Chapter 4

Centroid of an intuitionistic fuzzy number

In this chapter\(^1\), a formula for finding the centroid of an Intuitionistic Fuzzy Number is introduced and its properties are studied.

4.1 Introduction

Fuzzy sets defined on the set \(\mathbb{R}\) of real numbers are of special importance among various types of fuzzy sets. These sets under certain conditions can be viewed as fuzzy numbers. The term ‘fuzzy number’ is used to handle imprecise numerical quantities such as ‘close to 10’, ‘about 7’, ‘several’ etc. Such concepts are essential for characterizing state of fuzzy variable and thus play an important role in many applications such as decision making, optimization, economics etc. A general definition of fuzzy number is given by Dubois and Prade [39]. Yager

\(^1\)Some portions of this chapter have appeared in the journal *Notes on Intuitionistic Fuzzy Sets*, vol. 18, 2012, no.1, pp 19–24
[84] proposed a method which computes for each fuzzy number, a crisp measure, as its centroid. Other than Yager, a number of researchers like Murakami et al. [66], Cheng [33], Wang et al. [82] have also studied the concept of centroid of fuzzy numbers. Centroid of a fuzzy number \( f \) is its geometric center and is given by the formula
\[
\frac{\int_{-\infty}^{\infty} x f(x)dx}{\int_{-\infty}^{\infty} f(x)dx} \tag{32, 82}
\]

Burillo et al. [24] proposed definition of IFN and studied its properties. A number of researchers like Ban [14], P. Grzegorzewski [46], G. S. Mahapatra [63], Guha D. and D. Chakraborty [49], V. L. G. Nayagam [67] studied IFNs and analyzed its properties. Mitchell [64] considered the problem of ranking a set of IFNs to define a fuzzy rank and a characteristic vagueness factor for each IFN. Corresponding to every intuitionistic fuzzy number we introduce its crisp equivalent using its membership function and non membership function. The new scoring method has wide applications in various fields such as economics, intuitionistic fuzzy decision making etc.

### 4.2 Intuitionistic fuzzy numbers

Intuitionistic fuzzy numbers (IFNs) were studied in [14, 3, 15, 24, 49]. IFN is the generalization of fuzzy number and so it can be represented in the following manner.

**Definition 4.2.1.** [49] [Intuitionistic fuzzy numbers] An intuitionistic fuzzy subset

\[ A = \{ (x, \mu_A(x), \nu_A(x)) : x \in \mathbb{R} \} \]

of the real line \( \mathbb{R} \) is called an IFN if the following holds:

(i) There exists \( b \in \mathbb{R} \) such that \( \mu_A(b) = 1 \) and \( \nu_A(b) = 0 \)
(ii) $\mu_A$ is a continuous mapping from $\mathbb{R} \to [0, 1]$ and for every $x \in \mathbb{R}$, the relation $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ holds

(iii) The membership and non membership functions of $A$ is of the following form:

$$
\mu_A(x) = \begin{cases} 
0, & -\infty < x \leq a \\
f(x), & a \leq x \leq b \\
1, & x = b \\
g(x), & b \leq x \leq c \\
0, & c \leq x < \infty 
\end{cases}
$$

$$
\nu_A(x) = \begin{cases} 
1, & -\infty < x \leq e \\
h(x), & e \leq x \leq b, 0 \leq f(x) + h(x) \leq 1 \\
0, & x = b \\
k(x), & b \leq x \leq g, 0 \leq g(x) + k(x) \leq 1 \\
1, & g \leq x < \infty 
\end{cases}
$$

where $f, g, h, k$ are functions from $\mathbb{R} \to [0, 1]$, $f$ and $k$ are strictly increasing functions and $g$ and $h$ are strictly decreasing functions.

It is worth noting that each IFN $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in \mathbb{R} \}$ is a conjunction of two fuzzy numbers: $A^+$ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and $A^-$ with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$.

**Note.** Here $e \leq a$ and $c \leq g$. For $x \leq b$, if $e > a$, then there exists real number
m such that $a < m < e$,

$$\mu_A(x) + \nu_A(x) \leq 1 \Rightarrow f(m) \leq 0,$$

a contradiction. Hence $e \leq a$. Similarly $c \leq g$.

**Definition 4.2.2.** [Triangular intuitionistic fuzzy number] An IFN may be defined as a triangular intuitionistic fuzzy number (TIFN) if and only if its membership and non membership function takes the following form:

$$\mu_A(x) = \begin{cases} 
0, & x < a \\
\frac{x-a}{b-a}, & a \leq x \leq b \\
\frac{x-c}{c-b}, & b \leq x \leq c \\
0, & x > c 
\end{cases}$$

$$\nu_A(x) = \begin{cases} 
1, & x < e \\
\frac{e-x}{b-e}, & e \leq x \leq b \\
\frac{x-b}{g-b}, & b \leq x \leq g \\
1, & x > g.
\end{cases}$$

where $e \leq a$ and $c \leq g$. Symbolically TIFN $A$ is represented as $(a, b, c); (e, b, g)$.

### 4.3 Centroid of an intuitionistic fuzzy number

Here, we introduce a new fuzzy number $\rho$ and its centroid is taken as the centroid of the intuitionistic fuzzy number.
Definition 4.3.3. Let \( p : \mathbb{R} \to [0, 1] \) be defined by

\[
p(x) = \frac{(\mu(x) - \nu(x) + 1)}{2}
\]

where \( \mu \) and \( \nu \) are the membership functions of IFN \( A \).

Proposition 4.3.1. \( p \) is a fuzzy number.

Proof. Range of \( \mu - \nu \) lies in \([-1, 1]\).

\[
\begin{align*}
\mu(x) - \nu(x) &= \begin{cases} 
-1, & x < e \\
-h(x), & e \leq x < a \\
f(x) - h(x), & a \leq x < b \\
g(x) - k(x), & b \leq x < c \\
-k(x), & c \leq x < g \\
-1, & x \geq g 
\end{cases} \\
p(x) &= \begin{cases} 
0, & x < e \\
\frac{1-h(x)}{2}, & e \leq x < a \\
\frac{f(x)-h(x)+1}{2}, & a \leq x < b \\
\frac{g(x)-k(x)+1}{2}, & b \leq x < c \\
\frac{1-k(x)}{2}, & c \leq x < g \\
0, & x \geq g
\end{cases}
\end{align*}
\]

Range of \( p \) lies in \([0, 1]\).

\[
p(b) = 1.
\]
Therefore,

\[ p(x) = \begin{cases} 
1 & \text{for } x = b \\
l(x) & \text{for } x \in (-\infty, b) \\
r(x) & \text{for } x \in (b, \infty) 
\end{cases} \]

where

\[ l(x) = \begin{cases} 
0, & x < e \\
\frac{1-h(x)}{2}, & e \leq x < a \\
\frac{f(x)-h(x)+1}{2}, & a \leq x < b 
\end{cases} \]

\( l \) is a function from \((-\infty, b)\) to \([0, 1]\) continuous and monotonically increasing and

\[ r(x) = \begin{cases} 
g(x)-k(x)+1, & b < x \leq c \\
\frac{1-k(x)}{2}, & c < x \leq g \\
0, & x > g 
\end{cases} \]

\( r \) is a function from \((b, \infty)\) to \([0, 1]\), continuous and monotonically decreasing.

Hence \( p \) is a fuzzy number. \( \square \)

**Proposition 4.3.2.** Centroid of \( p \) is

\[
\frac{\int_{e}^{a} \frac{1-h(x)}{2} \, dx + \int_{a}^{b} \frac{f(x)-h(x)+1}{2} \, dx + \int_{b}^{c} \frac{g(x)-k(x)+1}{2} \, dx + \int_{c}^{g} \frac{1-k(x)}{2} \, dx}{\int_{e}^{a} \frac{1-h(x)}{2} \, dx + \int_{a}^{b} \frac{f(x)-h(x)+1}{2} \, dx + \int_{b}^{c} \frac{g(x)-k(x)+1}{2} \, dx + \int_{c}^{g} \frac{1-k(x)}{2} \, dx} \tag{4.1}
\]

**Remark 4.3.1.** (4.1) is taken as the centroid of the IFN \( A \).

**Remark 4.3.2.** If \( \nu_A(x) \equiv 1 - \mu_A(x) \), then IFN \( A \) is a fuzzy number. Then \( e = a, c = g \),

\[ h(x) = 1 - f(x) \quad \text{and} \quad k(x) = 1 - g(x). \]
Its centroid is

\[
\int_a^b f(x) \frac{(1-f(x))}{2} x dx + \int_b^c g(x) \frac{(1-g(x))}{2} x dx
\]

\[
\int_a^b f(x) \frac{(1-f(x))}{2} dx + \int_b^c g(x) \frac{(1-g(x))}{2} dx
\]

\[
\int_a^b f(x) dx + \int_b^c g(x) dx
\]

by (4.1). Hence formula for centroid of IFN introduced here is generalization of that of fuzzy number.

**Proposition 4.3.3.** The centroid of TIFN \((a, b, c); (e, b, g)\) is

\[
\frac{1}{3} \left[ \frac{(g-e)(b-2g-2e) + (c-a)(a+b+c) + 3(g^2 - e^2)}{g-e+c-a} \right]
\]

**Proof.** The fuzzy number \(p : \mathbb{R} \rightarrow [0, 1]\) defined by \(p(x) = \frac{\mu(x) - v(x) + 1}{2}\) for the TIFN \(A\) as given by \(A\) in Definition 4.2.2 is

\[
p(x) = \begin{cases} 
0 & x < e \\
\frac{x-b+1}{2} & e \leq x < a \\
\frac{x-a-b+1}{2} & a \leq x < b \\
\frac{x-a-b+c+1}{2} & b \leq x < c \\
\frac{x-c-b+1}{2} & c \leq x < g \\
0 & x > g.
\end{cases}
\]

Centroid of the TIFN is

\[
\frac{\int_e^a x \left( \frac{x-b+1}{2} \right) dx + \int_a^b x \left( \frac{x-a-b+1}{2} \right) dx + \int_b^c x \left( \frac{x-a-b+c+1}{2} \right) dx}{\int_e^a \left( \frac{x-b+1}{2} \right) dx + \int_a^b \left( \frac{x-a-b+1}{2} \right) dx + \int_b^c \left( \frac{x-a-b+c+1}{2} \right) dx + \int_c^g \left( \frac{x-c-b+1}{2} \right) dx}
\]

\[
\frac{\int_a^b x f(x) dx + \int_b^c x g(x) dx}{\int_a^b f(x) dx + \int_b^c g(x) dx}
\]

(4.2)
by (4.1).

\[
\int_{e}^{a} \frac{x(x - b)}{2(b - e)} \, dx = \frac{1}{2(b - e)} \int_{e}^{a} (x^2 - bx) \, dx
\]

\[
= \frac{1}{2(b - e)} \left[ \frac{x^3}{3} - \frac{bx^2}{2} \right]_{e}^{a}
\]

\[
= \frac{1}{12(b - e)} [2x^3 - 3bx^2]_e^{a}
\]

\[
= \frac{1}{12(b - e)} [2(a^3 - e^3) - 3b(a^2 - e^2)] \tag{4.3}
\]

\[
\int_{e}^{a} \frac{x}{2} \, dx = \frac{1}{4} (a^2 - e^2) \tag{4.4}
\]

\[
\int_{a}^{b} \frac{(x - a)}{2(b - a)} \, dx = \frac{1}{2(b - a)} \left[ \frac{x^3}{3} - \frac{ax^2}{2} \right]_{a}^{b}
\]

\[
= \frac{1}{12(b - a)} [2x^3 - 3ax^2]_a^{b}
\]

\[
= \frac{1}{12(b - a)} [2(b^3 - a^3) - 3a(b^2 - a^2)]
\]

\[
= \frac{1}{12} (2b^2 - ab - a^2) \tag{4.5}
\]

\[
\int_{a}^{b} \frac{(b - x)}{2(e - b)} \, dx = \frac{1}{2(e - b)} \left[ \frac{b^2}{2} - \frac{x^3}{3} \right]_{a}^{b}
\]

\[
= \frac{1}{12(e - b)} [3bx^2 - 2x^3]_a^{b}
\]

\[
= \frac{1}{12(e - b)} [3b(b^2 - a^2) - 2(b^3 - a^3)] \tag{4.6}
\]

\[
\int_{a}^{b} \frac{x}{2} \, dx = \frac{1}{4} (b^2 - a^2) \tag{4.7}
\]

\[
\int_{b}^{c} \frac{(x - c)}{2(b - c)} \, dx = \frac{1}{2(b - c)} \left[ \frac{x^3}{3} - \frac{cx^2}{2} \right]_{b}^{c}
\]

\[
= \frac{1}{12(b - c)} [2x^3 - 3cx^2]_b^{c}
\]

\[
= \frac{1}{12(b - c)} [2(c^3 - b^3) - 3c(c^2 - b^2)]
\]

\[
= \frac{1}{12} [-2b^2 + c^2 + bc] \tag{4.8}
\]
\[
\int_b^c \frac{(b-x)x}{2(g-b)} \, dx = \frac{1}{2(g-b)} \int_b^c (bx - x^2) \, dx \\
= \frac{1}{2(g-b)} \left[ bx^2 - \frac{x^3}{3} \right]_b^c \\
= \frac{1}{12(g-b)} \left[ 3bx^2 - 2x^3 \right]_b^c \\
= \frac{1}{12(g-b)} [3b(c^2 - b^2) - 2(c^3 - b^3)] \quad (4.9)
\]

\[
\int_b^c \frac{x}{2b} \, dx = \frac{1}{4}(c^2 - b^2) \quad (4.10)
\]

\[
\int_c^g \frac{x(x-b)}{2(b-g)} \, dx = \frac{1}{2(b-g)} \left[ \frac{x^3}{3} - \frac{bx^2}{2} \right]_c^g \\
= \frac{1}{12(b-g)} [2(g^3 - c^3) - 3b(g^2 - c^2)] \quad (4.11)
\]

\[
\int_c^g \frac{x}{2} \, dx = \frac{1}{4}(g^2 - c^2) \quad (4.12)
\]

Numerator of R.H.S. in equation (4.2) is got by adding equations (4.3) to (4.12).

Sum of the integrals in equations (4.3) to (4.12)

\[
= \frac{1}{12(b-e)} [2(a^3 - e^3) - 3b(a^2 - e^2)] + \frac{1}{12(g-b)} [-2(g^3 - c^3) + 3b(g^2 - c^2)] \\
+ \frac{1}{12(b-e)} [-3b(b^2 - a^2) - 2(b^3 - a^3)] + \frac{1}{12(g-b)} [3b(e^2 - b^2) - 2(e^3 - b^3)] \\
+ \frac{1}{4}(a^2 - e^2) + \frac{1}{4}(g^2 - e^2) + \frac{1}{4}(b^2 - a^2) + \frac{1}{4}(c^2 - b^2) \\
+ \frac{1}{12}(2b^2 - ab - a^2) + \frac{1}{12}(-2b^2 + c^2 + bc) \\
= \frac{1}{12(b-e)} [2(b^3 - e^3) - 3b(b^2 - e^2)] \\
+ \frac{1}{12(g-b)} [3b(g^2 - b^2) - 2(g^3 - b^3)] \\
+ \frac{1}{4}(g^2 - e^2) + \frac{1}{12}(e^2 - a^2 + b(c - a)] \\
= \frac{1}{12} [2(b^2 + e^2 + be) - 3b(b + e)]
\]
\[
\begin{align*}
+ \frac{1}{12} [3b(g + b) - 2(g^2 + b^2 + gb)] + \frac{1}{4}(g^2 - e^2) \\
+ \frac{1}{12} [(c - a)(a + b + c)] \\
= \frac{1}{12} [2e^2 - be + bg - 2g^2] + \frac{1}{12} [(c - a)(a + b + c)] + \frac{1}{4}(g^2 - e^2) \\
= \frac{1}{12} [(g - e)(b - 2g - 2e) + (c - a)(a + b + c)] + \frac{1}{4}(g^2 - e^2) 
\end{align*}
\] 

(4.13)

\[
\begin{align*}
\int_e^a \left( \frac{x - b}{b - e} + 1 \right) dx &= \frac{1}{b - e} \left[ \frac{x^2}{2} - bx \right]_e^a + [x]_e^a \\
&= \frac{1}{b - e} \left[ \frac{a^2 - e^2}{2} + b(e - a) \right] + a - e 
\end{align*}
\] 

(4.14)

\[
\begin{align*}
\int_c^g \left( \frac{x - b}{b - g} + 1 \right) dx &= \frac{1}{b - g} \left[ \frac{x^2}{2} - bx \right]_c^g + g - c \\
&= \frac{1}{b - g} \left[ \frac{g^2 - c^2}{2} + b(c - g) \right] + g - c 
\end{align*}
\] 

(4.15)

\[
\begin{align*}
\int_a^b \left( \frac{x - a}{b - a} - \frac{b - x}{b - e} + 1 \right) dx \\
&= \frac{1}{b - a} \left[ \frac{x^2}{2} - ax \right]_a^b + \frac{1}{e - b} \left[ bx - \frac{x^2}{2} \right]_a^b + b - a \\
&= \frac{1}{b - a} \left[ \frac{b^2 - a^2}{2} + a(a - b) \right] \\
&+ \frac{1}{e - b} \left[ b(b - a) + \frac{a^2 - b^2}{2} \right] + b - a 
\end{align*}
\] 

(4.16)
\[
\int_b^c \left( \frac{x - c}{b - c} - \frac{b - x}{b - g} + 1 \right) \, dx = \frac{1}{b - c} \left[ \frac{c^2 - b^2}{2} + c(b - c) \right] \\
+ \frac{1}{g - b} \left[ b(c - b) + \frac{b^2 - c^2}{2} \right] + c - b \quad (4.17)
\]

Sum of the integrals in equations (4.14) to (4.17)

\[
= \frac{1}{b - e} \left[ \frac{a^2 - e^2}{2} + be - ab \right] + a - e + \frac{1}{b - g} \left[ \frac{g^2 - c^2}{2} + b(c - g) \right] + g - c \\
+ \frac{1}{b - a} \left[ \frac{b^2 - a^2}{2} + a(a - b) \right] + \frac{1}{e - b} \left[ b(b - a) + \frac{a^2 - b^2}{2} \right] + b - a \\
+ \frac{1}{b - c} \left[ \frac{c^2 - b^2}{2} + c(b - c) \right] + \frac{1}{g - b} \left[ b(c - b) + \frac{b^2 - c^2}{2} \right] + c - b \\
= \frac{1}{b - e} \left[ \frac{a^2 - e^2}{2} + \frac{b^2 - a^2}{2} + be - ab + ab - b^2 \right] \\
+ \frac{1}{g - b} \left[ \frac{b^2 - e^2 + e^2 - g^2}{2} + be - b^2 + bg - bc \right] \\
+ \frac{1}{b - a} \left[ \frac{(b - a)(b + a)}{2} - a(b - a) \right] + \frac{1}{c - b} \left[ \frac{(b - c)(b + c)}{2} - c(b - c) \right] + g - e \\
= \frac{1}{b - e} \left[ \frac{b^2 - e^2}{2} + b(e - b) \right] + \frac{1}{g - b} \left[ \frac{b^2 - g^2}{2} + b(g - b) \right] \\
+ \frac{b + a}{2} - a + c - \frac{b + c}{2} + g - e \\
= \frac{c - a + g - e}{2} \quad (4.18)
\]

Denominator of RHS in equation (4.2)

\[
= \int_a^b \left( \frac{x - b}{b - e} + 1 \right) \, dx + \int_b^c \left( \frac{x - a - \frac{b - x}{b - e} + 1}{2} \right) \, dx \\
+ \int_b^c \left( \frac{x - c - \frac{b - x}{b - g} + 1}{2} \right) \, dx + \int_c^d \left( \frac{x - b}{b - g} + 1 \right) \, dx \\
= \frac{c - a + g - e}{4} \quad (4.19)
\]
Hence centroid of the TIFN is

\[
\frac{1}{12} [(g - e)(b - 2g - 2e) + (c - a)(a + b + c)] + \frac{3}{12}(g^2 - e^2)
\]

\[
= \frac{1}{3} \frac{(g - e)(b - 2g - 2e) + (c - a)(a + b + c) + 3(g^2 - e^2)}{g - e + c - a}
\] (4.20)

\[\square\]

**Remark 4.3.3.** If \(\nu_A(x) = 1 - \mu_A(x)\), then TIFN \((a, b, c); (e, b, g)\) will become the TFN \((a, b, c)\). Then \(e = a\) and \(g = c\).

By (4.20), Centroid of \((a, b, c) = \)

\[
\frac{1}{3} \left[ \frac{(c - a)(b - 2c - 2a) + (c - a)(a + b + c) + 3(c^2 - a^2)}{c - a + c - a} \right]
\]

\[
= \frac{a + b + c}{3}.
\]

**Proposition 4.3.4.** The centroid of TIFN \((a, b, c); (e, b, g)\) is \(b\) if \(b - a = c - b\) and \(b - e = g - b\).

**Proof.** Let \(b - a = c - b = k\) and \(b - e = g - b = l\), then \(c - a = 2k\), \(g - e = 2l\)

\[
b - 2g - 2e = e + l - 2(e + 2l) - 2e
\]

\[
= -3(e + l)
\]

\[
a + b + c = a + a + k + a + 2k = 3a + 3k
\]

\[
g + e = e + e + 2l = 2e + 2l.
\]

Centroid = \[
\frac{1}{3} \left[ \frac{(g - e)(b - 2g - 2e) + (c - a)(a + b + c) + 3(g^2 - e^2)}{g - e + c - a} \right]
\]
\[
= \frac{1}{3} \left[ \frac{2l \times -3(e + l) + 2k(3a + 3k) + 3 \times (2(e + l))2l}{2l + 2k} \right]
= \frac{-6l(e + l) + 6k(a + k) + 12l(e + l)}{6(l + k)}
= \frac{-6lb + 6kb + 12lb}{6(l + k)}
= \frac{6lb + 6kb}{6(l + k)} = \frac{6(l + k)b}{6(l + k)} = b.
\]

### 4.4 Significance of the proposed method

In the proposed formula for centroid, both membership grade and nonmembership grade are used to obtain a single score. If we obtain a membership score and a non membership score for IFNs, (as described in [68], IFNs are treated in a different manner there) ranking of IFNs fails if membership score of \(M_1 \leq\) membership score of \(M_2\) and non membership score of \(M_1 \leq\) non membership score of \(M_2\) where \(M_1\) and \(M_2\) are two IFNs. Getting a single score involving both membership and non membership grades overcome that situation.

This new scoring method has wide applications in various fields. In economics, for extending equilibrium in the fuzzy sense to intuitionistic fuzzy sense, these notions can be used.

A formula for finding centroid of an IFN is introduced and it is proved that it is a generalization of formula in the literature.

In the next chapter we are using this new formula for extending economic equilibrium to the intuitionistic fuzzy context.