CHAPTER 6

FIXED POINT THEOREMS IN ORDERED FUZZY CONE METRIC SPACE

6.1 INTRODUCTION

It is well known that Fuzzy metric space is an important generalization of metric space. Many authors has considered this problem and introduced in different ways. For instance, George and Veeramani [44] modified the concept of a Fuzzy metric space introduced by Karmosil and Michalek [66] and defined Hausdroff topology of a Fuzzy metric space. There exist considerable literature about fixed point properties, for mappings defined on Fuzzy metric space. Tarkan Oner et. al [97] introduced the idea of Fuzzy cone metric space which is a generalization of Fuzzy metric space by George and Veeramani [44]. Partially ordered set have many applications in computer languages, Game theory, Economics and many other fields [17]. Mathematical Analysis under the domain of partially ordered set evolved the fixed point theory to generalize and solved many results and problems in linear and nonlinear Analysis. T. Bag introduced Fuzzy cone metric space where the range of Fuzzy cone metric space is considered as $E^*(I)$ where $E$ is a given
Banach space and $E^*(I)$ denotes the set of all non-negative fuzzy real numbers defined on $E$. Motivated by above all we introduce an order in Fuzzy cone metric space and we obtain fixed point theorems in this chapter.

6.2 PRELIMINARIES

The definition and results quoted here are according to [97]

**Definition: 6.2.1:** Let $(X, M, \ast)$ is said to be Fuzzy cone metric space if $P$ is a cone in $E$, $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times \text{int}(P)$ satisfying the following conditions.

1. $M(x, y, t) > 0$

2. $M(x, y, t) = 1$ if and only if $x = y$

3. $M(x, y, t) = M(y, x, t)$

4. $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$

5. $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$

*If $E = R, P = [0, \infty)$and $a \ast b = ab$, then every Fuzzy cone metric space become a Fuzzy metric space.*
Example: 6.2.1: Let $E = \mathbb{R}^2$. Then

$P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \subset E$ is a normal cone with normal constant $K = 1$. Let $X = \mathbb{R}$,

$a \ast b = ab$ and $M : X^2 \times \text{int}(P) \to [0, 1]$ defined by $M = \frac{1}{e^{|x-y|}}$

Theorem: 6.2.1: Let $E$ be a real Banach space, $P$ is a cone in $E$, $(X, M, \ast)$ is a Fuzzy cone metric space. Let $x_0 \in X$, define a relation

$\preceq$ on $X$ as follows

$x \preceq y \iff M(x_0, x, t) \leq M(x_0, y, t)$

for $x, y \in X$ is a partial order relation on $(X, M, \ast)$.

Proof:

(i) Reflexive

Let $x \in X$ then $M(x_0, x, t) = M(x_0, x, t)$ therefore $x \preceq x$

(ii) Antisymmetry

Let $x, y \in X$ such that $x \preceq y$ and $y \preceq x$, therefore

$M(x_0, x, t) \leq M(x_0, y, t)$

$M(x_0, y, t) \leq M(x_0, x, t)$

which implies $M(x_0, x, t) = M(x_0, y, t)$ therefore $x = y$.

(iii) Transitive

Let $x, y, z \in X$ such that $x \preceq y$ and $y \preceq x$

$M(x_0, x, t) \leq M(x_0, y, t)$

$M(x_0, y, t) \leq M(x_0, z, t)$

which implies
\[ M(x_0, x, t) \leq M(x_0, z, t) \]

Therefore \( x \preceq z \) hence the relation is transitive. Then the Fuzzy cone metric space \((X, M, \ast)\) together with partial order relation \( \preceq \) is an Ordered fuzzy cone metric space \((X, M, \ast, \preceq)\).

\[ \square \]

**Example: 6.2.2:** Let \( E = R^2, P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \) is a normal cone with normal constant \( K = 1 \). Let \( X = R, a \ast b = ab \) and \( M : X^2 \times \text{int}(P) \to [0,1] \) defined by \( M(x, y, t) = \frac{1}{e^{\frac{|x-y|}{\|t\|}}} \) for all \( x, y \in X \) and \( 0 \ll t \). Fix \( x_0 \in X \). Define \( \preceq \) on \( X \) by \( x \preceq y \), \((x, y \in X)\) if \( M(x_0, x, t) \leq M(x_0, y, t) \)

**Proof:**

(i) Reflexive

\[ M(x_0, x, t) = \frac{1}{e^{\frac{|x_0-x|}{\|t\|}}} = M(x_0, x, t) \]

Therefore \( x \preceq x \) for all \( x \in X \)

(ii) Antisymmetric

Let \( x, y \in X \) such that \( x \preceq y \) and \( y \preceq x \). Then

\[ x \preceq y \]

\[ M(x_0, x, t) \leq M(x_0, y, t) \]

\[ \frac{1}{e^{\frac{|x_0-x|}{\|t\|}}} \leq \frac{1}{e^{\frac{|x_0-y|}{\|t\|}}} \]

\[ e^{\frac{|x_0-x|}{\|t\|}} \geq e^{\frac{|x_0-y|}{\|t\|}} \]

\[ \frac{|x_0-x|}{\|t\|} \geq \frac{|x_0-y|}{\|t\|} \]

\[ |x_0-x| \geq |x_0-y| \]

Similarly \( y \preceq x \) implies \( |x_0-y| \geq |x_0-x| \). Therefore

\[ |x_0-x| = |x_0-y| \]

Hence \( x = y \)
(iii) Transitive

Let \( x, y, z \in X \) such that \( x \preceq y, y \preceq z \).

This implies \( M(x_0, x, t) \leq M(x_0, y, t) \) and \( M(x_0, y, t) \leq M(x_0, z, t) \)
which implies \( M(x_0, x, t) \leq M(x_0, z, t) \). Therefore \( x \preceq z \). Hence \( X \)
is an Ordered fuzzy cone metric space.

\[ \Box \]

**Example: 6.2.3:** Let \( E = R, P = \{ k \in R : k \geq 0 \} \) be a cone in \( E, X = N, a \ast b = ab, M : X^2 \times \text{int}(P) \to [0, 1] \) defined by

\[
M(x, y, t) = \begin{cases} 
\frac{x}{y} & \text{if } x \leq y \\
\frac{y}{x} & \text{if } y \leq x 
\end{cases}
\]

for all \( x, y \in X \) and \( t \gg 0 \). Then \((X, M, \ast)\) is a Fuzzy cone metric space. Fix \( x_0 \in X \), define a relation \( \preceq \) as follows. For \( x, y \in X \),

\( x \preceq y \) if \( M(x_0, x, t) \leq M(x_0, y, t) \)

**Proof:**

(i) Reflexive

Let \( x \in X \) if \( x_0 \leq x \) then

\[
M(x_0, x, t) = \frac{x_0}{x} = M(x_0, x, t)
\]

If \( x_0 \geq x \) then \( M(x_0, x, t) = \frac{x}{x_0} = M(x_0, x, t) \)

Therefore \( x \preceq x \) for all \( x \in X \)

(ii) Antisymmetric

let \( x, y \in X \) such that \( x \preceq y \) and \( y \preceq x \)

\[
M(x_0, x, t) \leq M(x_0, y, t)
\]

\[
\frac{x_0}{x} \leq \frac{x_0}{y} \quad (\text{if } x_0 \leq x, y)
\]

\[
\frac{1}{x} \leq \frac{1}{y}
\]

\[
x \geq y
\]
Again

\[
M(x_0, y, t) \leq M(x_0, x, t)
\]
\[
\frac{x_0}{y} \leq \frac{x_0}{x} \quad (\text{if } x_0 \leq x, y)
\]
\[
\frac{1}{y} \leq \frac{1}{x}
\]
\[
y \geq x
\]

Hence we have \( x = y \). Similarly we can prove if \( x_0 \geq x, y \)

\((iii)\) Transitive

Let \( x, y, z \in X \) such that \( x \preceq y, y \preceq z \). This implies

\[
M(x_0, x, t) \leq M(x_0, y, t) \quad \text{and} \quad M(x_0, y, t) \leq M(x_0, z, t) .
\]

Hence

\[
M(x_0, x, t) \leq M(x_0, z, t) .
\]

Therefore \( x \preceq z \). Hence \( X \) is an Ordered fuzzy cone metric space. \( \square \)

**Example: 6.2.4:** Let \( E = R^2 \) be a real Banach space,

\( P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \) is a normal cone with normal constant \( K = 1 \). Let \( X = R \), \( a \ast b = ab \) and \( M : X^2 \times \text{int}(P) \to [0, 1] \) defined by

\[
M(x, y, t) = \frac{\min\{x, y\} + \|t\|}{\max\{x, y\} + \|t\|} \quad \text{for all } x, y \in X, t \in \text{int}(P).
\]

Define a relation \( \preceq \) as follows. For \( x, y \in X \), \( x \preceq y \) if

\[
M(x_0, x, t) \leq M(x_0, y, t)
\]

**Proof:** (i) Reflexive

Let \( x \in X \) if \( x_0 \leq x \) then

\[
M(x_0, x, t) = \frac{\min\{x_0, x\} + \|t\|}{\max\{x_0, x\} + \|t\|}
\]
\[
= \frac{x_0 + \|t\|}{x + \|t\|} \quad \text{if } (x_0 \leq x)
\]
\[
= M(x_0, x, t)
\]
If \( x_0 \geq x \) then
\[
M(x_0, x, t) = \frac{x + \| t \|}{x_0 + \| t \|} = M(x_0, x, t)
\]

Therefore \( x \preceq x \) for all \( x \in X \)

\( (ii) \) Antisymmetric

Let \( x, y \in X \) such that \( x \preceq y \) and \( y \preceq x \)

\[
\begin{align*}
M(x_0, x, t) &\leq M(x_0, y, t) \\
\min\{x_0, x\} + \| t \| &\leq \min\{x_0, y\} + \| t \| \\
\max\{x_0, x\} + \| t \| &\leq \max\{x_0, y\} + \| t \| \\
x_0 + \| t \| &\leq x_0 + \| t \| \\
\frac{1}{x + \| t \|} &\geq \frac{1}{y + \| t \|} \\
x + \| t \| &\geq y + \| t \| \\
\end{align*}
\]

Hence \( x \geq y \). Again

\[
\begin{align*}
M(x_0, y, t) &\leq M(x_0, x, t) \\
\min\{x_0, y\} + \| t \| &\leq \min\{x_0, x\} + \| t \| \\
\max\{x_0, y\} + \| t \| &\leq \max\{x_0, x\} + \| t \| \\
x_0 + \| t \| &\leq x_0 + \| t \| \\
\frac{1}{y + \| t \|} &\leq \frac{1}{x + \| t \|} \\
y + \| t \| &\geq x + \| t \| \\
\end{align*}
\]

Hence \( y \geq x \). Therefore \( x = y \). Similarly we can prove the result if \( x, y \geq x_0 \)

\( (iii) \) Transitive
Let \( x, y, z \in X \) such that \( x \preceq y, y \preceq z \). This implies
\[
M(x_0, x, t) \leq M(x_0, y, t) \quad \text{and} \quad M(x_0, y, t) \leq M(x_0, z, t).
\]
Hence \( M(x_0, x, t) \leq M(x_0, z, t) \). Therefore \( x \preceq z \). So \( X \) is an
Ordered fuzzy cone metric space. \( \square \)

**Example: 6.2.5:** Let \( E = R \) is a real Banach space,
\( P = \{ k \in R : k \geq 0 \} \) is a normal cone in \( E \) with \( K = 1 \). Let
\( X = R, a * b = ab \) and \( M : X^2 \times \text{int}(P) \to [0, 1] \) defined by
\[
M(x, y, t) = \frac{t}{t + |x - y|}
\]
for all \( x, y \in X, \ t \gg 0 \). Fix \( x_0 \in X \). Let
\( x, y \in X, \ x \preceq y \) if \( M(x_0, x, t) \leq M(x_0, y, t) \)

**Proof:** (i) Reflexive
Let \( x \in X, \ M(x_0, x, t) = \frac{t}{t + |x_0 - x|} = M(x_0, x, t) \). Therefore \( x \preceq x \)

(ii) Antisymmetric
let \( x, y \in X \) such that \( x \preceq y \) and \( y \preceq x \)
\[
M(x_0, x, t) \leq M(x_0, y, t)
\]
\[
\frac{t}{t + |x_0 - x|} \leq \frac{t}{t + |x_0 - y|}
\]
\[
t + |x_0 - x| \geq t + |x_0 - y|
\]
\[
|x_0 - x| \geq |x_0 - y|
\]
Again
\[
M(x_0, y, t) \leq M(x_0, x, t)
\]
\[
\frac{t}{t + |x_0 - y|} \leq \frac{t}{t + |x_0 - x|}
\]
\[
t + |x_0 - y| \geq t + |x_0 - x|
\]
\[
|x_0 - y| \geq |x_0 - x|
\]
Therefore $|x_0 - y| = |x_0 - x|$ which implies $x = y$

(iii) Transitive

Let $x, y, z \in X$ such that $x \preceq y$, $y \preceq z$

This implies $M(x_0, x, t) \leq M(x_0, y, t)$ and $M(x_0, y, t) \leq M(x_0, z, t)$.

Hence $M(x_0, x, t) \leq M(x_0, z, t)$. So $x \preceq z$. Hence $X$ is an Ordered fuzzy cone metric space.

**Proposition: 6.2.1:** Let $(X, M, *, \preceq)$ be an Ordered fuzzy cone metric space. If $f : X \rightarrow X$ is a continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then $f$ has a fixed point in $X$.

**Proof:** Given $x_0 \in X$, and $x_0 \preceq fx_0$. Define $x_n = fx_{n-1}$, for $n = 1, 2, \ldots$. Therefore

$x_1 = fx_0, x_0 \preceq fx_0, fx_0 \preceq fx_1, fx_1 \preceq fx_2, \ldots$.

That is $x_1 \preceq x_2 \preceq x_3 \preceq \ldots$. Therefore $\{x_n\}$ is non decreasing.

Hence $M(x_0, x_1, t) \leq M(x_0, x_2, t) \leq M(x_0, x_3, t) \leq \ldots \ldots$.

Thus $\{M(x_0, x_n, t)\}$ is increasing and bounded.

Hence $\{M(x_0, x_n, t)\}$ converges. Therefore $\lim_{n \to \infty} M(x_0, x_n, t) = 1$.

Therefore $x_n \rightarrow x_0$. Hence $f(x_n) \rightarrow f(x_0)$. Therefore $x_{n+1} \rightarrow f(x_0)$.

Hence $f(x_0) = x_0$, that is $x_0$ is a fixed point of $f$.  

6.3 FIXED POINT THEOREMS FOR CONTINUOUS MAP IN ORDERED FUZZY CONE METRIC SPACE

**Theorem: 6.3.1:** Let $(X, M, *, \preceq)$ be an Ordered fuzzy cone metric space which is complete and $P$ is a normal cone with normal
constant $K$. Let $f : X \to X$ is continuous, nondecreasing, $x_0 \in X$ such that $x_0 \preceq fx_0$ and satisfies the condition

$$M(fx, fy, t) \geq \min\{M(x, fx, t), M(y, fy, t), M(x, y, t), M(y, fx, t), M(x, fy, t)\} \quad (6.1)$$

Then $f$ has a unique fixed point.

**Proof:**

Given $x_0 \in X$ such that $x_0 \preceq fx_0$. Define $\{x_n\}$ as $x_n = f(x_{n-1}), n = 1, 2, \ldots$. Then

$x_0 \preceq fx_0 = x_1 \preceq x_2 \preceq x_3 \ldots \ldots \because f$ is non decreasing. We shall prove $\{x_n\}$ is a Cauchy sequence by induction method. We have

Case (1)

$$M(x_2, x_1, t) = M(fx_1, fx_0, t) \geq M(x_1, fx_1, t) \quad \text{by inequality (6.1)}$$

$$= M(x_1, x_2, t) \geq M(x_2, x_0, \frac{t}{2}) \cdot M(x_0, x_1, \frac{t}{2}) \geq M(x_0, x_0, \frac{t}{2}) \cdot M(x_0, x_0, \frac{t}{2}) = 1 \cdot 1 = 1$$
Case (2)

\[ M(x_2, x_1, t) = M(f x_1, f x_0, t) \]
\[ \geq M(x_0, x_1, t) \quad \text{by inequality (6.1)} \]
\[ = M(x_1, x_0, t) \]
\[ \geq M(x_0, x_0, t) \]
\[ = 1 \]

Case (3)

\[ M(x_2, x_1, t) = M(f x_1, f x_0, t) \]
\[ \geq M(x_1, x_0, t) \quad \text{by inequality (6.1)} \]
\[ = M(x_0, x_1, t) \]
\[ M(x_2, x_1, t) \geq M(x_0, x_0, t) \]
\[ = 1 \]

Case (4)

\[ M(x_2, x_1, t) = M(f x_1, f x_0, t) \]
\[ \geq M(x_0, f x_1, t) \quad \text{by inequality (6.1)} \]
\[ = M(x_0, x_2, t) \]
\[ = M(x_2, x_0, t) \]
\[ \geq M(x_0, x_0, t) \]
\[ = 1 \]
Case (5)

\[ M(x_2, x_1, t) = M(f x_1, f x_0, t) \]
\[ \geq M(x_1, f x, t) \quad \text{by inequality (6.1)} \]
\[ = M(x_1, x_1, t) \]
\[ = 1 \]

Thus the result is true for \( n = 2 \). Assume \( M(x_m, x_{m-1}, t) \geq 1 \). We shall prove the result is true for \( n = m + 1 \)

Case (1)

\[ M(x_{m+1}, x_m, t) = M(f x_m, f x_{m-1}, t) \]
\[ \geq M(x_m, f x_m, t) \quad \text{by inequality (6.1)} \]
\[ = M(x_m, x_{m+1}, t) \]
\[ = M(x_{m+1}, x_m, t) \]
\[ \geq M(x_{m+1}, x_0, t) * M(x_0, x_m, t) \]
\[ \geq M(x_0, x_0, t) * M(x_0, x_0, t) \]
\[ = 1 * 1 = 1 \]

Case (2)

\[ M(x_{m+1}, x_m, t) = M(f x_m, f x_{m-1}, t) \]
\[ \geq M(x_{m-1}, f x_{m-1}, t) \quad \text{by (6.1)} \]
\[ = M(x_{m-1}, x_m, t) \]
\[ \geq M(x_m, x_{m-1}, t) \]
\[ \geq 1 \]
Case (3)

\[ M(x_{m+1}, x_m, t) = M(fx_m, fx_{m-1}, t) \]
\[ \geq M(x_{m-1}, x_m, t) \text{ by inequality (6.1)} \]
\[ = M(x_m, x_{m-1}, t) \]
\[ \geq 1 \]

Case (4)

\[ M(x_{m+1}, x_m, t) = M(fx_m, fx_{m-1}, t) \]
\[ \geq M(x_{m-1}, fx_m, t) \text{ by inequality (6.1)} \]
\[ = M(x_{m-1}, x_{m+1}, t) \]
\[ = M(x_{m+1}, x_{m-1}, t) \]
\[ \geq M(x_{m+1}, x_0, \frac{t}{2}) \ast M(x_0, x_{m-1}, \frac{t}{2}) \]
\[ \geq M(x_0, x_0, \frac{t}{2}) \ast M(x_0, x_0, \frac{t}{2}) \]
\[ = 1 \ast 1 = 1 \]

Case (5)

\[ M(x_{m+1}, x_m, t) = M(fx_m, fx_{m-1}, t) \]
\[ \geq M(x_m, fx_{m-1}, t) \text{ by inequality (6.1)} \]
\[ = M(x_m, x_m, t) \]
\[ = 1 \]

Hence by induction \( M(x_{n+1}, x_n, t) \geq 1 \). But \( M(x_{n+1}, x_n, t) \) cannot be greater than 1. Therefore \( M(x_{n+1}, x_n, t) = 1 \).
Now we shall prove $M(x_{n+p}, x_n, t) = 1$ for every positive integer $p$. Proof is based on induction. We have proved the result is true for $p = 1$. Assume the result is true for $p = k$. That is $M(x_{n+k}, x_n, t) = 1$. We have

$$M(x_{n+k+1}, x_n, t) \geq M(x_{n+k+1}, x_{n+k}, \frac{t}{2}) \cdot M(x_{n+k}, x_n, \frac{t}{2}) \geq 1 \cdot 1 = 1$$

Therefore $M(x_{n+k+1}, x_n, t) = 1$. Hence by induction $M(x_{n+p}, x_n, t) = 1$. Therefore $M(x_{n+p}, x_n, t) \to 1$ as $n, p \to \infty$

So $\{x_n\}$ is a Cauchy sequence in $X$. But $X$ is complete. Therefore there exist an $x \in X$ such that $x_n \to x$. Again $f$ is continuous. Hence $f(x_n) \to f(x)$. Therefore $f(x) = x$. Hence $x$ is a fixed point of $f$.

To prove the uniqueness. Let $y \in X$ such that $f(y) = y$.

Case (1)

$$M(x, y, t) = M(fx, fy, t) \geq M(x, fx, t) \text{ by inequality (6.1)}$$

$$= M(x, x, t) = 1$$

Therefore $x = y$.

Case (2)

$$M(x, y, t) = M(fx, fy, t) \geq M(y, fy, t) \text{ by inequality (6.1)}$$

$$= M(y, y, t) = 1$$
Therefore $x = y$.

Case (3)

\[ M(x, y, t) = M(fx, fy, t) \]
\[ \geq M(x, y, t) \quad \text{by inequality (6.1)} \]
\[ \geq M(x, x_0, \frac{t}{2}) \cdot M(x_0, y, \frac{t}{2}) \]
\[ \geq M(x_0, x_0, \frac{t}{2}) \cdot M(x_0, x_0, \frac{t}{2}) \]
\[ = 1 \cdot 1 = 1 \]

Therefore $x = y$.

Case (4)

\[ M(x, y, t) = M(fx, fy, t) \]
\[ \geq M(y, fx, t) \quad \text{by inequality (6.1)} \]
\[ = M(y, x, t) \]
\[ = M(x, y, t) \]
\[ \geq M(x, x_0, \frac{t}{2}) \cdot M(x_0, y, \frac{t}{2}) \]
\[ \geq M(x_0, x_0, \frac{t}{2}) \cdot M(x_0, x_0, \frac{t}{2}) \]
\[ = 1 \cdot 1 = 1 \]

Therefore $x = y$.

Case (5)

\[ M(x, y, t) = M(fx, fy, t) \]
\[ \geq M(x, fy, t) \quad \text{by inequality (6.1)} \]
\[ = M(x, y, t) \]
Therefore \( x = y \). Thus \( f \) has a unique fixed point.

**Example: 6.3.1:** Let \( X = N, E = R, P = \{ k \in R : k \geq 0 \} \) is a normal cone in \( E \), \( a * b = ab \) and \( M : X^2 \times \text{int}(P) \to [0, 1] \) defined by

\[
M(x, y, t) = \begin{cases} 
\frac{x}{y} & \text{if } x \leq y \\
\frac{y}{x} & \text{if } y \leq x 
\end{cases}
\]

for any \( x, y \in X \). Then \( X \) is an Ordered fuzzy cone metric space.

Define \( f : X \to X \) by \( f(x) = x^2, x \in X \). Then \( f \) is continuous and non decreasing. Let \( x_0 = 1 \), then \( fx_0 = 1 \)

\[
M(x_0, x_0, t) = M(1, 1, t) = 1
\]

\[
M(x_0, fx_0, t) = M(1, 1, t) = 1
\]

Therefore \( M(x_0, x_0, t) \leq M(x_0, fx_0, t) \). Hence \( x_0 \preceq fx_0 \)

Suppose \( x \leq y \)

\[
M(fx, fy, t) = M(x^2, y^2, t) = \frac{x^2}{y^2} = \frac{x}{y}
\]

\[
M(x, y, t) = \frac{x}{y}
\]

\[
M(x, fx, t) = M(x, x^2, t) = \frac{x}{x^2} = \frac{1}{x}
\]

\[
M(y, fy, t) = M(y, y^2, t) = \frac{y}{y^2} = \frac{1}{y}
\]

\[
M(x, fy, t) = M(x, y^2, t) = \frac{x}{y^2}
\]
\[
M(y, fx, t) = M(y, x^2, t) = \begin{cases} 
\frac{x^2}{y} & \text{if } x^2 \leq y \\
\frac{y}{x^2} & \text{if } y \leq x^2 
\end{cases}
\]

Therefore

\[
\min\{M(x, y, t), M(x, fx, t), M(y, fy, t), M(x, fy, t), M(y, fx, t)\} = \min\{\frac{x}{y}, \frac{1}{x}, \frac{x}{y^2}, \frac{x^2}{y}\}
\]

\[
= \frac{x}{y^2}
\]

Again \(\frac{x^2}{y^2} \geq \frac{x}{y^2}\)

Therefore

\[
M(fx, fy, t) \geq \min\{M(x, y, t), M(x, fx, t), M(y, fy, t), M(x, fy, t), M(y, fx, t)\}
\]

Next suppose \(x \geq y\)

\[
M(fx, fy, t) = M(x^2, y^2, t)
\]

\[
= \frac{y^2}{x^2}
\]

\[
M(x, y, t) = \frac{y}{x}
\]

\[
M(x, fx, t) = M(x, x^2, t)
\]

\[
= \frac{x}{x^2} = \frac{1}{x}
\]

\[
M(y, fy, t) = M(y, y^2, t)
\]

\[
= \frac{y}{y^2} = \frac{1}{y}
\]

\[
M(y, fx, t) = M(y, x^2, t)
\]

\[
= \frac{y}{x^2}
\]
\[ M(x, fy, t) = M(x, y^2, t) = \begin{cases} \frac{y^2}{x} & \text{if } y^2 \leq x \\ \frac{x}{y^2} & \text{if } y^2 \geq x \end{cases} \]

Therefore

\[
\min \{M(x, y, t), M(x, fx, t), M(y, fy, t), M(x, fy, t) \}
= \min \left\{ \frac{y^2}{x}, \frac{1}{x}, \frac{1}{y}, \frac{y^2}{x^2}, \frac{x}{y^2} \right\}
= \frac{y}{x^2}
\]

Again \( \frac{y^2}{x^2} \geq \frac{y}{x^2} \)

Therefore

\[
M(fx, fy, t) \geq \min \{M(x, y, t), M(x, fx, t), M(y, fy, t), M(x, fy, t), M(y, fx, t) \}
\]

Thus \( f \) satisfies all the conditions of the theorem. Therefore \( f \) has a fixed point. Clearly \( x = 1 \) is a fixed point of \( f \).

**Theorem: 6.3.2:** Let \( (X, M, *, \preceq) \) be an Ordered fuzzy cone metric space, \( P \) is a cone in \( E \), Let \( x_0 \in X \) such that \( x_0 \preceq Tx_0 \), where \( T : X \to X \) is continuous, nondecreasing and satisfies

\[
M(T^n x, T^n y, t) \geq \min \{M(x, y, t), M(T^n x, x, t), M(T^n x, y, t), M(x, T^n y, t) \} \quad (6.2)
\]

for every \( x, y \in X, n \in N \). Then \( T \) has a unique fixed point.

**Proof:**
Given $x_0 \in X$ and $x_0 \preceq Tx_0$, $T$ is nondecreasing

So $x_0 \preceq Tx_0 \preceq T^2x_0 \preceq ... \preceq T^nx_0 \preceq ...$

Let $x_n = T^nx_0$, $n \in \mathbb{N}$. Then $x_0 \preceq x_1 \preceq x_2 \preceq ...$

Now we shall prove \{\{x_n\}\} is a Cauchy sequence.

Case (1)

$$M(x_{n+p}, x_n, t) = M(T^{n+p}x_0, T^nx_0, t)$$
$$= M(T^n(T^px_0), T^nx_0, t)$$
$$\geq M(T^nTx_0, x_0, t)$$
$$\geq M(x_0, x_0, t) = 1$$

Case (2)

$$M(x_{n+p}, x_n, t) = M(T^{n+p}x_0, T^nx_0, t)$$
$$= M(T^n(T^px_0), T^nx_0, t)$$
$$\geq M(T^n(T^px_0), T^px_0, t)$$
$$\geq M(T^{n+p}x_0, x_0, t) \frac{t}{2}) * M(x_0, T^px_0, \frac{t}{2})$$
$$\geq M(x_0, x_0, t) * M(x_0, x_0, t)$$
$$= 1 * 1 = 1$$

Case (3)

$$M(x_{n+p}, x_n, t) = M(T^{n+p}x_0, T^nx_0, t)$$
$$= M(T^n(T^px_0), T^nx_0, t)$$
$$\geq M(T^n(T^px_0), x_0, t)$$
\begin{align*}
M(x_{n+p}, x_n, t) &= M(T^{n+p}x_0, x_0, t) \\
&\geq M(x_0, x_0, t) = 1
\end{align*}

Case (4)

\begin{align*}
M(x_{n+p}, x_n, t) &= M(T^{n+p}x_0, T^nx_0, t) \\
&= M(T^n(T^px_0), T^nx_0, t) \\
&\geq M(T^px_0, T^nx_0, t) \\
&\geq M(T^px_0, x_0, t)*M(x_0, T^nx_0, t) \\
&\geq M(x_0, x_0, t)*M(x_0, x_0, t) \\
&= 1*1 = 1
\end{align*}

Thus in all possible cases \(M(x_{n+p}, x_n, t) = 1\).

Hence \(\lim\limits_{n,p \to \infty} M(x_{n+p}, x_n, t) = 1\). Therefore \(\{x_n\}\) is a Cauchy sequence in \(X\). But \(X\) is complete. Hence there exist an \(x \in X\) such that \(x_n \to x\). Given \(T\) is continuous. Therefore \(Tx_n \to Tx\). That is \(x_{n+1} \to Tx\). Therefore \(Tx = x\). Hence \(x\) is a fixed point of \(T\).

Next we shall prove uniqueness. Let \(x, y \in X\) such that \(Tx = x\) and \(Ty = y\). Then \(T^nx = x\) and \(T^ny = y\) for every \(n \in N\)

\begin{align*}
M(x, y, t) &= M(T^n x, T^n y, t) \\
&\geq M(T^n x, x_0, t)*M(x_0, T^n y, t) \\
&= M(x, x_0, t)*M(x_0, y, t)
\end{align*}

Therefore
\[ M(x, y, t) \geq M(x_0, x_0, t) \star M(x_0, x_0, t) \]
\[ = 1 \star 1 = 1 \]

Therefore \( x = y \). \( \square \)

**Example: 6.3.2:** Let \( X = N, E = R, P = \{ k \in R : k \geq 0 \} \) is a normal cone in \( E \), \( a \star b = ab \) and \( M : X^2 \times \text{int}(P) \to [0,1] \) defined by

\[ M(x, y, t) = \begin{cases} 
\frac{x}{y} & \text{if } x \leq y \\
\frac{y}{x} & \text{if } y \leq x
\end{cases} \]

for all \( x, y \in X \) and \( t \gg 0 \). Then \( X \) is an Ordered fuzzy cone metric space. Define \( T : X \to X \) by \( T(x) = x^2, x \in X \). Then \( T \) is continuous and non decreasing. Let \( x_0 = 1 \), then \( Tx_0 = 1 \)

\[ M(x_0, x_0, t) = M(1, 1, t) = 1 \]
\[ M(x_0, Tx_0, t) = M(1, 1, t) = 1 \]

Therefore \( M(x_0, x_0, t) \leq M(x_0, Tx_0, t) \). Hence \( x_0 \preceq Tx_0 \)

Suppose \( x \leq y \)

\[ M(T^n x, T^n y, t) = M(x^{2n}, y^{2n}, t) \]
\[ = \frac{x^{2n}}{y^{2n}} \]
\[ M(x, y, t) = \frac{x}{y} \]
\[ M(T^n x, x, t) = M(x^{2n}, x, t) \]
\[ = \frac{x}{x^{2n}} \]

And
\[ M(x, T^n y, t) = M(x, y^n, t) \]
\[ = \frac{x}{y^{2n}} \]

\[ M(T^n x, y, t) = M(x^{2n}, y, t) = \begin{cases} \frac{x^{2n}}{y} & \text{if } x^{2n} \leq y \\ \frac{y}{x^{2n}} & \text{if } x^{2n} \geq y \end{cases} \]

Therefore

\[
\min\{M(x, y, t), M(T^n x, x, t), M(T^n x, y, t), M(x, T^n y, t)\} = \min\\{\frac{x}{y}, \frac{x}{x^{2n}}, \frac{x^{2n}}{y}, \frac{x}{y^{2n}}\} = \frac{x}{y^{2n}}
\]

Again \(\frac{x^{2n}}{y^{2n}} \geq \frac{x}{y^{2n}}\)

Therefore

\[
M(T^n x, T^n y, t) \geq \min\{M(x, y, t), M(x, T^n x, t), M(x, T^n y, t), M(y, T^n x, t)\}
\]

Next suppose \(x \geq y\)

\[
M(T^n x, T^n y, t) = M(x^{2n}, y^{2n}, t) \]
\[ = \frac{y^{2n}}{x^{2n}} \]
\[
M(x, y, t) = \frac{y}{x} 
\]
\[
M(x, T^n x, t) = M(x, x^{2n}, t) \]
\[ = \frac{x}{x^{2n}} \]

Again
\[ M(x, T^n y, t) = M(x, y^{2n}, t) = \begin{cases} 
\frac{y^{2n}}{x} & \text{if } y^{2n} \leq x \\
\frac{x}{y^{2n}} & \text{if } y^{2n} \geq x
\end{cases} \]

And

\[ M(y, T^n x, t) = M(y, x^{2n}, t) = \frac{y}{x^{2n}} \]

\[ M(x, T^n y, t) = M(x, y^{2n}, t) = \begin{cases} 
\frac{y^{2n}}{x} & \text{if } y^{2n} \leq x \\
\frac{y}{x^{2n}} & \text{if } y^{2n} \geq x
\end{cases} \]

Therefore

\[ \min\{M(x, y, t), M(x, T^n x, t), \\
M(x, T^n y, t), M(y, T^n x, t)\} = \min\left\{\frac{y}{x}, \frac{x}{x^{2n}}, \frac{y}{x^{2n}}, \frac{y^{2n}}{x}\right\} = \frac{y}{x^{2n}} \]

Again \( \frac{y^{2n}}{x^{2n}} \geq \frac{y}{x^{2n}} \)

Therefore

\[ M(T^n x, T^n y, t) \geq \min\{M(x, y, t), M(x, T^n x, t), \\
M(x, T^n y, t), M(y, T^n x, t)\} \]

Thus \( f \) satisfies all the conditions of the theorem. Therefore \( f \) has a fixed point. Clearly \( x = 1 \) is a fixed point of \( f \).
6.4 COMMON FIXED POINT THEOREMS FOR CONTINUOUS MAP IN ORDERED FUZZY CONE METRIC SPACE

Theorem: 6.4.1: Let \((X, M, *, \preceq)\) is an Ordered fuzzy cone metric space which is complete. \(f, g : X \to X\) are continuous, nondecreasing and satisfies the conditions (i) \(x_0 \preceq fx_0, x_0 \preceq gx_0\), where \(x_0 \in X\) (ii) \(f, g\) commute and (iii) \(M(fx, gy, t) \geq \min\{M(x, fx, t), M(y, gy, t)\}\), for \(x, y \in X\). Then \(f\) and \(g\) have a common fixed point.

Proof: Let \(x_0 \in X\) be arbitrary. Define \(\{x_n\}\) as follows \(x_{2n+1} = fx_{2n}\) and \(x_{2n+2} = gx_{2n+1}\) for \(n = 0, 1, 2, \ldots\). Now we shall prove \(\{x_n\}\) is a Cauchy sequence by induction method.

Case (1)

\[
M(x_2, x_1, t) = M(gx_1, fx_0, t) = M(fx_0, gx_1, t) \geq M(x_0, fx_0, t) \geq M(x_0, x_0, t) = 1
\]

Case (2)

\[
M(x_2, x_1, t) = M(fx_0, gx_1, t)
\]

That is

\[
M(x_2, x_1, t) \geq M(x_1, gx_1, t) \geq M(x_1, x_1, t) = 1
\]
Because we have $x_0 \leq gx_0$ and $f$ is continuous.

Therefore $fx_0 \leq fgx_0$.

Again $f$, $g$ commute. Hence $x_1 \leq gfx_0$. That is $x_1 \leq gx_1$.

Assume $M(x_k, x_{k-1}, t) \geq 1$ for some positive integer $k$. We shall prove $M(x_{k+1}, x_k, t) \geq 1$.

Assume $k$ is even. Let $k = 2n$ where $n$ is an integer.

$$M(x_{k+1}, x_k, t) = M(x_{2n+1}, x_{2n}, t)$$
$$= M(fx_{2n}, gx_{2n-1}, t)$$
$$= M(f(gx_{2n-1}), g(fx_{2n-2}), t)$$

Let $u = gx_{2n-1}$, $v = fx_{2n-2}$ then $x_0 \leq u$, $x_0 \leq v$.

Therefore $M(x_{k+1}, x_k, t) = M(fu, gv, t)$

Case (1)

$$M(x_{k+1}, x_k, t) = M(fu, gv, t)$$
$$\geq M(u, fu, t)$$
$$\geq M(u, x_0, \frac{t}{2}) \ast M(x_0, fu, \frac{t}{2})$$
$$\geq M(x_0, x_0, \frac{t}{2}) \ast M(x_0, x_0, \frac{t}{2})$$
$$= 1 \ast 1 = 1$$

Case (2)

$$M(x_{k+1}, x_k, t) = M(fu, gv, t)$$
$$\geq M(v, gv, t)$$
Therefore

\[ M(x_{k+1}, x_k, t) \geq M(v, x_0, \frac{t}{2}) \times M(x_0, g v, \frac{t}{2}) \]
\[ \geq M(x_0, x_0, \frac{t}{2}) \times M(x_0, x_0, \frac{t}{2}) \]
\[ = 1 \times 1 = 1 \]

Next suppose \( k \) is odd. Let \( k = 2n + 1 \).

Case (1)

\[ M(x_{k+1}, x_k, t) = M(x_{2n+2}, x_{2n+1}, t) \]
\[ = M(g x_{2n+1}, f x_{2n}, t) \]
\[ \geq M(g(f x_{2n}), f(g x_{2n-1}), t) \]

Let \( w = f x_{2n} \) and \( p = g x_{2n-1} \). Therefore

\[ M(x_{k+1}, x_k, t) = M(gw, fp, t) \]
\[ = M(fp, gw, t) \]
\[ \geq M(p, fp, t) \]
\[ \geq M(p, x_0, \frac{t}{2}) \times M(x_0, fp, \frac{t}{2}) \]
\[ \geq M(x_0, x_0, \frac{t}{2}) \times M(x_0, x_0, \frac{t}{2}) \]
\[ = 1 \times 1 = 1 \]

Case (2)

\[ M(x_{k+1}, x_k, t) = M(fp, gw, t) \]
\[ \geq M(w, gw, t) \]
Therefore

\[
M(x_{k+1}, x_k, t) \geq M(w, x_0, \frac{t}{2}) \ast M(x_0, gw, \frac{t}{2}) \\
\geq M(x_0, x_0, \frac{t}{2}) \ast M(x_0, x_0, \frac{t}{2}) \\
= 1 \ast 1 = 1
\]

Therefore \(M(x_{k+1}, x_k, t) \geq 1\) for every positive integer \(k\).
Now we shall prove \(\{x_n\}\) is a Cauchy sequence. It is enough to prove
\(M(x_{n+p}, x_n, t) \to 1\) as \(n, p \to \infty\). We shall prove \(M(x_{n+p}, x_n, t) = 1\) by induction on \(P\).
When \(p = 1\), \(M(x_{n+1}, x_n, t) = 1\) which we have proved.
Assume the result is true for \(p = k\). Therefore \(M(x_{n+k}, x_n, t) = 1\).
Next we shall prove \(M(x_{n+k+1}, x_n, t) = 1\).
Suppose \(n, n + k\) are even and \(n = 2r, n + k = 2m\) where \(m, r\) are integers.

\[
M(x_{n+k+1}, x_n, t) = M(x_{2m+1}, x_{2r}, t) \\
= M(f(x_{2m}), g(x_{2r-1}), t) \\
= M(fgx_{2m-1}, gfx_{2r-2}, t)
\]

Let \(\theta = gx_{2m-1}, \phi = fx_{2r-2}\). Then \(x_0 \preceq \theta, x_0 \preceq \phi\). Therefore
\(M(x_{n+k+1}, x_n, t) = M(f\theta, g\phi, t)\). As above we can prove
\(M(x_{n+k+1}, x_n, t) \geq 1\) in all possible cases. Therefore by mathematical induction
\(M(x_{n+p}, x_n, t) = 1\). Hence \(\{x_n\}\) is a Cauchy sequence in \(X\). But \(X\) is complete so there exist an \(x \in X\) such that \(x_n \to x\). Continuity
of $f$ and $g$ implies $fx_{2n+1} \rightarrow fx$, $gx_{2n} \rightarrow gx$. Thus $fx = gx = 1$.

Therefore $x$ is a common fixed point of $f$ and $g$. By a similar proof we can prove this common fixed point is unique. □

**Example: 6.4.1:** Let $X = N$ is an Ordered fuzzy cone metric space with $E = R$, $P = \{r \in R : r \geq 0\}$, $a \ast b = ab$ and

$M : X^2 \times \text{int}(P) \rightarrow [0,1]$ defined by $M(x,y,t) = \frac{1}{e^{-\frac{|x-y|}{\|t\|}}}$, $x,y \in N$, $t \in \text{int}P$. Define $f,g : X \rightarrow X$ by $f(x) = x^2$, $g(x) = 2x - 1$.

Take $x_0 = 1$ then $f(x_0) = 1$, $g(x_0) = 1$. Therefore $M(x_0,x_0,t) = 1, M(x_0,fx_0,t) = 1, M(x_0,gx_0,t) = 1$. Hence $x \preceq fx_0$ and $x \preceq gx_0$.

\[
M(fx,gx,t) = M(x^2,2x-1,t)
\]
\[
= \frac{1}{e^{-\frac{|x^2-2x+1|}{\|t\|}}}
\]
\[
= \frac{1}{e^{-\frac{(x-1)^2}{\|t\|}}}
\]

$M(fx,x,t) = M(x^2,x,t)$
\[
= \frac{1}{e^{-\frac{x^2-x}{\|t\|}}}
\]
\[
= \frac{1}{e^{-\frac{|x-1|}{\|t\|}}}
\]

$M(gx,x,t) = \frac{1}{e^{-\frac{|2x-1-x|}{\|t\|}}}$
\[
= \frac{1}{e^{-\frac{|x-1|}{\|t\|}}}
\]

\[
\min\{M(fx,x,t), M(gx,x,t)\} = \min\{\frac{1}{e^{-\frac{|x-1|}{\|t\|}}}, \frac{1}{e^{-\frac{|x-1|}{\|t\|}}})\}
\]
\[
= \frac{1}{e^{-\frac{|x-1|}{\|t\|}}}
\]

Therefore
\[
\frac{1}{e^{(x-1)^2/\|t\|^2}} \geq \frac{1}{e^{\|x\| \|x-1\|/\|t\|}}
\]

\[
M(fx, gx, t) \geq \min\{M(fx, x, t), M(gx, x, t)\}
\]

Further \(f\) and \(g\) are continuous and nondecreasing. Thus by the theorem \(f\) and \(g\) has a common fixed point. Clearly \(x = 1\) is a common fixed point.
\[ \frac{1}{e^{\frac{(x-1)^2}{\|t\|}}} \geq \frac{1}{e^{\frac{|x|-1}{\|t\|}}} \]

\[ M(f^x, g^x, t) \geq \min\{M(f^x, x, t), M(g^x, x, t)\} \]

Further \( f \) and \( g \) are continuous and nondecreasing. Thus by the theorem \( f \) and \( g \) has a common fixed point. Clearly \( x = 1 \) is a common fixed point.