CHAPTER - II

A Study of Fuzzy Relations

Abstract:

The concept of fuzzy set is generalisation of the concept of ordinary (crisp) set. Similarly the concept of fuzzy relations is generalisation of concept of relation, in set theory. Applications of fuzzy relations are wide spread and important. In present paper a study on fuzzy relations have been done for fuzzy binary relations. A generalization to n-ary relations is straightforward.

Introduction:

Fuzzy relations are fuzzy sub sets of $X \times Y$, i.e. mappings from $X \rightarrow Y$.

They have been studied by a number of research workers, in particular by Zadeh (1965, 1971).
Kaufmann [1975] and Rosenfeld [1975].
We make a parallel study of different type of
relations of ordinary set theory with reference to fuzzy set
theory.

**Def. 2.1 : A binary fuzzy relation :**

A fuzzy set $\tilde{R} = \{(x,y), \mu_{\tilde{R}}(x,y) : (x, y) \in X \times Y \}$.

is called a binary fuzzy relation on $X \times Y$.

**Def. 2.2 : n' ary fuzzy relation :**

An n' ary fuzzy relation is a fuzzy set on $X_1 \times X_2 \times 
X_3 \ldots \times X_n$.

Example 2.1.

Let $X = Y = R$ (The set of real numbers), $\tilde{R}$ :

= "Considerably larger than". The membership function of the
fuzzy relation, which is, of course, a fuzzy set on $X \times Y$ can then
be
\[ \mu_{\tilde{R}}(x, y) = \begin{cases} 
0, & \text{for } x \leq y \\
\frac{(x - y)}{10y}, & \text{for } y < x \leq 11y \\
1, & \text{for } x > 11y 
\end{cases} \]

For discrete support's fuzzy relations can also be defined by matrices.

Example 2.2.

Let \( X = (x_1, x_2, x_3) \) and \( Y = (y_1, y_2, y_3, y_4) \)

\[ \tilde{R} = "x \text{ considerably larger than } y". \]

and \( \bar{Z} = "y \text{ very close to } x" \)
Def. 2.3: Let $X, Y \subseteq \mathbb{R}$ and

$$\tilde{A} = \{ x, \mu_{\tilde{A}}(x) \} : x \in X$$

and $\tilde{B} = \{ y, \mu_{\tilde{B}}(y) \} : y \in Y$ be two fuzzy sets.

Then $\tilde{R} = \{(x,y), \mu_{\tilde{R}}(x,y)\} : (x,y) \in X \times Y$ is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{A}}(x), \quad \forall (x,y) \in X \times Y$$

and $\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{B}}(y), \quad \forall (x,y) \in X \times Y$

i.e. $\mu_{\tilde{R}}(x,y) \leq \min \{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y) \}$

Def. 2.4: Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product space. Then union and intersection of $\tilde{R}$ and $\tilde{Z}$ is defined by

$$\mu_{\tilde{R} \cup \tilde{Z}}(x,y) = \max \{ \mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y) \}, \quad (x,y) \in X \times Y$$

$$\mu_{\tilde{R} \cap \tilde{Z}}(x,y) = \min \{ \mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y) \}, \quad (x,y) \in X \times Y$$
Example 2.3:

Let $\tilde{R}$ and $\tilde{Z}$ be the two fuzzy relations defined in Example 2.2. The union of $\tilde{R}$ and $\tilde{Z}$, which can be interpreted as "x considerably larger or very close to y" is then given by

$$
\begin{array}{cccc}
  & y_1 & y_2 & y_3 & y_4 \\
 x_1 & .8 & 1 & .9 & .7 \\
 x_2 & .9 & .8 & .5 & .7 \\
 x_3 & .9 & 1 & .8 & .8 \\
\end{array}
$$

The intersection of $\tilde{R}$ and $\tilde{Z}$ is represented by

$$
\begin{array}{cccc}
  & y_1 & y_2 & y_3 & y_4 \\
 x_1 & .4 & 0 & .1 & .6 \\
 x_2 & 0 & .4 & 0 & 0 \\
 x_3 & .3 & 0 & .7 & .5 \\
\end{array}
$$
Def. 2.5:

Let \( \tilde{R} \) be a fuzzy binary relation.

The first projection of \( R \) is then defined as

\[
\tilde{R}^{(1)} = \{ (x, \max_y \mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y \}
\]

The second projection is defined as

\[
\tilde{R}^{(2)} = \{ (y, \max_x \mu_{\tilde{R}}(x,y)) : (x,y) \in X \times Y \}
\]

And the total projection as

\[
\tilde{R}^{(T)} = \max_x \max_y \{ \mu_{\tilde{R}}(x,y) : (x,y) \in X \times Y \}
\]

Example 2.4. Let \( \tilde{R} \) be a fuzzy relation defined by the following relational matrix. The first, second and total projections are
then shown at the appropriate places below:

\[
\begin{array}{cccccc}
 y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
 x_1 & .1 & .2 & .4 & .8 & 1 & 1 \\
 x_2 & .2 & .4 & .8 & 1 & .8 & 1 \\
 x_3 & .4 & .8 & 1 & .8 & .4 & 1 \\
\end{array}
\]

Ind Projection \([\mu_R(y)]\) 
\[
.4 \quad .8 \quad 1 \quad 1 \quad 1 \quad .8 \quad 1
\]

Total projection \(\tilde{R}^{(T)}\).

Definition 2.6.

**Inverse of binary fuzzy relation:**

The inverse of a binary fuzzy relation \(\tilde{R}\) on \(X \times X\), denoted by \(\tilde{R}^{-1}\), is a fuzzy relation on \(Y \times X\) defined by

\[
\tilde{R}^{-1}((y, x)) = \tilde{R}((x, y))
\]

Definition 2.7.

**Reflexivity:**

Let \(\tilde{R}\) be a fuzzy relation in \(X \times X\), then \(\tilde{R}\) is called reflexive.
If $\mu_{\tilde{R}}(x, x) = 1, \forall x \in X$

Example 2.5:

Let $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4)$

The following relation "y is close to x" is reflexive

<table>
<thead>
<tr>
<th></th>
<th>y_1</th>
<th>y_2</th>
<th>y_3</th>
<th>y_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>x_2</td>
<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x_3</td>
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<td>7</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>x_4</td>
<td>0</td>
<td>1</td>
<td>4</td>
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</tbody>
</table>

Definition 2.8: $\text{Symmetry} \Rightarrow \tilde{R} (y, x)$

A fuzzy relation $\tilde{R}$ is called symmetric

if $\tilde{R}(x, y) = \tilde{R}(y, x)$

The relation $\tilde{R} (x, y)$:

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>x_1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>.1</td>
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<tr>
<td>x_2</td>
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<td>.3</td>
</tr>
<tr>
<td>x_3</td>
<td>0</td>
<td>.2</td>
<td>.8</td>
<td>.8</td>
</tr>
<tr>
<td>x_4</td>
<td>.1</td>
<td>.3</td>
<td>.8</td>
<td>1</td>
</tr>
</tbody>
</table>
Definition 2.9:

**Transitivity:**

A fuzzy relation \( \tilde{R} \) is called (max-min.) transitive if \( \tilde{R} \circ \tilde{R} \subseteq \tilde{R} \)

Let the fuzzy relation \( \tilde{R} \) be defined as

\[
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  x_1 & .2 & 1 & .4 & .4 \\
  x_2 & 0 & .6 & .3 & 0 \\
  x_3 & 0 & 1 & .3 & 0 \\
  x_4 & .1 & 1 & 1 & .1 \\
\end{array}
\]

Then \( \tilde{R} \circ \tilde{R} \) is

\[
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  x_1 & .2 & .6 & .4 & .2 \\
  x_2 & 0 & .6 & .3 & 0 \\
  x_3 & 0 & .6 & .3 & 0 \\
  x_4 & .1 & 1 & .3 & .1 \\
\end{array}
\]

Here we see that \( \mu_{\tilde{R} \circ \tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, y) \) holds for all \( x, y \in X \). Therefore relation \( \tilde{R} \) is transitive.

**Conclusions:**

The combinations of the above property, give
1. If $\tilde{R}$ is symmetric and transitive, then $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x)$ for all $x, y \in X$.

2. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$.

3. If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive.

Def. 2.10.

**Fuzzy Partial Ordering relation:**

A fuzzy relation $P$ on $X \times X$ is called fuzzy partial ordering relation on $X \times X$ if it is reflexive, Perfectly anti symmetric and min. transitive on $X \times X$.

Def. 2.11.

**Tolerance Relation:**

A fuzzy relation on $X \times X$ is called tolerance relation if it is reflexive and symmetric fuzzy relation on $X \times X$. 
References


2. Rosenfeld (1975); Fuzzy relations and fuzzy graph.

In Zadeh et all, p. 77-96.

