CHAPTER 2

FIXED POINT THEOREMS IN CONE METRIC SPACE

2.1 INTRODUCTION

The Recently discovered application of ordered topological vector space, normal cones and topical function in optimization theory have generate a lot of interest and research in ordered topological vector space. Recently, Huang and Zhang [53] introduced Cone metric space which is a generalization of metric space by replacing the real numbers with ordered Banach space and obtained some fixed point theorems of mappings satisfying different contractive condition with the assumption of normality of a cone. Subsequently various authors [2], [58], [68], [91], [98], etc have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones.

Posteriorly, some of the mentioned results were obtained by Sh. Rezapour and R. Hamlbarani [83] omitting the assumption of normality on the cone. On the other hand, a few authors obtained coincidence and common fixed point theorems, similar to the area in Abbas and Junck [2], but for a more general class of almost
contraction, by restricting the ambient space to the case of normal metric space. Motivated by [83] we obtain different fixed point theorems with generalized contractive conditions.

2.2 PRELIMINARIES

Here we recall some definition and results of [53] which are used in the subsequent results.

Let $E$ be real Banach space, $P$ be a cone in $E$ with $\text{int}P \neq \emptyset$ and $\leq$ is a partial ordering with respect to $P$.

**Definition: 2.2.1:** Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

1. $0 \leq d(x, y)$ for every $x, y \in X, d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for every $x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$

Then $(X, d)$ is a Cone metric space.

Clearly Cone metric space is a generalization of metric space.

**Lemma: 2.2.1:** Let $(X, d)$ be a Cone metric space. $P$ be a normal cone with normal constant $K$. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. 
Lemma: 2.2.2: Let $f$ and $g$ be weakly compatible self maps of a set $X$. If $f$ and $g$ have a unique point of coincidence $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

We generalize the results in [94], [99] by omitting the condition of normality.

2.3 FIXED POINT THEOREMS IN CONE METRIC SPACE

Theorem: 2.3.1: Let $(X, d)$ be a complete Cone metric space. Suppose that the sequence of mappings \( \{T_n\} : X \to X \) satisfy

\[
d(T_i^m x, T_j^m y) \leq \alpha_{i,j} d(x, y)
\]

for some positive integer $m$ and for all $i, j = 1, 2, \cdots$ and $x, y \in X$ where $\alpha_{i,j}$ and $k$ are constants with $0 < \alpha_{i,j} < k < 1$. Then the \( \{T_n\} \) has a unique common fixed point in $X$.

Proof: Let $x_0$ be an arbitrary point in $X$ and

\[
x_1 = T_1^mx_0, \ x_2 = T_2^mx_1, = \cdots
\]

Then $d(x_1, x_2) = d(T_1^mx_0, T_2^mx_1) \leq \alpha_{1,2} d(x_0, x_1)$. Again $d(x_2, x_3) = d(T_2^mx_1, T_3^mx_2) \leq \alpha_{2,3} d(x_1, x_2) \leq \alpha_{1,2} \alpha_{2,3} d(x_0, x_1)$ and so on. Therefore by induction

\[
d(x_n, x_{n+1}) \leq \prod_{i=1}^{n} \alpha_{i,i+1} d(x_0, x_1) < k^n d(x_0, x_1)
\]

So for $n \geq m$ we have
\[ d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \]
\[ \leq \prod_{i=1}^{n-1} \alpha_{i,i+1} d(x_0, x_1) + \cdots + \prod_{i=1}^{m} \alpha_{i,i+1} d(x_0, x_1) \]
\[ \leq (k^{n-1} + \cdots + k^m) d(x_0, x_1) \]
\[ \leq \frac{k^m}{1-k} d(x_0, x_1) \]
\[ d(x_n, x_m) \leq \frac{k^m}{1-k} d(x_0, x_1) \quad (2.2) \]

Let \( c \gg 0 \), then there exist a \( \delta > 0 \) such that \( c + N_\delta(0) \) is a subset of \( P \) where \( N_\delta(0) = \{ y \in E : \| y \| < \delta \} \). Since \( k < 1 \) there exist a positive integer \( N \) such that
\[ \| \frac{k^m}{1-k} d(x_0, x_1) \| < \delta, \text{ for every } m \geq N. \]

Therefore
\[ \frac{k^m}{1-k} d(x_0, x_1) \in N_\delta(0) \subseteq P, \text{ for every } m \geq N. \]

Hence
\[ -\frac{k^m}{1-k} d(x_0, x_1) \in N_\delta(0), \text{ for every } m \geq N. \]

That is
\[ c - \frac{k^m}{1-k} d(x_0, x_1) \in c + N_\delta(0) \subseteq P, \text{ for every } m \geq N. \]

That is
\[ \frac{k^m}{1-k} d(x_0, x_1) \ll c, \text{ for every } m \geq N. \]

Thus by inequality (2.2) \( d(x_n, x_m) \ll c \) for \( n, m \geq N \). Therefore \( \{x_n\} \) is a Cauchy sequence in \( X \) which is complete. Therefore there exist \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to \infty \). So there exist a positive
integer $N_1$ such that $d(x_n, x^*) \ll \frac{c}{2}$ for every $n \geq N_1$. For some positive integer $m$

$$d(x^*, T_i^m x^*) \leq d(x^*, x_n) + d(x_n, T_i^m x^*)$$

$$\leq d(x^*, x_n) + d(T_i^m x_n, T_i^m x^*)$$

$$\leq d(x^*, x_n) + \alpha_{n,i} d(x_{n-1}, x^*)$$

$$\leq d(x^*, x_n) + k d(x_{n-1}, x^*)$$

$$< d(x^*, x_n) + d(x_{n-1}, x^*)$$

Therefore $d(x^*, T_i^m x^*) \ll \frac{c}{2} + \frac{c}{2} = c$ for every $n \geq N_1$. Thus $d(x^*, T_i^m x^*) \ll \frac{c}{m}$ for every $m \geq 1$. That is $\frac{c}{m} - d(x^*, T_i^m x^*) \in P$ for every $m \geq 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and $P$ is closed $-d(x^*, T_i^m x^*) \in P$. But $d(x^*, T_i^m x^*) \in P$. This is possible only if $d(x^*, T_i^m x^*) = 0$, hence $T_i^m x^* = x^*$. So $x^*$ is a periodic point of $T_i$. If $y^*$ is another periodic point of $T_i$ then

$$d(x^*, y^*) = d(T_i^m x^*, T_i^m y^*) \leq \alpha_{i,j} d(x^*, y^*) \leq k d(x^*, y^*)$$

That is

$$(1 - k) d(x^*, y^*) \leq 0$$

Since $1 - k > 0$, we have $d(x^*, y^*) = 0$ which implies $x^* = y^*$. Hence $x^*$ is a unique periodic point of $T_i$. Consider,

$$T_i x^* = T_i (T_i^m x^*)$$

$$= T_i^m (T_i x^*)$$

Thus $T_i x^*$ is also a periodic point of $T_i$. Since the periodic point is
unique $T_ix^* = x^*$. Hence $x^*$ is a unique common fixed point of the sequence $\{T_n\}$. □

**Theorem: 2.3.2:** Let $(X,d)$ be a complete Cone metric space with respect to a cone $P$ contained in a real Banach space $E$. Let $T$ be a surjective self map of $X$ satisfying

**case (i)**

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Ty) + \gamma d(x, y)$$

for all $x, y \in X$, $2\alpha + \beta + \gamma \in [0, 1)$

**case (ii)**

$$d(Tx, Ty) \leq \alpha d(y, Tx) + \beta d(x, Tx) + \gamma d(x, y)$$

for all $x, y \in X$, $\alpha + \gamma, \beta + \gamma \in [0, 1)$

**case (iii)**

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y)$$

for all $x, y \in X$, $\alpha + \beta + \gamma \in [0, 1)$

where $\alpha, \beta, \gamma \geq 0$. Then $T$ has a unique fixed point.

**Proof:** We shall prove the theorem for case (i), case (ii) and case (iii) separately. First take case (i)

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Ty) + \gamma d(x, y)$$

for all $x, y \in X$, $2\alpha + \beta + \gamma \in [0, 1)$
Let $x_0$ be an arbitrary point in $X$. Since $Tx_0 \in X$ and $T$ is surjective we can choose a point $x_1 \in X$ such that $Tx_0 = x_1$. In this way we can define a sequence $\{x_n\}$ as $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$. If $x_n = x_{n+1}$ for some $n$ then we see that $x_n$ is a fixed point of $T$, therefore we suppose that no two consecutive terms of $\{x_n\}$ are equal.

Now we shall prove $\{x_n\}$ is a Cauchy sequence in case (i).

We have

\[
d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})
\leq \alpha d(x_n, Tx_{n+1}) + \beta d(x_{n+1}, Tx_{n+1})
+ \gamma d(x_n, x_{n+1})
= \alpha d(Tx_{n-1}, Tx_{n+1}) + \beta d(Tx_n, Tx_{n+1})
+ \gamma d(Tx_{n-1}, Tx_n)
\leq \alpha d(Tx_{n-1}, Tx_n) + \alpha d(Tx_n, Tx_{n+1})
+ \beta d(Tx_n, Tx_{n+1}) + \gamma d(Tx_{n-1}, Tx_n)
= (\alpha + \gamma) d(Tx_n, Tx_{n-1})
+ (\alpha + \beta) d(Tx_n, Tx_{n+1})
\leq (\alpha + \gamma) d(x_{n+1}, x_n) + (\alpha + \beta) d(x_{n+1}, x_{n+2})
\]

\[
(1 - \alpha - \beta) d(x_{n+1}, x_{n+2}) \leq (\alpha + \gamma) d(x_{n+1}, x_n)
\]

\[
d(x_{n+1}, x_{n+2}) \leq \frac{\alpha + \gamma}{1 - \alpha - \beta} d(x_{n+1}, x_n)
\]

Let $\frac{\alpha + \gamma}{1 - \alpha - \beta} = k$ then $k < 1$

Therefore

\[
d(x_{n+1}, x_{n+2}) \leq k d(x_{n+1}, x_n) \text{ for every } n\]
Hence
\[ d(x_n, x_{n+1}) \leq k^n d(x_1, x_0) \]

Let \( n > m \)
\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\
\leq (k^{n-1} + k^{n-2} + \cdots + k^m) d(x_1, x_0) \\
d(x_n, x_m) < \frac{k^m}{1-k} d(x_1, x_0) \tag{2.3}
\]

Let \( \epsilon \gg 0 \), then there is a \( \delta > 0 \) such that \( \epsilon + N_\delta(0) \subseteq P \) where
\[ N_\delta(0) = \{ y \in E : \|y\| < \delta \} \]. Since \( k < 1 \) there is a positive integer \( N \) such that \( \|\frac{k^m}{1-k} d(x_1, x_0)\| \leq \delta \) for every \( m \geq N \).

Therefore
\[ \frac{k^m}{1-k} d(x_0, x_1) \in N_\delta(0) \]

Hence
\[ -\frac{k^m}{1-k} d(x_0, x_1) \in N_\delta(0) \]
Therefore
\[ \epsilon - \frac{k^m}{1-k} d(x_0, x_1) \in \epsilon + N_\delta(0) \subseteq P \]

That is
\[ \frac{k^m}{1-k} d(x_0, x_1) \ll \epsilon \text{ for } m \geq N \]

Hence by inequality (2.3) \( d(x_n, x_m) \ll \epsilon \) for all \( n, m \geq N \). Therefore \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete there is an
$x \in X$ such that \{x_{n}\} converges to $x$.

Now we shall prove $x$ is a fixed point of $T$. We have

\[
d(x, Tx) \leq d(x, x_{n}) + d(x_{n}, Tx) \\
\leq d(x, x_{n}) + d(Tx_{n-1}, Tx) \\
\leq d(x, x_{n}) + \alpha d(x_{n-1}, Tx) + \beta d(x, Tx) + \gamma d(x_{n-1}, x)
\]

Letting $n \to \infty$ we get

\[
d(x, Tx) \leq \alpha d(x, Tx) + \beta d(x, Tx)
\]

That is

\[
(1 - \alpha - \beta) d(x, Tx) \leq 0
\]

Since $\alpha + \beta < 1$, $d(x, Tx) = 0$. Therefore $Tx = x$. Thus $x$ is a fixed point of $T$ in case (i).

Now we shall prove this fixed point is unique. If possible $z$ is another fixed point of $T$.

So $Tz = z$, we have

\[
d(x, z) = d(Tx, Tz) \\
\leq \alpha d(x, Tz) + \beta d(z, Tz) + \gamma d(x, z) \\
= \alpha d(x, z) + \beta d(z, z) + \gamma d(x, z)
\]

\[
(1 - \alpha - \gamma) d(x, z) \leq 0
\]

Since $\alpha + \gamma < 1$, $d(x, z) = 0$. Hence $x = z$.

Now we shall prove \{x_{n}\} is a Cauchy sequence in case (ii)

We have
\[ d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \]
\[ \leq \alpha d(x_n, Tx_{n-1}) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_{n-1}, x_n) \]
\[ = (\beta + \gamma) d(x_{n-1}, x_n) \]
\[ d(x_n, x_{n+1}) = k d(x_{n-1}, x_n) \text{ where } k = \beta + \gamma < 1 \text{ for every } n \]

Therefore \( d(x_n, x_{n+1}) \leq k^n d(x_1, x_0) \).

Let \( n > m \) then

\[ d(x_n, x_m) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \]
\[ \leq (k^{n-1} + \cdots + k^m) d(x_1, x_0) \]
\[ d(x_n, x_m) < \frac{k^m}{1-k} d(x_1, x_0) \]

Then as in case (i) we can see that \( \{x_n\} \) is a Cauchy sequence in \( X \).

But \( X \) is complete therefore there exist an \( x \in X \) such that \( \{x_n\} \) converges to \( x \).

Now we shall prove \( x \) is a fixed point of \( T \). By triangle inequality

\[ d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \]
\[ = d(x, x_n) + d(Tx_{n-1}, Tx) \]
\[ \leq d(x, x_n) + \alpha d(x, Tx_{n-1}) + \beta d(x_{n-1}, Tx_{n-1}) \]
\[ + \gamma d(x_{n-1}, x) \]

Letting \( n \to \infty \)

\[ d(x, Tx) \leq (1 + \alpha + \beta + \gamma) d(x, x) \]
\[ = 0 \]
Therefore $d(x, Tx) = 0$. Hence $Tx = x$, which shows $x$ is fixed point of $T$.

Now we shall prove this fixed point $x$ is unique. If possible let $z$ be another fixed point of $T$, then $Tz = z$. By case (ii) we have

$$d(x, z) = d(Tx, Tz) \leq \alpha d(z, Tx) + \beta d(x, Tx) + \gamma d(x, z) = \alpha d(z, x) + \beta d(x, x) + \gamma d(x, z)$$

$$(1 - \alpha - \gamma) d(x, z) \leq 0$$

But $\alpha + \gamma < 1$. Hence the uniqueness.

Now for case (iii)

$$d(x_n, x_{n+1}) = d(Tx_n, Tx_{n-1}) \leq \alpha d(x_n, Tx_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, x_{n-1})$$

Therefore

$$d(x_n, x_{n+1}) = \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n-1})$$

$$(1 - \alpha) d(x_n, x_{n+1}) \leq (\beta + \gamma) d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{\beta + \gamma}{1 - \alpha} d(x_{n-1}, x_n)$$

Let $k = \frac{\beta + \gamma}{1 - \alpha}$, then $k < 1$.

Therefore

$$d(x_n, x_{n+1}) \leq k d(x_n, x_{n-1}) \text{ for every } n$$
Hence

\[ d(x_n, x_{n+1}) \leq k^n d(x_1, x_0) \]

Then as in case (i) we can prove \( \{x_n\} \) is a Cauchy sequence in \( X \). But \( X \) is complete. So there exist an \( x \in X \) such that \( \{x_n\} \) converges to \( x \).

We shall prove \( x \) is a fixed point of \( T \).

By triangle inequality

\[
\begin{align*}
d(x, Tx) & \leq d(x, x_n) + d(x_n, Tx) \\
& = d(x, x_n) + d(Tx_{n-1}, Tx) \\
& \leq d(x, x_n) + \alpha d(x_{n-1}, Tx_{n-1}) + \beta d(x, Tx) + \gamma d(x_{n-1}, x) \\
& = d(x, x_n) + \alpha d(x_{n-1}, x_n) + \beta d(x, Tx) + \gamma d(x_{n-1}, x)
\end{align*}
\]

That is

\[(1 - \beta) d(x, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, x_n) + \gamma d(x_{n-1}, x)\]

Letting \( n \to \infty \), we get \((1 - \beta) d(x, Tx) \leq 0\). But \( \beta < 1 \). Hence \( Tx = x \). Thus \( x \) is a fixed point of \( T \).

Now we shall prove this fixed point \( x \) is unique. If possible let \( z \) be another fixed point of \( T \). Then \( Tz = z \).

\[
\begin{align*}
d(x, z) & = d(Tx, Tz) \\
& \leq \alpha d(x, Tx) + \beta d(z, Tz) + \gamma d(x, z) \\
& = \alpha d(x, x) + \beta d(z, z) + \gamma d(x, z) \\
(1 - \gamma) d(x, z) & \leq 0
\end{align*}
\]
But $\gamma < 1$. Therefore $x = z$. Hence the theorem. \hfill \Box

The next theorem gives unique common fixed point for continuous mappings which commute.

### 2.4 COMMON FIXED POINT THEOREMS FOR COMMUTING MAPS

**Theorem: 2.4.1:** Let $(X, d)$ be a complete Cone metric space. Suppose that the commuting mappings $f, g : X \to X$ are such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$

$$d(fx, fy) \leq \lambda d(gx, gy) \quad (2.4)$$

If the range of $g$ contains the range of $f$ and if $g$ is continuous, then $f$ and $g$ have a unique common fixed point.

**Proof:** Let $x_0 \in X$, be arbitrary since $f(x_0) \in f(X) \subseteq g(X)$ we can choose $x_1$ such that $f(x_0) = g(x_1)$. Let $y_0 = f(x_0) = g(x_1)$. Now $x_1 \in X$, since $f(x_1) \in f(X) \subseteq g(X)$ we can choose $x_2 \in X$ such that $f(x_1) = g(x_2)$. Let $y_1 = f(x_1) = g(x_2)$. Continuing like this having chosen $x_n$ we choose $x_{n+1} \in X$ such that $y_n = f(x_n) = g(x_{n+1})$. By inequality $(2.4)$ we have

$$d(y_n, y_{n-1}) = d(fx_n, fx_{n-1})$$

$$\leq \lambda d(gx_n, gx_{n-1})$$

$$\leq \lambda d(y_{n-1}, y_{n-2})$$
Therefore
\[
d(y_n, y_{n-1}) \leq \lambda^2 d(y_{n-2}, y_{n-3}) \\
\leq \ldots \\
d(y_n, y_{n-1}) \leq \lambda^{n-1} d(y_1, y_0)
\]

Now we shall prove \(\{y_n\}\) is a Cauchy sequence. Let \(n > m\)
\[
d(y_n, y_m) \leq d(y_n, y_{n-1}) + \cdots + d(y_{m+1}, y_m) \\
\leq \lambda^{n-1}d(y_1, y_0) + \cdots + \lambda^m d(y_1, y_0) \\
= (\lambda^m + \cdots + \lambda^{n-1})d(y_1, y_0) \\
d(y_n, y_m) < \frac{\lambda^m}{1 - \lambda}d(y_1, y_0) \quad (2.5)
\]

Let \(c \gg 0\), then there exist a \(\delta > 0\) such that \(c + N_\delta(0)\) is a subset of \(P\) where \(N_\delta(0) = \{y \in E : \|y\| < \delta\}\).

Since \(\lambda < 1\) there exist a positive integer \(N\) such that
\[
\| \frac{\lambda^m}{1 - \lambda}d(x_0, x_1) \| < \delta \text{ for every } m \geq N
\]

Therefore
\[
\frac{\lambda^m}{1 - \lambda}d(x_0, x_1) \in N_\delta(0) \text{ for every } m \geq N
\]

Hence
\[
-\frac{\lambda^m}{1 - \lambda}d(x_0, x_1) \in N_\delta(0) \text{ for every } m \geq N
\]

Thus
\[
c - \frac{\lambda^m}{1 - \lambda}d(x_0, x_1) \in c + N_\delta(0) \subseteq P \text{ for every } m \geq N
\]
That is
\[
\frac{\lambda^m}{1 - \lambda} d(x_0, x_1) \ll c \text{ for every } m \geq N
\]
Therefore by inequality (2.5) \(d(y_n, y_m) \ll c\) for \(n, m \geq N\).
Therefore \(\{y_n\}\) is a Cauchy sequence.
Since \(X\) is complete there is some \(z\) in \(X\) such that
\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_{n+1} = z. \text{ Given } g \text{ is continuous and } f, g \text{ commute.}
\]
Therefore
\[
g z = g \left[ \lim_{n \to \infty} g x_{n+1} \right] = \lim_{n \to \infty} g^2 x_{n+1}
\]
And
\[
g z = g \left[ \lim_{n \to \infty} f x_n \right] = \lim_{n \to \infty} g f x_n = \lim_{n \to \infty} f g x_n
\]
By inequality (2.4) \(d(f g x_n, f z) \leq \lambda d(g^2 x_n, g z)\).
Taking the limit as \(n \to \infty\) we obtain
\[
d(g z, f z) \leq \lambda d(g z, g z)
\]
That is \(d(g z, f z) \leq 0\). Therefore \(g z = f z\).
Again from inequality (2.4)
\[
d(f x_n, f z) \leq \lambda d(g x_n, g z)
\]
Taking the limit as \(n \to \infty\), we get
\[
d(z, f z) \leq \lambda d(z, g z) = \lambda d(z, f z)
\]
Since \(0 < \lambda < 1\), \(d(z, f z) = 0\), which implies \(f z = z\). Thus we have \(f z = g z = z\). Hence \(z\) is a common fixed point of \(f\) and \(g\).
Now we shall prove $z$ is unique. Let $z_1$ be another common fixed point of $f$ and $g$. Then

$$d(z, z_1) = d(fz, fz_1) \leq \lambda d(gz, gz_1) = \lambda d(z, z_1)$$

Since $0 < \lambda < 1$, $d(z, z_1) = 0$ which implies $z = z_1$. Hence the theorem. $\square$

The following theorem gives the common fixed point for four mapping.

**Theorem: 2.4.2:** Let $S$ and $I$ be commuting mappings, $T$ and $J$ be commuting mappings of a complete Cone metric space $(X, d)$ into itself satisfying

$$d(Sx, Ty) \leq \lambda d(Ix, Jy) \text{ for all } x, y \in X, \text{ where } 0 < \lambda < 1 \ (2.6)$$

If $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$ and if $I$ and $J$ are continuous, then all $S, T, I$ and $J$ have a unique common fixed point.

**Proof:** Let $x_0$ in $X$ be arbitrary. Since $Sx_0 \in S(X) \subseteq J(X)$ there exist $x_1 \in X$ such that $Sx_0 = Jx_1$. Again $Tx_1 \in T(X) \subseteq I(X)$. Therefore there exist $x_2 \in X$ such that $Tx_1 = Ix_2$. Continuing like this we can choose $x_{n+1} \in X$ such that $Sx_{2n} = Jx_{2n+1}$ and $x_{n+2} \in X$ such that $Tx_{2n+1} = Ix_{2n+2}, \ n = 0, 1, 2, \cdots$.

Let $y_{2n} = Jx_{2n+1} = Sx_{2n}$ and $y_{2n+1} = Ix_{2n+2} = Tx_{2n+1}$. Now

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \lambda d(Ix_{2n}, Jx_{2n+1})$$

$$= \lambda d(x_{2n-1}, y_{2n})$$
Continuing in the same manner, we obtain that

\[ d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}), \text{ for } n \geq 2 \]

Let \( n > 2 \)

\[ d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}) \leq \cdots \leq \lambda^{n-1} d(y_1, y_0) \]

As in the proof of previous theorems we can prove \( \{y_n\} \) is a Cauchy sequence. But \((X, d)\) is complete, there exist a \( y \in X \) such that

\[
\lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Jx_{2n+1} = \lim_{n \to \infty} Ix_{2n+2} = \lim_{n \to \infty} Tx_{2n+1} = y
\]

By the continuity of \( I \)

\[
\lim_{n \to \infty} I^2x_{2n+2} = Iy
\]

Again \( S \) and \( I \) commute so

\[
\lim_{n \to \infty} SIx_{2n} = \lim_{n \to \infty} ISx_{2n} = Iy
\]

From inequality (2.6)

\[
d(SIx_{2n}, Tx_{2n+1}) \leq \lambda d(I^2x_{2n}, Jx_{2n+1})
\]

Taking the limit as \( n \to \infty \) we get \( d(Iy, y) \leq \lambda d(Iy, y) \).

This implies \( Iy = y \). Since \( J \) is continuous \( \lim_{n \to \infty} J^2x_{2n+1} = Jy \).

\( T \) and \( J \) commute therefore

\[
\lim_{n \to \infty} TJx_{2n+1} = \lim_{n \to \infty} JTx_{2n+1} = Jy
\]

Similarly we obtain

\[
d(Sx_{2n}, TJx_{2n+1}) \leq \lambda d(Ix_{2n}, J^2x_{2n+1})
\]
Letting $n \to \infty$ we get $d(y, Jy) \leq \lambda d(y, Jy)$.

Since $0 < \lambda < 1$, $d(y, Jy) = 0$ which implies $Jy = y$.

Again from inequality (2.6)

$$d(Sy, Tx_{2n+1}) \leq \lambda d(Iy, Jx_{2n+1})$$

Letting $n \to \infty$, we get $d(Sy, y) \leq d(y, y) = 0$

Therefore $d(Sy, y) = 0$, which implies $Sy = y$.

Again from inequality (2.6)

$$d(Sy, Ty) \leq \lambda d(Iy, Jy) = \lambda d(y, y) = 0$$

Therefore $d(Sy, Ty) = 0$ which implies $Sy = Ty$. Thus we have $Sy = Ty = Iy = Jy = y$. That is $y$ is a common fixed point of $S$, $T$, $I$ and $J$.

To prove $y$ is unique, let $x$ be another fixed point in $X$ of $S, T, I$ and $J$. Therefore $Sx = Tx = Ix = Jx = x$. Then

$$d(x, y) = d(Sx, Ty) \leq \lambda d(Ix, Jy) \leq \lambda d(x, y)$$

Since $0 < \lambda < 1$, $d(x, y) = 0$ which implies $x = y$. Hence the proof.

\[\square\]

Even the two maps are not commuting if they are weakly compatible we can obtain the common fixed point for the two mappings.
2.5 COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPS

Theorem: 2.5.1: Let \((X,d)\) be a Cone metric space, \(P\) is a cone in \(E\). Suppose mappings \(f, g : X \to X\) satisfy

Case (i)

\[
d(fx, fy) \leq \lambda d(gx, gy), \text{ for all } x, y \in X, \lambda \in (0, 1) \tag{2.7}
\]

Case (ii)

\[
d(fx, fy) \leq \lambda (d(fx, gx) + d(fy, gy)), \text{ for all } x, y \in X, \lambda \in (0, \frac{1}{2}) \tag{2.8}
\]

Case (iii)

\[
d(fx, fy) \leq \lambda (d(fx, gy) + d(fy, gx)), \text{ for all } x, y \in X, \lambda \in (0, \frac{1}{2}) \tag{2.9}
\]

If the range of \(g\) contains the range of \(f\) and \(g(X)\) is a complete subspace of \(X\), then \(f\) and \(g\) have a unique point of coincidence. Moreover if \(f\) and \(g\) are weakly compatible, \(f\) and \(g\) have a unique common fixed point.

Proof:

Let \(x_0\) be an arbitrary point in \(X\).

Since \(f(x_0) \in f(X) \subseteq g(X)\) we can choose a point \(x_1 \in X\) such that \(f(x_0) = g(x_1)\) continuing this process having chosen \(x_n \in X\) we obtain \(x_{n+1} \in X\) such that \(f(x_n) = g(x_{n+1})\).
Let \( y_n = f(x_n) = g(x_{n+1}) \). Then we shall prove
\[
d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}), \quad n = 2, 3, 4, \ldots \text{ in all the three cases (i), (ii) and (iii).}
\]
First take case (i), by inequality (2.7), we have
\[
d(fx, fy) \leq \lambda d(gx, gy)
\]
for every \( x, y \in X, \lambda \in (0, 1) \). Now
\[
d(y_n, y_{n-1}) = d(gx_{n+1}, gx_n) \\
= d(fx_n, fx_{n-1}) \\
\leq \lambda d(gx_n, gx_{n-1}) \\
= \lambda d(y_{n-1}, y_{n-2}), \quad n = 2, 3, \ldots
\]
For the case (ii)
\[
d(fx, fy) \leq \lambda (d(fx, gx) + d(fy, gy)) \text{ for every } x, y \in X, \lambda \in (0, \frac{1}{2})
\]
We have
\[
d(y_n, y_{n-1}) = d(gx_{n+1}, gx_n) \\
= d(fx_n, fx_{n+1}) \\
\leq \lambda (d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1})) \\
= \lambda (d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})) \\
(1 - \lambda) d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}) \\
d(y_n, y_{n-1}) \leq \frac{\lambda}{1 - \lambda} d(y_{n-1}, y_{n-2})
\]
Let \( h = \frac{\lambda}{1 - \lambda} \in (0, 1) \) where \( \lambda \in (0, \frac{1}{2}) \).
Hence
\[
d(y_n, y_{n-1}) \leq h d(y_{n-1}, y_{n-2})
\]
For the case \((iii)\)

\[
d(f_x, f_y) \leq \lambda (d(f_x, g_y) + d(f_y, g_x))
\]

for every \(x, y \in X, \lambda \in (0, \frac{1}{2})\)

\[
d(y_n, y_{n-1}) = d(gx_{n+1}, gx_n) \\
= d(fx_n, fx_{n-1}) \\
\leq \lambda \left(d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n)\right) \\
= \lambda \left(d(y_n, y_{n-2}) + d(y_{n-1}, y_{n-1})\right) \\
d(y_n, y_{n-1}) = \lambda d(y_n, y_{n-2})
\]

Therefore

\[
\leq \lambda (d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})) \\
(1 - \lambda) d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}) \\
d(y_n, y_{n-1}) \leq \frac{\lambda}{1-\lambda} d(y_{n-1}, y_{n-2})
\]

Since \(\lambda \in (0, \frac{1}{2})\), let \(h = \frac{\lambda}{1-\lambda} \in (0, 1)\).

We get

\[
d(y_n, y_{n-1}) \leq h d(y_{n-1}, y_{n-2})
\]

Then for all the three cases

\[
d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2})
\]

for every \(n\). Thus we obtain

\[
d(y_n, y_{n-1}) \leq \lambda^{n-1} d(y_1, y_0)
\]
As in the proof of previous theorems we can prove \( \{y_n\} \) is a Cauchy sequence. Since \( g(X) \) is complete there is a \( y \in g(X) \) such that 
\[
y_n \to y \text{ as } n \to \infty.
\]
Since \( y \in g(X) \) there is a \( z \in X \) such that \( y = g(z) \).

Now we shall prove \( z \) is a point of coincidence of \( f \) and \( g \) in case (i).

From inequality (2.7)
\[
d(gx_n, fz) = d(fx_{n-1}, fz) \leq \lambda d(gx_{n-1}, gz)
\]
Letting \( n \to \infty \) we get
\[
d(gz, fz) \leq \lambda d(gz, gz) = 0
\]
Therefore \( d(gz, fz) = 0 \) which implies \( gz = fz \). Thus \( z \) is a point of
coincidence of \( f \) and \( g \). Claim this point of coincidence is unique. For assume that there is another point of coincidence \( z_1 \) in \( X \) such
that \( fz_1 = gz_1 = y_1 \). Then \( d(gz_1, gz) = d(fz_1, fz) \leq \lambda d(gz_1, gz) \).
As \( 0 < \lambda < 1 \), \( d(gz_1, gz) = 0 \) which implies \( gz_1 = gz \). Hence
\( gz_1 = gz = fz = fz_1 = y_1 = y \). By lemma (2.2.2) \( f \) and \( g \) have a
unique common fixed point.

Now we shall prove \( z \) is a point of coincidence of \( f \) and \( g \) in
case (ii). From inequality (2.8)
\[
d(gx_n, fz) = d(fx_{n-1}, fz) \leq \lambda (d(fx_{n-1}, gx_{n-1}) + d(fz, gz))
\]
Letting \( n \to \infty \)
\[
d(gz, fz) \leq \lambda (d(gz, gz) + d(fz, gz)) = \lambda d(fz, gz)
\]
Given \( 0 < \lambda < 1 \), \( d(fz, gz) = 0 \) hence \( fz = gz \).

To prove that \( f \) and \( g \) have a unique point of coincidence in case
(ii). For assume there exist another point of coincidence $z_1$ in $X$ such that $fz_1 = gz_1 = y_1$.
Now
\[
d(gz_1, gz) = d(fz_1, f z) \\
\leq \lambda (d(f z, gz) + d(fz_1, gz_1)) \\
= \lambda (0 + 0) \\
= 0
\]

Therefore $d(gz_1, gz) = 0$ which implies $gz_1 = gz$.
Hence $gz_1 = gz = f z = fz_1 = y = y_1$. By lemma (2.2.2), $f$ and $g$ have a unique common fixed point.
Now we shall prove $f$ and $g$ has a point of coincidence $z$ in case (iii).
From inequality (2.9) we get
\[
d(gx_n, fz) = d(fx_{n-1}, f z) \leq \lambda (d(fx_{n-1}, gz) + d(gx_{n-1}, f z))
\]
Letting $n \to \infty$ we get
\[
d(gz, fz) \leq \lambda (d(gz, gz) + d(gz, fz)) = \lambda d(gz, fz)
\]
since $0 < \lambda < 1$ it follows that $gz = fz$.
To prove in case (iii) $f$ and $g$ have a unique point of coincidence.
Let $z_1$ in $X$ such that $fz_1 = gz_1 = y_1$. Now
\[
d(gz_1, gz) = d(fz_1, f z) \\
\leq \lambda (d(f z_1, gz) + d(f z, gz_1)) \\
= \lambda (d(gz_1, gz) + d(gz, gz_1)) \\
= 2\lambda d(gz_1, gz)
\]
From this we get $gz = gz_1$. Hence $gz_1 = gz = fz = f z_1 = y = y_1$. By lemma (2.2.2), $f$ and $g$ have a unique common fixed point. Hence the theorem. □