3, \ldots\}\). For \(k = 1\), these polynomials reduce to the Laguerre
polynomials \(L_\alpha(x)\), and their special cases when \(k = 2\) were encoun-
tered earlier by Spencer and Fano [19] in certain calculations
involving the penetration of gamma rays through matter, and were
subsequently discussed by Preiser [16]. Furthermore, we have [10,
p. 303]

\[
\int_0^\infty x^\nu e^{-x} Y_\nu(x; k) Z_\nu(x; k) \, dx = \frac{\Gamma(k \nu + \alpha + 1)}{\nu!} \delta_{ij},
\quad \forall i, j \in \{0, 1, 2, \ldots\},
\]

(1.1)

which exhibits the fact that the polynomial sets \(\{Y_\nu(x; k)\}\) and
\(\{Z_\nu(x; k)\}\) are biorthogonal with respect to the weight function \(x^\alpha e^{-x}\)
over the interval \((0, \infty)\), it being understood that \(\alpha > -1\), \(k\) is a
positive integer, and \(\delta_{ij}\) is the Kronecker delta.

An explicit expression for the polynomials \(Z_\alpha(x; k)\) was given by
Konhauser in the form [10, p. 304, Eq. (5)]

\[
Z_\alpha(x; k) = \frac{\Gamma(k \alpha + \alpha + 1)}{\alpha!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{k \nu}}{\Gamma(k \nu)}.
\]

(1.2)

As for the polynomials \(Y_\alpha(x; k)\), Carlitz [3] subsequently showed that
[op. cit., p. 427, Eq. (9)]

\[
Y_\alpha(x; k) = \frac{1}{\alpha!} \sum_{n=0}^{\infty} \frac{x^{k \nu}}{\alpha!} \sum_{\nu=0}^{\infty} (-1)^n \frac{\Gamma(n + \alpha + 1)}{\nu!} \frac{(j + \alpha + 1)}{k}.
\]

(1.3)

where \((\lambda)_{\nu}\) is the Pochhammer symbol defined by
\[ (\lambda)_{n} = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \]

(1.4) \[ \begin{cases} 1, \text{ if } n = 0, \lambda \neq 0, \\ (\lambda(\lambda + 1) \cdots (\lambda + n - 1), \forall n \in \{1, 2, 3, \cdots\}. \end{cases} \]

The object of the present paper is to show that several interesting properties of the biorthogonal polynomials \( Y_\alpha^n(x; k) \) and \( Z_\alpha^n(x; k) \) follow fairly readily from relatively more familiar results by applying the explicit expressions (1.2) and (1.3). A number of properties thus derived are believed to be new, and others were proven in the literature by employing markedly different techniques.

2. The biorthogonal polynomials \( Y_\alpha^n(x; k) \). We begin by recalling the polynomials \( G_\alpha^n(x, r, p, k) \) which were introduced by Srivastava and Singhal [24] in an attempt to provide an elegant unification of the various known generalizations of the classical Hermite and Laguerre polynomials. These polynomials are defined by the generalized Rodrigues formula [op. cit., p. 75, Eq. (1.3)]

\[ G_\alpha^n(x, r, p, k) = \frac{x^{kn}}{n!} \exp(\mu x^2) (x^{k+1} D_x)^n \{x^r \exp(-\mu x^2)\}, \]

(2.1) where \( D_x = d/dx \), and the parameters \( \alpha, k, p \) and \( r \) are unrestricted, in general. We also have the explicit polynomial expression [24, p. 77, Eq. (2.1)]

\[ G_\alpha^n(x, r, p, k) = \frac{k^n}{n!} \sum_{i=0}^{n} \binom{p x^2}{i} \sum_{j=0}^{i} \binom{r j + \alpha}{j} \frac{(-1)^j}{k^j}. \]

(2.2) On comparing (2.2) with Carlitz's result (1.3), we at once get the known relationship [23, p. 315, Eq. (83)]

\[ Y_\alpha^n(x; k) = k^{-n} G_\alpha^{n+1}(x, 1, 1, k), \alpha > -1, \quad k = 1, 2, 3, \cdots, \]

(2.3) which evidently enables us to derive the following properties of the Konhauser biorthogonal polynomials \( Y_\alpha^n(x; k) \) by suitably specializing those of the Srivastava-Singhal polynomials \( G_\alpha^n(x, r, p, k) \).

I. Rodrigues' formula. In (2.1) we set \( p = r = 1 \), replace \( \alpha \) by \( \alpha + 1 \), and appeal to the relationship (2.3). We thus obtain

\[ Y_\alpha^n(x; k) = \frac{x^{kn+1} e^{x^2} (x^{k+1} D_x)^n \{x e^x\} \exp(-\mu x^2)}{k^n n!}, \]

(2.4) where, by definition, \( \alpha > -1 \) and \( k \) is now restricted to be a positive integer.

Alternatively, we may recall that [15, p. 802, Eq. (2.6)]
\[
(2.5) \quad Y_n^\alpha(x; k) = \frac{x^{k+n-\alpha-1}e^{x}}{\Gamma(n)} \left[ e^{-x}e^{-(s^{(n+1)}/k)} \right]_{x=s^k},
\]
which indeed is equivalent to
\[
(2.6) \quad Y_n^\alpha(x; k) = \frac{x^{k+n-\alpha-1}e^{x}}{\Gamma(n)} \left[ e^{-x}e^{-(s^{(n+1)/k})} \right]_{x=s^k},
\]
since
\[
(2.7) \quad (x^k D_x)^n[g(x)] = x^{n+1}D_x^n[x^{-1}g(x)]
\]
for every non-negative integer \(n\).

Now we set \(s = x^k\) and \(s^k D_x \rightarrow k^{-}x^{k+1}D_x\) in (2.6), and the Rodrigues formula (2.4) follows at once.

Incidentally, the Rodrigues type representation (2.4) is due to Calvez et Gémin (2, p. A41, Eq. (1)); it is stated slightly differently in a recent paper by Patil and Thakare (12, p. 921, Eq. (1.2)).

II. Recurrence relations. In view of the relationship (2.3), the known results (24, p. 80, Eq. (4.3), (4.4), (4.5) and (4.6)) readily yield
\[
(2.8) \quad k(n + 1)Y_n^\alpha(x; k) = xD_x Y_n^\alpha(x; k) + (kn + \alpha - k + 1)Y_n^\alpha(x; k),
\]
\[
(2.9) \quad D_x Y_n^\alpha(x; k) = Y_n^\alpha(x; k) - Y_n^{\alpha+1}(x; k),
\]
\[
(2.10) \quad (\alpha - k + 1)Y_n^\alpha(x; k) = xY_n^{\alpha+1}(x; k) + (n + 1)kY_n^{\alpha+1}(x; k)
\]
and
\[
(2.11) \quad k(n + 1)Y_n^{\alpha+1}(x; k) = (kn + \alpha + 1)Y_n^\alpha(x; k) - xY_n^{\alpha+1}(x; k).
\]

The recurrence relation (2.8) was given earlier by Konhauser (10, p. 398, Eq. (16)), while (2.9), (2.10) and (2.11) are believed to be new. Notice, however, that by eliminating the term \(xY_n^{\alpha+1}(x; k)\) between (2.10) and (2.11) we obtain
\[
(2.12) \quad Y_n^{\alpha+1}(x; k) = Y_n^\alpha(x; k),
\]
which is equivalent to the familiar generalization (cf. [10], p. 311) of a well-known recurrence relation for the Laguerre polynomials [18, p. 203, Eq. (8)].

III. Operational formulas. Making use of the relationship (2.3), we can specialize the Srivastava-Singhal results (24, p. 85, Eq. (7.5) and (7.6)) to obtain the following operational formulas involving the biorthogonal polynomials \(Y_n^\alpha(x; k)\):
\[
(2.13) \quad \int_0^1 (\delta + \alpha + jk - x + 1) = k\alpha n! \sum_{j=0}^n \frac{(ka^x)^{-j}}{j!} Y_n^{-j}(x; k)(x^{k+1}D_x)^j.
\]
and

\begin{equation}
Y(x; k) = \frac{1}{k^n} \prod_{j=0}^{n-1} (j \alpha + j/k - x + 1),
\end{equation}

where $\delta = xDx$.

IV. Generating functions. From the known results [24, p. 78, Eq. (3.2); p. 79, Eq. (3.4) and (3.6)], due to Srivastava and Singhal [24], it readily follows on appealing to (2.3) that

\begin{equation}
\sum_{n=0}^{\infty} Y^a_n(x; k)t^n = (1 - t)^{-\lfloor x + 1 \rfloor/k} \exp(x[1 - (1 - t)^{-1/k}] ),
\end{equation}

\begin{equation}
\sum_{n=0}^{\infty} Y_n^{a-k}(x; k)t^n = (1 + t)^{\lfloor x - k + 1 \rfloor/k} \exp(x[1 - (1 + t)^{1/k}] ),
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} \binom{m + n}{n} Y^a_{m+n}(x; k)t^n = (1 - t)^{-m - \lfloor x + 1 \rfloor/k} \exp(x[1 - (1 - t)^{-1/k}] ) Y^a_n(x[1 - t]^{-1/k}; k),
\end{equation}

where $m$ is a non-negative integer.

Furthermore, by using the definition (2.1) and the aforementioned result [24, p. 79, Eq. (3.4)], it is not difficult to derive the generating function

\begin{equation}
\sum_{n=0}^{\infty} \binom{m + n}{n} G_n^{a, k}(x, r, p, k)t^n
\end{equation}

\begin{equation}
= (1 + kt)^{\lfloor x - k + 1 \rfloor/k} \exp((x[1 + (kt)^{1/k}] )
\end{equation}

\begin{equation}
G_n^{a}(x(1 + kt)^{-1}, r, p, k), \quad k \neq 0,
\end{equation}

which, for $p = r = 1$, yields a generalization of (2.16) in the form:

\begin{equation}
\sum_{n=0}^{\infty} \binom{m + n}{n} Y_n^{a-k}(x; k)t^n
\end{equation}

\begin{equation}
= (1 + t)^{\lfloor a - k + 1 \rfloor/k} \exp(x[1 + t]^{1/k}] )
\end{equation}

\begin{equation}
\times Y_n^{a}(x(1 + t)^{-1}; k), \quad \forall m \in \{0, 1, 2, \ldots\},
\end{equation}

where, by definition, $k$ is a positive integer.

The generating function (2.15) was derived earlier by Carlitz [3, p. 426, Eq. (8)], while (2.16), (2.17) and (2.19) are due to Calvez et Génin [2]. In fact, (2.15) and (2.17) were also given independently by Prabhakar [15, p. 801, Eq. (2.3); p. 803, Eq. (3.3)].

Incidentally, in view of the known generating function [24, p. 78, Eq. (3.2)] and Lagrange's expansion in the form [13, p. 146,
Problem 207:

\[
\frac{f(\xi)}{1 - t \phi'(\xi)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_f^{n}[f(\xi) \phi(\xi)^n] = 0
\]

where

\[
\zeta = t \phi(\xi), \quad \phi(0) \neq 0
\]

it is fairly easy to show that

\[
\sum_{n=0}^{\infty} G_n^{(\alpha + \beta x)}[x + ny]^{1/r} = \frac{(1 - u)^{\alpha x} \exp(px[1 - (1 - u)^{-r/\beta}])}{1 - k^{-1} u(1 - u)^{1/\beta - 1} r py^{-1} (1 - u)^{-r/\beta}}, \quad k \neq 0
\]

or, equivalently,

\[
\sum_{n=0}^{\infty} G_n^{(\alpha + \beta x)}[x + ny]^{1/r} = \frac{(1 + v)^{\alpha x} \exp(px[1 - (1 + v)^{-r/\beta}])}{1 - k^{-1} v[\beta - r py^{-1} (1 + v)^{-r/\beta}].} \quad k \neq 0
\]

where \(u\) and \(v\) are functions of \(t\) defined implicitly by

\[
u = kt(1 - u)^{-r/\beta} \exp(py^{-1} (1 - (1 - u)^{-r/\beta})\), \quad u(0) = 0
\]

and

\[
v = kt(1 + v)^{(r + 1)/k} \exp(py^{-1} (1 - (1 + v)^{-r/\beta})), \quad v(0) = 0.
\]

In their special cases when \(p = r = 1\), (2.22) and (2.23) obviously yield the following generating functions for the Konhauser polynomials \(Y_n(x; k)\):

\[
\sum_{n=0}^{\infty} Y_n^{(\alpha + \beta x)}(x + ny; k) t^n = \frac{(1 - \xi)^{-r(1 + 1)/k} \exp(x [1 - (1 - \xi)^{-r/\beta}])}{1 - k^{-1} \xi(1 - \xi)^{-1}[\beta - y(1 - \xi)^{-r/\beta}]},
\]

where \(\xi\) is a function of \(t\) defined implicitly by

\[
\xi = t(1 - \xi)^{-r/\beta} \exp(y[1 - (1 - \xi)^{-r/\beta}]), \quad \xi(0) = 0;
\]

\[
\sum_{n=0}^{\infty} Y_n^{(\alpha + \beta x)}(x + ny; k) t^n = \frac{(1 + \eta)^{r(1 + 1)/k} \exp(x [1 - (1 + \eta)^{-r/\beta}])}{1 - k^{-1} \eta[\beta - y(1 + \eta)^{-r/\beta}]},
\]

where \(\eta\) is a function of \(t\) given implicitly by

\[
\eta = t(1 + \eta)^{(r + 1)/k} \exp(y[1 - (1 + \eta)^{-r/\beta}]), \quad \eta(0) = 0.
\]

For \(y = 0\), the generating functions (2.26) and (2.28) are essentially equivalent to the Calvez-Gënin result [2, p. A41, Eq. (2)]. (Indeed, their reductions to (2.15) when \(\beta = y = 0\) and to (2.16) when
\( \beta = -k \) and \( y = 0 \) are immediate.) On the other hand, their special cases when \( k = 1 \), involving Laguerre polynomials, were given recently by Carlitz [4, p. 525, Eq. (5.2) and (5.5)].

From the Srivastava-Singhal result [24, p. 78, Eq. (3.2)] we further have

\[
(2.30) \quad (x^r D_x^m \exp(-px^r) G_{r+\alpha}^{\alpha}(x, r, p, k)) = (-rp)^m \exp(-px^r) G_{r+\alpha(m)}^{\alpha}(x, r, p, k), \quad m \geq 0,
\]

and

\[
(2.31) \quad G_{r+\alpha(m)}^{\alpha}(x^r + y^r) \frac{t^r}{r!} = \sum_{j=0}^{\infty} G_{j+\alpha(m)}^{\alpha}(x, r, p, k) G_{j+\alpha}^{\alpha}(y, r, p, k),
\]

which, for \( p = r = 1 \), yield the known results

\[
(2.32) \quad D_x^r [e^{Y_{r+\alpha(m)}^\alpha}(x; k)] = (-1)^m e^{Y_{r+\alpha(m)}^\alpha(x; k)}, \quad m \geq 0
\]

and

\[
(2.33) \quad Y_{r+\alpha(m)}^\alpha(x + y; k) = \sum_{j=0}^{\infty} Y_{j+\alpha(m)}^\alpha(x; k) Y_{j+\alpha}^\alpha(y; k),
\]

due to Génin et Calvez [8, p. A34, Eq. (6); p. A33, Eq. (2)]. [For (2.33) see also [15, p. 803, Eq. (3.2)].]

Applying (2.30) in conjunction with Taylor's theorem, we obtain yet another new generating function in the form:

\[
(2.34) \quad \sum_{n=0}^{\infty} G_{r+\alpha(n)}^{\alpha}(x, r, p, k) \frac{t^n}{n!} = e^{\frac{t^r}{r!}} G_{r}^{\alpha}(x; k), \quad m, n \geq 0,
\]

which, in view of the relationship (2.3), reduces at once to the Génin-Calvez result [8, p. A34, Eq. (7)]

\[
(2.35) \quad \sum_{n=0}^{\infty} Y_{r+\alpha(n)}^\alpha(x; k) \frac{t^n}{n!} = e^{\frac{t^r}{r!}} Y_{r}^\alpha(x; k), \quad m, n \geq 0.
\]

We conclude this part by recording the following special case of a known result given by Srivastava and Singhal [24, p. 84, Eq. (7.3)]:

\[
(2.36) \quad Y_{\alpha}^{\alpha}(x; k) = \sum_{j=0}^{\infty} \binom{\alpha + (\alpha - \beta)/k}{j} Y_{\alpha-j}^{\alpha}(x; k),
\]

which is due to Prabhakar [15, p. 802, Eq. (3.1)]; for \( k = 1 \), (2.36) yields a well-known property of the Laguerre polynomials [18, p. 209, Eq. (2)].

Incidentally, the well-known special case \( y = 0 \) of (2.28) [with \( \beta \) replaced trivially by \( \beta \)], and an erroneous version of the Génin-Calvez result (2.35), were rederived in a recent paper by B.K. Karande and K.R. Patil [Indian J. Pure Appl. Math. 12 (1981),
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222–225; especially see p. 224, Eq. (12), and p. 223, Eq. (6) without any reference to the relevant earlier papers [2], [7] and [8].

V. Mixed multilateral generating functions. The generating-function relationships (2.17) and (2.19) enable us to apply the results of Srivastava and Lavoie [23], and we are led rather immediately to the following interesting variations of a general bilateral generating function [op. cit., p. 319, Eq. (107)]:

\[
\sum_{m=0}^{\infty} Y_m^{a,k}(x; k) A_n(y, \ldots, y_N; z)t^n
\]

(2.37)

\[
= (1 - t)^{-(k - a + 1)/k} \exp \{x[1 - (1 - t)^{1/k}] \}
\]

\[
\times F[x(1 - t)^{-1/k}; y, \ldots, y_N; zt^{1/(1 - t)^q}],
\]

and

\[
\sum_{m=0}^{\infty} Y_m^{a,k}(x; k) A_n(y, \ldots, y_N; z)t^n
\]

(2.38)

\[
= (1 + t)^{-(a - k + 1)/k} \exp \{x[1 - (1 + t)^{-1/k}] \}
\]

\[
\times G[x(1 + t)^{1/k}; y, \ldots, y_N; zt^{-1/(1 + t)^q}],
\]

where

(2.39) \quad F[x; y, \ldots, y_N; z] = \sum_{m=0}^{\infty} c_n Y_m^{a,k}(x; k) A_n(y, \ldots, y_N) z^n,

(2.40) \quad G[x; y, \ldots, y_N; z] = \sum_{m=0}^{\infty} c_n Y_m^{a,k}(x; k) A_n(y, \ldots, y_N) z^n,

\[c_n \neq 0\] are arbitrary complex constants, \(m \geq 0\) and \(q \geq 1\) are integers, and, in terms of the non-vanishing functions \(A_n(y, \ldots, y_N)\) of \(N\) variables \(y, \ldots, y_N, N \geq 1,\)

(2.41) \quad A_n(y, \ldots, y_N; z) = \sum_{j=0}^{n} \binom{m + n}{n - qj} c_j A_n(y, \ldots, y_N) z^j

By assigning suitable values to the arbitrary coefficients \(c_n\), it is fairly straightforward to derive, from the general formulas (2.37) and (2.38), a considerably large variety of bilateral generating functions for the polynomials \(Y_m^{a,k}(x; k)\) and \(Y_m^{a,k}(x; k)\), respectively. On the other hand, in every situation in which the multivariable function \(A_n(y, \ldots, y_N)\) can be expressed as a suitable product of several simpler functions, we shall be led to an interesting class of mixed multilateral generating functions for the Konhauser polynomials considered and, of course, for the Laguerre polynomials when \(k = 1\), and for the polynomial systems studied by Spencer and Fano [19] and Preiser [16] when \(k = 2\).
VI. *Further finite sums.* The results to be presented here are in addition to the finite summation formulas (2.33) and (2.36) and their general forms involving the Srivastava-Singhal polynomials \( G_n^{(p)}(x, r, p, k) \). Indeed, from the known generating functions \([24, p. 78, \text{Eq. (3.2)}; \text{p. 79, Eq. (3.4)}]\) it is readily observed that

\[
(2.42) \quad G_n^{(s)}(x, r, p, k) = \sum_{j=0}^{n} (-k)^j \binom{n - 1}{j} G_{n-j}^{(s-k+1x)}(x, r, p, k),
\]

\[
(2.43) \quad G_n^{(s)}(x, r, p, k) = \sum_{j=0}^{n} k^j \binom{n - 1}{j} G_{n-j}^{(s-k+j)}(x, r, p, k)
\]

and

\[
(2.44) \quad G_n^{(s)}(x, r, p, k) = \sum_{j=0}^{n} k^j \binom{r - 1}{j} G_{n-j}^{(s-k+j)}(x, r, p, k),
\]

which, on setting \( p = r = 1 \) and appealing to (2.3), yield the following new results involving the Konhauser polynomials \( Y_n^s(x; k) \):

\[
(2.45) \quad Y_n^s(x; k) = \sum_{j=0}^{n-1} (-1)^j \binom{n - 1}{j} Y_{n-j}^{s-k+k}(x; k),
\]

\[
(2.46) \quad Y_n^s(x; k) = \sum_{j=0}^{n-1} \binom{r - 1}{j} Y_{n-j}^{s-k+j}(x; k)
\]

and

\[
(2.47) \quad Y_n^s(x; k) = \sum_{j=0}^{n} \binom{r - 1}{j} Y_{n-j}^{3s+j}(x; k),
\]

respectively.

This last formula (2.47) is analogous to the earlier result (2.36).

3. The biorthogonal polynomials \( Z_n^\pm(x; k) \). Since the parameter \( k \) in (1.2) is restricted, by definition, to take on positive integer values, by the well-known multiplication theorem for the \( \Gamma \)-function we have

\[
(3.1) \quad \Gamma(kj + \alpha + 1) = \Gamma(\alpha + 1) \prod_{i=1}^{k} \left( \frac{\alpha + \frac{i}{k}}{k} \right), \quad j = 0, 1, 2, \ldots,
\]

where \( \lambda_n^\pm \) is given by (1.4). From (1.2) and (3.1) we obtain the hypergeometric representation

\[
(3.2) \quad Z_n^\pm(x; k) = \frac{(\alpha + 1)x}{n!} _2F_1[-n; (\alpha + 1)/k, (\alpha + k)/k; (x/k)^2],
\]

which can alternatively be used to derive the following properties
of the biorthogonal polynomials \( Z_n(x; k) \) by simply specializing those of the generalized hypergeometric function

\[
^{*}F^{\mu}[\alpha_i, \cdots, \alpha_p; \beta_i, \cdots, \beta_q; z] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)^{m}}{\prod_{j=1}^{q} (\beta_j)^{m}} m! z^n
\]

where \( \beta_j \neq 0, -1, -2, \cdots, \quad \forall \ j \in [1, \cdots, q] \).

I. **Differential equations.** Denoting the first member of the preceding equation (3.3) by \( F \), we have the well-known hypergeometric differential equation \([18, \text{p. 77}, \text{Eq. (2)}]\)

\[
[\theta \prod_{j=1}^{p} (\theta + \beta_j - 1) \cdot z \prod_{j=1}^{q} (\theta + \alpha_j)] \theta' = 0, \quad \theta \equiv q + 1,
\]

where, for convenience, \( \theta = zD_x \).

In (3.4) we set \( p = 1, q = k, \alpha = (z/k)^{\ast}, \theta = k^{-\ast} \delta \), where \( \delta = zD_x \), and apply the hypergeometric representation (3.2). We thus obtain a differential equation satisfied by the polynomials \( Z_n(x; k) \) in the form:

\[
\left\{ \prod_{j=1}^{p} (\delta + \alpha - k + j) \right\} \delta Z_n(x; k) = x^{\ast}(\delta - kn)Z_n(x; k).
\]

Recalling that \( \text{cf., e.g., [26, p. 310, \text{Eq. (19)}]} \)

\[
f(\delta + \alpha) \{ y(x) \} = x^{-\ast} f(\delta) \{ x^{\ast} y(x) \}, \quad \delta = zD_x,
\]

it is easily verified that

\[
\prod_{j=1}^{p} (\delta + \alpha - k + j) \{ y(x) \} = x^{-\ast} D_x \{ x^{\ast} y(x) \},
\]

and the differential equation (3.5) obviously reduces to its equivalent form \([10, \text{p. 306, \text{Eq. (10)}}]\)

\[
D_x \{ x^{\ast} z D_x Z_n(x; k) \} = x^{\ast}(zD_x - kn)Z_n(x; k).
\]

II. **Recurrence relations.** It is well known that \( \text{cf., e.g., [11, p. 279, \text{Problem 20}}]\)

\[
D_x \{ F_{\mu}[\alpha_i, \cdots, \alpha_p; \beta_i, \cdots, \beta_q; z] \}
\]

\[
= \frac{\alpha_1 \cdots \alpha_p}{\beta_1 \cdots \beta_q} F_{\mu}[\alpha_i + 1, \cdots, \alpha_p + 1; \beta_i + 1, \cdots, \beta_q + 1; z],
\]

whence, by setting \( p = 1, q = k, \alpha = (z/k)^{\ast}, \beta = (k/z)^{k-\ast}D_x \), and applying (3.2), we have

\[
D_x Z_n(x; k) = -kx^{k-1} Z_{n+1}^{k+1}(x; k),
\]

or, more generally,
\[ (x^{-k}D_x)^m Z_n(x; k) = (-k)^m Z_{n+m}(x; k), \quad n \geq m \geq 0. \]

Similarly, from the known results ([18, p. 82, Eq. (12), (13) and (15)]; see also [17]), involving the generalized hypergeometric function \(\text{(3.3)}\), we readily obtain the following mixed recurrence relations:

\[ XD_n Z_n(x; k) = kn Z_n(x; k) - \frac{k \Gamma(kn + \alpha + 1)}{\Gamma(k(n - 1) + \alpha + 1)} Z_{n+1}(x; k), \]

\[ XD_n Z_n(x; k) = (kn + \alpha) Z_{n+1}(x; k) - \alpha Z_n(x; k), \]

\[ Z_n(x; k) - Z_{n+1}(x; k) = \frac{k \Gamma(kn + \alpha)}{\Gamma(k(n - 1) + \alpha + 1)} Z_{n+1}(x; k). \]

It is not difficult to verify that the recurrence relation (3.14) results from (3.12) and (3.13) by eliminating their common term \(xD_n Z_n(x; k)\). If, however, we eliminate this derivative term in (3.12) or (3.13) by using (3.10) instead, we shall arrive at the recurrence relations

\[ x^r Z_n^{*+r}(x; k) = (kn + \alpha + 1) Z_n(x; k) - (n + 1) Z_{n+1}(x; k) \]

and

\[ kx^r Z_n^{*+r}(x; k) = \alpha Z_{n+1}(x; k) - (kn + \alpha + 1) Z_{n+1}(x; k). \]

Formulas\(^1\) (3.10) and (3.12) were given earlier by Konhauser [10, p. 306, Eq. (8); p. 305, Eq. (6)], (3.14) is due to Génin et Calvez [7, p. A1565, Eq. (5)], while (3.15) was derived by Prabhakar [14, p. 215, Eq. (2.6)] by using a contour integral representation for \(Z_n(x; k)\).

For a direct proof of (3.15), we observe from (1.2) that

\[ x^r Z_n^{*+r}(x; k) = \frac{\Gamma(k(n + 1) + \alpha + 1)}{\Gamma(k(n - 1) + \alpha + 1)} \sum_{j=0}^{n} \frac{(-1)^j}{j! \Gamma(k+j+1)} \frac{x^{k+j+1}}{\Gamma(kj+\alpha+1)} \]

\[ = \frac{\Gamma(k(n+1)+\alpha+1)}{\Gamma(k(n-1)+\alpha+1)} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^j}{\Gamma(kj+\alpha+1)}, \]

and since

\[ \binom{n}{j-1} = \binom{n}{j} - \binom{n+1}{j}, \quad 0 \leq j \leq n + 1, \]

it follows that

\(^1\) The pure recurrence relation (3.16) appears erroneously in a recent paper by K.R. Patil and N.K. Thakare [J. Mathematical Phys. 18 (1977), 1724-1726; especially see p. 1725].

\(^2\) It may be of interest to mention here that the known results (3.10) and (3.15) were rederived, using Prabhakar's version [14, p. 214, Eq. (2.2)] of the generating function (3.20) of this paper, by B. Nath [Kyungpook Math. J. 14 (1974), 81-82].
\begin{align*}
\sum_{n=0}^{\infty} & (\frac{\lambda}{\beta})_n F_{\ell}\left[-n, \alpha_1, \ldots, \alpha_\ell; \beta_1, \ldots, \beta_\ell; \beta; z\right]t^n \\
\quad = & (1 - t)^{-i} F_{\ell}\left[\lambda; \frac{\alpha + 1}{k}, \ldots, \frac{\alpha + k}{k}; \frac{x^t}{(t - 1)k^t}\right], \quad |t| < 1.
\end{align*}

Both (3.17) and (3.18) are stated by Erdélyi et al. [6, p. 267, q. (22) and (25)], and their various generalizations have appeared in the literature (cf., e.g., [20, p. 68, Eq. (3.9) and (3.10)].

By specializing (3.17) and (3.18) in view of the hypergeometric representation (3.2) for \(Z_n(x; k)\), we at once get the generating functions

\begin{align*}
\sum_{n=0}^{\infty} & (\frac{\lambda}{\beta})_n Z_n(x; k) t^n \\
= & (1 - t)^{-i} F_{\ell}\left[\lambda; \frac{\alpha + 1}{k}, \ldots, \frac{\alpha + k}{k}; \frac{x^t}{(t - 1)k^t}\right], \quad |t| < 1
\end{align*}

and

\begin{align*}
\sum_{n=0}^{\infty} & Z_n(x; k) \frac{t^n}{(\alpha + 1)_n} \\
= & e^t k^{\alpha - 1} \left[\frac{\alpha + 1}{k}, \ldots, \frac{\alpha + k}{k}; -\left(\frac{x}{k}\right)^t\right], \quad |t| < 1.
\end{align*}

respectively.

The generating function (3.19) is due essentially to Génin et Calvez [7, p. A1564, Eq. (3)], while (3.20) was given by Srivastava.
[21, p. 490, Eq. (7)]; the latter appears also, with an obvious typographical error, in a recent paper [12, p. 922]. In fact, both (3.19) and (3.20) were given (in disguised forms) by Prabhakar [14, p. 218, Eq. (4.1); p. 214, Eq. (2.2)]. Notice that the so-called generalized Mittag-Leffler function \( E_{\nu}^{\alpha}(z) \) and the “Bessel-Maitland” function \( \phi(k, \alpha + 1; z) \), occurring in Prabhakar’s results just cited, are indeed the familiar hypergeometric functions \( F_{1}^{1} \) and \( F_{1}^{1} \), respectively, \( k \) being a positive integer. More precisely, we have, for \( k = 1, 2, 3, \ldots \),

\[
E_{\nu}^{\alpha+1}(x) = \sum_{m=0}^{\infty} \frac{\left(\lambda\right)_{m} z^{m}}{m! \Gamma(km + \alpha + 1)} \tag{3.21}
\]

\[
= \frac{1}{\Gamma(\alpha + 1)} F_{1}^{1} \left[ \alpha + 1, \frac{\alpha + k}{k} \left( \frac{z}{k} \right)^{k} \right]
\]

and

\[
\phi(k, \alpha + 1; z) = \sum_{m=0}^{\infty} \frac{z^{m}}{m! \Gamma(km + \alpha + 1)} \tag{3.22}
\]

\[
= \frac{1}{\Gamma(\alpha + 1)} F_{1}^{1} \left[ \alpha + 1, \frac{\alpha + k}{k} \left( \frac{z}{k} \right)^{k} \right],
\]

by appealing to the well-known multiplication theorem for the \( F_{1}^{1} \)-function.

Next we consider the double series

\[
\sum_{m=0}^{\infty} z^{m} \sum_{n=0}^{\infty} \binom{m + n}{n} Z_{n+1}(x; \kappa) \frac{t^{n}}{\Gamma(km + \alpha + 1)}
\]

\[
- \sum_{n=0}^{\infty} \frac{Z_{n}(x; \kappa)}{\Gamma(\alpha + 1)_{k_{n}}} \sum_{m=0}^{\infty} \binom{n}{m} \frac{t^{m} z^{m}}{m!} \frac{(x + t)^{m}}{(\alpha + 1)_{k_{n}}}
\]

\[
eq e^{x \cdot t} F_{1}^{1} \left[ \alpha + 1, \frac{\alpha + k}{k}, \frac{(x + t)^{k}}{(\alpha + 1)_{k_{n}}} \right], \text{ by (3.20)},
\]

\[
= \sum_{n=0}^{\infty} \frac{(t^{k})^{n}}{n! (\alpha + 1)_{k_{n}}} \sum_{m=0}^{\infty} \binom{n + \nu}{m} \frac{(x + t)^{m}}{m! (\alpha + 1)_{k_{n}}} z^{m} \tag{3.23}
\]

and, on equating the coefficients of \( z^{m} \), we have the generating relation

\[
\sum_{n=0}^{\infty} \binom{m + n}{n} Z_{n+1}(x; \kappa) \frac{t^{n}}{\Gamma(km + \alpha + 1)}
\]

\[
= \sum_{n=0}^{\infty} \binom{n}{m} \frac{t^{n-m}}{m! (\alpha + 1)_{k_{n}}} \frac{(x + t)^{m-n}}{\Gamma(\alpha + 1)_{k_{n}}} F_{1}^{1} \left[ n + 1; n - m + 1; \nu \right],
\]

Incidentally, the generalized Bessel function \( \phi(\alpha, \beta; z) \) was introduced by E. Maitland Wright [27, p. 72, Eq. (1.3)]; see also Erdélyi et al. [6, p. 211, Eq. (27)].
which holds true for every non-negative integer \( m \).

Alternatively, this last generating relation (3.23) may be derived as a special case of our earlier result [20, p. 68, Theorem 3]. Of course, it is not difficult to develop a direct proof of (3.23) without using the generating function (3.20).

For \( m = 0 \), (3.23) evidently reduces to the familiar generating function (3.20). Its special case when \( k = 1 \) leads to what is obviously contained in the following limiting form of a known result [22, p. 152, Eq. (19)]:

\[
\sum_{n=0}^{\infty} \binom{m+n}{n} L_{n+1}^{(a)}(x) t^n \left( \frac{\lambda}{\mu} \right)^n
= \binom{\alpha + m}{m} e^{t \Psi_2[\alpha + m + 1; \mu, \alpha + 1; t, -x]},
\]

where \( \Psi_2 \) is a (Humbert's) confluent hypergeometric function of two variables defined by [1, p. 126]

\[
\Psi_2[\alpha; c, c'; x, y] = \sum_{n=0}^{\infty} \binom{(a)_{n+c}}{(c)'_n} \frac{x^n}{m!} \frac{y^n}{n!}.
\]

Formula (3.24) follows from the known generating function [22, p. 152, Eq. (19)] by writing \( t/\lambda \) in place of \( t \) and then letting \( \lambda \to \infty \). Furthermore, if we replace \( t \) in (3.24) by \( \mu t \) and let \( \mu \to \infty \), we shall arrive at the well-known generating function [18, p. 211, Eq. (9)]

\[
\sum_{n=0}^{\infty} \binom{m+n}{n} L_{n+1}^{(a)}(x) t^n = (1 - t)^{-m-\alpha-1} \exp \left( \frac{-\alpha t}{1 - t} \right)
\times L_{m}^{(\alpha)} \left( \frac{x}{1 - t} \right), \quad m = 0, 1, 2, \cdots,
\]

which follows also from (2.17) when \( k = 1 \).

IV. Multilinear generating functions. By making use of the hypergeometric representation (3.2), a number of new multilinear generating functions for the product

\[
Z_n^a(y; k), \cdots Z_n^a(y; k_r),
\]

analogous to the corrected version of the Patil-Thakare result [12, p. 921, Eq. (2.1)]], can be derived by suitably specializing a general formula given earlier by Srivastava and Singhal [25, p. 1244, Eq. (24)] for a product of several generalized hypergeometric polynomials. We omit the details involved.

V. Finite summation formulas. In view of the exponential
generating function (3.20), Theorem 1 (p. 64) of Srivastava [20] will apply to the biorthogonal polynomials $Z_n^\alpha(x; k)$, and we thus have

$$
(3.28) \quad Z_n^\alpha(x; k) = (x/y)^{kn} \sum_{j=0}^{n} \binom{\alpha + kn}{kj} (kn - kj)! \frac{y^k - x^k}{x^k} Z_{n-j}^\alpha(y; k),
$$

or, equivalently,

$$
(3.29) \quad Z_n^\alpha(x; k) = (x/y)^{kn} \sum_{j=0}^{n} \binom{\alpha + kn}{kn - kj} (kn - kj)! \frac{y^k - x^k}{x^k} Z_{n-j}^\alpha(y; k).
$$

The summation formula (3.28) can indeed be derived directly (cf. [21, p. 490, § 4]). It can also be rewritten in the form [op. cit., p. 491, Eq. (12)]:

$$
(3.30) \quad Z_n^\alpha(\mu x; k) = \sum_{j=0}^{n} \binom{kn + \alpha}{kj} (kn - kj)! f^{(n-j)}(1 - f^k)^j Z_{n-j}^\alpha(x; k),
$$

which obviously provides us with an elegant multiplication formula for the biorthogonal polynomials $Z_n^\alpha(x; k)$.

VI. Laplace transforms. Employing the usual notation for Laplace's transform, viz

$$
(3.31) \quad \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \quad \text{Re} (s - \sigma) > 0,
$$

where $f \in L(0, R)$ for every $R > 0$, and $f(t) = O(e^{\sigma t})$, $t \to \infty$, we have

$$
(3.32) \quad \mathcal{L}\{t^n Z_n^\alpha(x t; k); s\} = \frac{(\alpha + 1)_s \Gamma(\beta + 1)}{s^{\alpha+1} n!}
\times \sum_{i=1}^{\infty} \left[ \frac{-n}{k}, \frac{\beta + 1}{k}, \ldots, \frac{\beta + k}{k}, \frac{\alpha + 1}{k}, \ldots, \frac{\alpha + k}{k}; \left( \frac{x}{s} \right)^k \right],
$$

provided that $\text{Re} (s) > 0$ and $\text{Re} (\beta) > -1$.

The Laplace transform formula (3.32) can be derived fairly easily from the hypergeometric representation (3.2) by using readily available tables. In the special case when $\beta = \alpha$, it simplifies at once to the elegant form [14, p. 217, Eq. (3.7)]:

$$
(3.33) \quad \mathcal{L}\{t^n Z_n^\alpha(x t; k); s\} = \frac{\Gamma(kn + \alpha + 1)}{s^{\alpha+1} n!} (s^k - x^k)^n,
$$

where, as before, $\text{Re} (s) > 0$ and (by definition) $\alpha > -1$. 

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A MULTILINEAR GENERATING FUNCTION
FOR THE KONHAUSER SETS
OF BIORTHOGONAL POLYNOMIALS
SUGGESTED BY THE LAGUERRE POLYNOMIALS

H. M. SRIVASTAVA

The polynomial sets \( \{ Y_n^\alpha(x; k) \} \) and \( \{ Z_n^\alpha(x; k) \} \), discussed by Joseph D. E. Konhauser, are biorthogonal over the interval \((0, \infty)\) with respect to the weight function \( x^ae^{-x} \), where \( \alpha > -1 \) and \( k \) is a positive integer. The object of the present note is to develop a fairly elementary method of proving a general multilinear generating function which, upon suitable specializations, yields a number of interesting results including, for example, a multivariable hypergeometric generating function for the multiple sum:

\[
\sum_{n_1, \ldots, n_r = 0}^\infty (m + n_1 + \cdots + n_r)! Y_{n_1, \ldots, n_r}^\alpha(x; k) \prod_{i=1}^r \left( \frac{Z_{n_i}^\alpha(y_i; x_i) u_i^{\beta_i}}{(1 + \beta_i) \gamma_{n_i}} \right),
\]

involving the Konhauser biorthogonal polynomials; here, by definition,

\( \alpha > -1; \quad \beta_i > -1; \quad k, s_i = 1, 2, 3, \ldots; \quad \forall i \in \{1, \ldots, r\}. \)

1. Introduction. Joseph D. E. Konhauser ([5]; see also [4]) introduced two interesting classes of polynomials: \( Y_n^\alpha(x; k) \) a polynomial in \( x \), and \( Z_n^\alpha(x; k) \) a polynomial in \( x^k, \alpha > -1 \) and \( k = 1, 2, 3, \ldots \). For \( k = 1 \), these polynomials reduce to the classical Laguerre polynomials \( L_n^{(\alpha)}(x) \), and for \( k = 2 \) they were encountered earlier by Spencer and Fano [8] in the study of the penetration of gamma rays through matter and were discussed subsequently by Preiser [7]. Also [5, p. 303]

\[
(1) \quad \int_0^\infty x^\alpha e^{-x} Y_n^\alpha(x; k) Z_n^\alpha(x; k) \, dx = \frac{\Gamma(kn + \alpha + 1)}{n!} \delta_{mn}, \quad \forall m, n \in \{0, 1, 2, \ldots\},
\]
so that the Konhauser polynomial sets \( \{ Y_n^\alpha(x; k) \} \) and \( \{ Z_n^\alpha(x; k) \} \) are biorthogonal over the interval \((0, \infty)\) with respect to the weight function \(x^\alpha e^{-x}\), where \(\alpha > -1\), \(k\) is a positive integer, and \(\delta_{mn}\) is the Kronecker delta.

The following explicit expression for the polynomials \(Z_n^\alpha(x; k)\) was given by Konhauser [5, p. 304, Eq. (5)]:

\[
(2) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}.
\]

Subsequently, Carlitz pointed out that [2, p. 427, Eq. (9)]

\[
(3) \quad Y_n^\alpha(x; k) = \frac{1}{n!} \sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{l + \alpha + 1}{k}\right)_n x^j,
\]

where \((\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)\).

In a recent paper [10] we derived various properties of (for example) the Konhauser biorthogonal polynomials \(Y_n^\alpha(x; k)\) by suitably specializing those of the Srivastava-Singhal polynomials \(G_n^{(a)}(x, h, p, k)\) which are defined by the generalized Rodrigues formula [14, p. 75, Eq. (1.3)]

\[
(4) \quad G_n^{(a)}(x, h, p, k) = \frac{x^{-kn - \alpha} \exp(px^h)}{h^n} \cdot \left( x^{k+1} D_x \right)^n \{ x^\alpha \exp(-px^h) \}, \quad D_x = \frac{d}{dx},
\]

and given explicitly by [14, p. 77, Eq. (2.1)]

\[
(5) \quad G_n^{(a)}(x, h, p, k) = \frac{k^n}{n!} \sum_{j=0}^{n} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl + \alpha}{k}\right)_n,
\]

where the parameters \(\alpha, h, k\) and \(p\) are unrestricted, in general. In fact, by comparing (5) with Carlitz's result (3), we at once deduce the known relationship [13, p. 315, Eq. (83)]

\[
(6) \quad Y_n^\alpha(x; k) = k^{-\alpha} G_n^{(\alpha+1)}(x, 1, 1, k), \quad \alpha > -1; \ k = 1, 2, 3, \ldots,
\]

which was of fundamental importance in our paper [10].

The object of the present note is first to give a rather elementary proof of a general multilinear generating function for the Srivastava-Singhal polynomials \(G_n^{(a)}(x, h, p, k)\). We then show how this multilinear generating function can be further generalized and applied to derive a
number of interesting results including, for example, a multivariable
hypergeometric generating function for the multiple sum (•) involving the
product of several Konhauser biorthogonal polynomials. Our main result
is contained in the following

THEOREM. For a bounded multiple sequence \( \{ \Lambda(n_1, \ldots, n_r) \} \) of arbitrary
complex numbers, let

\[
\mathcal{H}[n_1, \ldots, n_r; y_1, \ldots, y_r] = \sum_{j_1 = 0}^{n_r / m_r} \sum_{j_2 = 0}^{n_r / m_r} \frac{(-n_1)_{m_1 j_1}}{j_1!} \cdots \frac{(-n_r)_{m_r j_r}}{j_r!} \Lambda(j_1, \ldots, j_r) y_1^{j_1} \cdots y_r^{j_r},
\]

where \( m_1, \ldots, m_r \) are positive integers. Also let \( \Delta_r \) be defined by

\[
\Delta_r = 1 - \sum_{i=1}^{r} u_i, \quad r = 1, 2, 3, \ldots.
\]

Then, for every nonnegative integer \( m \),

\[
\sum_{n_1, \ldots, n_r = 0}^{\infty} (m + n_1 + \cdots + n_r)! G^{(a)}_{m+n_1 + \cdots + n_r}(x, h, p, k)
\cdot \mathcal{H}[n_1, \ldots, n_r; y_1, \ldots, y_r] \left( \frac{u_1/k}{n_1!} \cdots \frac{u_r/k}{n_r!} \right)^{n}
= k^m \exp \left( \frac{px^k}{k} \right) \Delta_r^{-m-a/k}
\sum_{n_1, \ldots, n_r = 0}^{\infty} \left( \frac{hn + \alpha}{k} \right)_{m+n_1 + \cdots + n_r} \left( \frac{1}{n!} \right) \Lambda(n_1, \ldots, n_r) \left[ -\frac{px^k}{\Delta_r^{a/k}} \right]^n
\cdot \prod_{i=1}^{r} \left( \frac{(-u_i/\Delta_r)^m y_i}{n_i!} \right)^{n_i}, \quad k \neq 0,
\]

provided that the multiple series on the right-hand side of (9) has a meaning,
and

\[
|u_1 + \cdots + u_r| < 1.
\]

2. Proof of the theorem. For convenience, let \( \Omega(u_1, \ldots, u_r) \) denote
the left-hand side of (9), and set

\[
N = n_1 + \cdots + n_r \quad \text{and} \quad J = m_1 j_1 + \cdots + m_r j_r.
\]
Applying the explicit representation (5) and the definition (7), we find that

\begin{equation}
\Omega(u_1, \ldots, u_r) = k^m \sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{u_1^{n_1} \cdots u_r^{n_r}}{n_1! \cdots n_r!} \cdot \sum_{j=0}^{m+N} \frac{(px^h)^j}{j!} \sum_{l=0}^{m} (-1)^l \binom{j}{l} \left( \frac{hl + \alpha}{k} \right)^{m+N}
\end{equation}

\begin{align*}
&\cdot \prod_{i=1}^{r} \left( \sum_{j=0}^{\lfloor n_i/m_i \rfloor} (-1)^{m_i} \frac{y_i}{j!(n_i - m_ij)_i!} \right) \Lambda(j_1, \ldots, j_r) \\
&= k^m \sum_{j_1, \ldots, j_r = 0}^{\infty} \Lambda(j_1, \ldots, j_r) \prod_{i=1}^{r} \left( \frac{(-u_i)^{m_i}y_i}{j!} \right) \\
&\cdot \sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{u_1^{n_1} \cdots u_r^{n_r}}{n_1! \cdots n_r!} \sum_{j=0}^{m+N+j} \frac{(px^h)^j}{j!} \\
&\cdot \sum_{j=0}^{J} (-1)^l \binom{j}{l} \left( \frac{hl + \alpha}{k} \right)^{m+N+j}.
\end{align*}

Now we appeal to the series identity [9, p. 4, Eq. (12)]

\begin{equation}
\sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{f(n_1 + \cdots + n_r)u_1^{n_1} \cdots u_r^{n_r}}{n_1! \cdots n_r!} = \sum_{n=0}^{\infty} \frac{f(n)(u_1 + \cdots + u_r)^n}{n!},
\end{equation}

and (12) becomes

\begin{equation}
\Omega(u_1, \ldots, u_r) = k^m \sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{(u_1 + \cdots + u_r)^n}{n!} \cdot \prod_{i=1}^{r} \left( \frac{(-u_i)^{m_i}y_i}{j!} \right) \sum_{j=0}^{m+N+j} \frac{(px^h)^j}{j!} \\
\cdot \sum_{j=0}^{J} (-1)^l \binom{j}{l} \left( \frac{hl + \alpha}{k} \right)^{m+N+j},
\end{equation}

where \( J \) is defined, as before, by (11).
The innermost sum in (14) is the jth difference of a polynomial of
degree \( m + n + J \) in \( \alpha \); it is nil when \( j > m + n + J. \) Thus we have

\[
\sum_{j=0}^{m+n+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{J} (-1)^l \binom{J}{l} \frac{\binom{hl + \alpha}{k}}{l!} m+n+j \\
= \sum_{l=0}^{\infty} \frac{hl + \alpha}{k} \frac{(px^h)^l}{l!} \sum_{j=0}^{\infty} \frac{(px^h)^j}{j!} m+n+j \\
= \exp(px^h) \sum_{l=0}^{\infty} \frac{hl + \alpha}{k} \frac{(-px^h)^l}{l!} m+n+j,
\]

and substituting this expression in (14), and applying the binomial expansion to sum the resulting n-series, we finally obtain

\[
(15) \ \Omega(u_1, \ldots, u_r) = k^m \exp(px^h) \Delta_r^{-m-a/k} \\
\sum_{l, l, \ldots, l=0}^{\infty} \left( \frac{hl + \alpha}{k} \right) m+n+j \frac{1}{l!} \Delta(l_1, \ldots, l_r) \left( \frac{px^h}{\Delta_r^{a/k}} \right)^l \\
\prod_{i=1}^{r} \left\{ \frac{(-u_i/\Delta_r)^m y_i}{j!} \right\}, \quad k \neq 0,
\]

where \( \Delta_r \) and \( J \) are given by (8) and (11), respectively, and the inequality in (10) is assumed to hold.

The right-hand sides of (9) and (15) are essentially the same. This evidently completes the proof of our theorem under the hypothesis that the various interchanges of the order of summation are permissible by absolute convergence of the series involved. Thus, in general, our theorem holds true whenever each member of (9) has a meaning.

**Remark.** Our method of derivation can be applied mutatis mutandis in order to prove the following generalization of the multilinear generating function (9):

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} (m + n_1 + \cdots + n_r)! \mathfrak{F}_{m+n_1+\cdots+n_r}(x, \ h, \ p, \ k) \\
\cdot \mathfrak{F}_{n_1}^r \left( \frac{u_1/k}{n_1!}, \ldots, \frac{u_r/k}{n_r!} \right) \\
= k^m \exp(px^h) \Delta_r^{-m-a/k} \\
\sum_{n_1, \ldots, n_r=0}^{\infty} \binom{hn + \alpha}{k} m+n_1+\cdots+n_r, \\
\cdot \frac{\xi_n}{n!} \Delta(n_1, \ldots, n_r) \left( \frac{px^h}{\Delta_r^{a/k}} \right)^r \prod_{i=1}^{r} \left\{ \frac{(-u_i/\Delta_r)^m y_i}{n_i!} \right\}, \quad k \neq 0,
\]
where, in terms of the bounded sequence \( \{ \xi_n \} \) of arbitrary complex numbers,

\[
\Phi^{(a)}(x, h, p, k) = \frac{k^n}{n!} \sum_{j=0}^{\infty} \left( \frac{px^k}{j!} \right)^j \sum_{l=0}^{j} \frac{(-1)^l}{l!} \xi_l \left( \frac{hl + \alpha}{k} \right)^n,
\]

which obviously reduces to the Srivastava-Singhal equation (5) when \( \xi_l = 1, l \geq 0 \).

3. Applications. By assigning suitable special values to the arbitrary coefficients \( \Lambda(j_1, \ldots, j_r) \), the multiple sum in (7) can indeed be expressed in terms of the generalized Lauricella hypergeometric function of \( r \) variables [11, p. 454]. Thus, following the various notations and conventions explained fairly fully by Srivastava and Daoust ([11, p. 545 et seq.]; see also [12]), we obtain from our theorem the multivariable hypergeometric generating function:

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} (m + n_1 + \cdots + n_r)G^{(a)}_{m+n_1+\cdots+n_r}(x, h, p, k)
\]

\[
= \frac{A^1 + B^1; \cdots; 1 + B^{(r)}}{C^1 + D^1; \cdots; D^{(r)}} 
\]

\[
\left[ \begin{array}{c}
\left( a ; \theta_1, \ldots, \theta^{(r)} \right) \left[ \begin{array}{c}
\left( b ; \phi \right) \left[ \begin{array}{c}
\left( c ; \psi, \psi^{(r)} \right) \left[ \begin{array}{c}
\left( d^{(r)} ; \delta^{(r)} \right)
\end{array}\right]
\end{array}\right]
\end{array}\right]
\end{array}\right)
\]

\[
= k^m \left( \frac{a}{k} \right)^m \exp(px^k) \Delta, \quad m = a/k F^1 + A^1; B^1; \cdots; B^{(r)}
\]

\[
C^1 + D^1; \cdots; D^{(r)}
\]

\[
\left[ \begin{array}{c}
\left( m + a/k ; h/k, m_1, \ldots, m_r \right) \left[ \begin{array}{c}
\left( c ; 0, \theta^{(r)} \right) \left[ \begin{array}{c}
\left( d^{(r)} ; \delta^{(r)} \right)
\end{array}\right]
\end{array}\right]
\end{array}\right)
\]

\[
= 0, \quad h/k > 0, \Delta_r \text{ is given by (8), and}
\]

\[
\Xi_0 = -\frac{px^h}{\Delta^{h/k}_r}, \quad \Xi_i = y_i \left( -\frac{p}{\Delta_r} \right)^{m_i}, \quad i = 1, \ldots, r.
\]

Next we set \( A = C = 0 \) in (18) and, for convenience, let each of the positive coefficients \( \phi^{(i)}, j = 1, \ldots, B^{(i)}; \delta^{(i)}, j = 1, \ldots, D^{(i)} (i = 1, \ldots, r) \) equal 1. Denoting the array of parameters

\[
(-n_i + j - 1)/m_i, \quad j = 1, \ldots, m_i,
\]
by \( \Delta(m; -n) \), \( i = 1, \ldots, r \), we thus find from (18) that

\[
(20) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} (m + n_1 + \cdots + n_r) G_{m+n_1+\cdots+n_r}^{(a)}(x, h, p, k) =
\prod_{i=1}^{r} \left\{ \frac{\Delta(m; -n_i)}{\prod_{j=0}^{i-1} F_{0}^{(i)} \left[ \Delta(m; -n_i), (b^{(i)}); \left( \frac{u_i}{k} \right)^{n_i} \right]} \right\}
= k^m \left( \frac{\alpha}{k} \right)_m \exp \left( p x^k \right) \Delta_{-a/k}^{m-a/k},
\]

\[
\begin{array}{cccc}
F_1 & 0; B_{(r)} & \left[ m + a/k; h/k, m_1, \ldots, m_r \right]; & \vdots \\
0 & 1; D_{(r)} & \vdots & \vdots \\
0 & 1 & \vdots & \vdots \\
\end{array}
\]

\[
[(b': 1); \ldots; (b'(i): 1); \xi_0, \xi_1, \ldots, \xi_r], \quad k \neq 0,
\]

where \( h/k \geq 0 \), \( \Delta \), is given by (8), and \( \xi_0, \xi_1, \ldots, \xi_r \) are defined by (19).

Obviously, this last formula (20) generates the product of \( r \) generalized hypergeometric polynomials; it is a generalization of several known results due to Srivastava and Singh [15].

For special values of the parameters, the Srivastava-Singhal polynomials \( G_{m}^{(a)}(x, h, p, k) \) can be reduced to the classical Hermite and Laguerre polynomials and their various generalizations studied in the literature (cf. [14, p. 76]). Furthermore, the generalized hypergeometric polynomials occurring in (20) can be specialized to several important classes of hypergeometric polynomials including, for example, the classical Hermite polynomials and their various generalizations as those considered by Gould and Hopper [3, p. 58]

\[
(21) \quad g_{m}^{n}(x, \lambda) = \sum_{j=0}^{[n/m]} \frac{n!}{j!(n-mj)!} \lambda^j x^{n-mj}
= x^m \frac{F_0 \left[ \Delta(m; -n); \left( \frac{-m}{x} \right)^{m} \right]}{a^{m} \prod_{j=0}^{m} F_{0}^{(j)} \left[ \Delta(m; -n); \left( \frac{-m}{x} \right)^{m} \right]},
\]

and by Brfman [1, p. 186]

\[
(22) \quad B_{m}^{(a)} \left[ \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_r; x \right]
= x^{m} \frac{F_0 \left[ \Delta(m; -n); \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_r; x \right]}{a^{m} \prod_{j=0}^{m} F_{0}^{(j)} \left[ \Delta(m; -n); \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_r; x \right]},
\]

where, as in (20), \( \Delta(m; -n) \) abbreviates the array of \( m \) parameters

\[
(-n + j - 1)/m, \quad j = 1, \ldots, m,
\]
$m$ being an arbitrary positive integer. The details involved in these derivations of known or new multilinear generating functions from (20) may be left as an exercise to the interested reader.

Yet another interesting application of our theorem would result when in (18) we set

\[
\begin{align*}
  & h = p = 1, \quad A = B^{(i)} = C = D^{(i)} - 1 = 0, \\
  & a_i^{(i)} = 1 + \beta_i, \quad \delta_i^{(i)} = s_i, \quad m_i = 1, \quad i = 1, \ldots, r,
\end{align*}
\]

replace $\alpha$ by $\alpha + 1$, and $y_i$ by $y_i^n, i = 1, \ldots, r$, and appeal to the relationship (6) and to the explicit representation (2). We thus obtain our desired multilinear generating function for the Konhauser biorthogonal polynomials in the form:

\[
\begin{align*}
  \sum_{n_1, \ldots, n_r = 0} \infty (m + n_1 + \cdots + n_r)! Y_{m+n_1+\cdots+n_r}^n(x; k) \\
  \prod_{i=1}^r \left( Z_{n_i}^{s_i}(y_i; s_i) \frac{u_i^{n_i}}{(1 + \beta_i)_{s_i n_i}} \right) \\
  \left( \frac{\alpha + 1}{k} \right)_m e^{x \Delta_r^{m-\alpha+1}/k} \\
  = \left( \frac{\alpha + 1}{k} \right)_m e^{x \Delta_r^{m-\alpha+1}/k} \cdot \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{u_1}{\Delta_1^{s_1}}, \quad \frac{u_2}{\Delta_1^{s_2}}, \quad \cdots, \quad \frac{u_r y_i^{s_i}}{\Delta_r^{s_i}}, \quad \cdots, \quad \frac{u_r y_i^{s_r}}{\Delta_r},
\end{align*}
\]

where, by definition,

\[
\alpha > -1; \quad \beta_i > -1; \quad k, s_i = 1, 2, 3, \ldots; \quad \forall i \in \{1, \ldots, r\}.
\]

A seriously erroneous version of a special case of the multilinear generating function (23), when $s_i = \cdots = s_r = s$, was proven earlier by Patil and Thakare [6] who incidentally used a markedly different method. In fact, (23) with $k = s_1 = \cdots = s_r = 1$ is a well-known result (involving the classical Laguerre polynomials) due to Srivastava and Singhal [15, p. 1239, Eq. (5)].

Since $s_i, \ldots, s_r$ are, by definition, positive integers, the multilinear generating function (23) would follow also as an obvious special case of (20).
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REFERENCES


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