CHAPTER VI

Fuzzy integration and fuzzy differentiation

6.1 Abstract:

On the lines of Lebesgue measures and Lebesgue integrals, fuzzy measure is considered and a definition of fuzzy integral which is a generalization of Lebesgue integral is given. Also in analogy to fuzzy integration, fuzzy differentiation is defined. Differentiation of fuzzy functions is considered. We have considered fuzzy differentiation of a differentiable function.

6.2 Fuzzy Integration:

Introduction:

One of the first concepts of a fuzzy integral was put forward by Sugeno [1972, 1977]; who considered fuzzy measures and suggested a definition of a fuzzy integral which is a
generalization of Lebesgue integral: "From the view point of functionals, fuzzy integrals are merely a kind of non linear functionals while Lebesgue integrals are linear one's" [Sugeno 1977, p.92].

We shall focus our attention on approaches along the lines of Riemann Integrals. The main references for the following are Dubois and Prade [1980a, 1982b], Aumann [1965] and Nguyen [1978].

The classical concept of integration of a real valued function over a closed interval is generalized in two ways.

(i) The function can be fuzzy function which is to be integrated over a crisp interval, or

(ii) Function is crisp and interval is fuzzy.

6.3 **Integration of a fuzzy function over a crisp interval**:

Let the fuzzy function be L.R. type,
We shall therefore assume that, $\tilde{f}(x) = [f(x), s(x), t(x)]_{LR}$

This is a fuzzy number in LR representation for all $x \in [a, b]$; $f$, $s$, $t$ are assumed to be positive integrable functions on $[a, b]$.

Dubois and Prade [1980a, p. 109] have shown that under these conditions

$$\tilde{f}(a, b) = \left( \int_a^b f(x)dx, \int_a^b s(x)dx, \int_a^b t(x)dx \right)_{LR}$$

It is then sufficient to integrate the mean value and spread functions of $\tilde{f}(x)$ over $[a,b]$ and the result will be again LR fuzzy number.

6.1 Ex. Let fuzzy function be $\tilde{f}(x) = [f(x), s(x), t(x)]_{LR}$

with the mean function.

$f(x) = x^2$, the spread functions $s(x) = x/4$ and $t(x) = x/2$

$L(x) = 1/1+x^2$, $R(x) = 1/1+2|x|$
To determine the integral from $a = 0$ to $b = 4$; that is to compute \( \int_{1}^{4} f \).

According to above formula we compute

\[
\int_{a}^{b} f(x) \, dx = \int_{1}^{4} x^2 \, dx = 2 \\
\int_{a}^{b} s(x) \, dx = \int_{1}^{4} x/4 \, dx = 1.875 \\
\int_{a}^{b} t(x) \, dx = \int_{1}^{4} x/2 \, dx = 3.75
\]

This yields the fuzzy number

\( \tilde{f}(a, b) = (21, 1.875, 3.75)_{LR} \) as the value of the fuzzy integral.

One interesting property of fuzzy integrals is as under:

6.4 Th. Let \( \tilde{f} \) and \( \tilde{g} \) be fuzzy functions whose supports are bounded.

Then

\[
\int_{t} (\tilde{f} \oplus \tilde{g}) \supseteq \int_{t} \tilde{f} \oplus \int_{t} \tilde{g} \quad \ldots \quad (1)
\]
6.4 Integration of a (crisp) Real valued function over a fuzzy interval :-

Now, we consider a case for which Dubois and Prade [1982a, p.106] proposed a quite interesting solution:

A fuzzy domain of the real line $\mathbb{R}$ is assumed to be bounded by two normalized convex fuzzy sets the membership functions of which are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, respectively.

$\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ can be interpreted as degrees of confidence to which $x$ can be taken to be a lower or upper bound of interval.
6.5 Definition :-

Let \( f \) be a real-valued function which is integrable in the interval \([a_0, b_0]\), then according to the extension principle the membership function of the integral \( j = \int f \) is given by

\[
\mu_{\tilde{j}} f(z) = \operatorname{Sup} \left\{ \operatorname{Min} \mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y) \right\}
\]

\[
Z = \int_{x}^{y} f
\]

6.2 Ex. Let \( \tilde{a} = \{ (4, .8), (5, 1), (6, .4) \} \)

\( \tilde{b} = \{ (6, .7), (7, 1), (8, .2) \} \)

\( f(x) = 2, \quad x \in [a_0, b_0] = [4, 8] \)

Then \( \int f(x) \, dx = \int_{a}^{b} 2 \, dx = 2 \int_{a}^{b} \)
The detailed computational results are:

<table>
<thead>
<tr>
<th>[a, b]</th>
<th>( \int_{a}^{b} 2 , dx )</th>
<th>( \min { \mu_{X}(a), \mu_{X}(b) } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[4, 6]</td>
<td>4</td>
<td>.7</td>
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<tr>
<td>[4, 7]</td>
<td>6</td>
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<td>[4, 8]</td>
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<td>[5, 6]</td>
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<td>[5, 7]</td>
<td>4</td>
<td>1.0</td>
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<td>[6, 8]</td>
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</table>

Hence choosing maximum of the membership values for each value of the integral, yields:

\[
\int f = \{ (0, .4), (2, .7), (4, 1), (6, 8), (8, 2) \}
\]

6.1 Proposition

Let \( f \) and \( g \) be two functions \( f, g : I \to \mathbb{R} \), integrable on \( I \)
\[
\begin{array}{|c|c|c|}
\hline
[a, b] & \int_a^b g(x) \, dx & \min\{\mu_{x_a}(a), \mu_{x_b}(b)\} \\
\hline
[1, 3] & 2 & .7 \\
[1, 4] & 0 & .8 \\
[1, 5] & -4 & .3 \\
[2, 3] & 0 & .7 \\
[2, 4] & -2 & 1.0 \\
[2, 5] & -6 & .3 \\
[3, 3] & 0 & .4 \\
[3, 4] & -2 & .4 \\
[3, 5] & -6 & .3 \\
\hline
\end{array}
\]

Hence choosing max. value of membership function for each value of the integral.

\[
\int f = \{ (0, .4), (2, .7), (4, .4), (6, 1), (10, .3), (12, .3) \}
\]

\[
\int g = \{ (-6, .3), (-4,.3), (-2, 1), (0,.8), (2,.7) \}
\]
\[ \begin{array}{|c|c|c|} \hline [a, b] & \int_a^b (f+g) & \min \{ \mu_x(a), \mu_x(b) \} \\ \hline [1, 3] & 4 & .7 \\ [1, 4] & 6 & .8 \\ [1, 5] & 8 & .3 \\ [2, 3] & 2 & .7 \\ [2, 4] & 4 & 1.0 \\ [2, 5] & 6 & .3 \\ [3, 3] & 0 & .4 \\ [3, 4] & 2 & .4 \\ [3, 5] & 4 & .3 \\ \hline \end{array} \]

Hence choosing max. value of membership function for each value of the integral.

\[ \int (f+g) = \{ (0, .4), (2, .7), (4, 1), (6, .8), (8, .3) \} \]

Applying the formula for the extended addition according to the extension principle.

\[ \int_a^b f \oplus \int_a^b g = \{ (-6, 3), (-4, 3), (-2, 3), (0, 4), (2, 7), (4, 1), (6, 8), (8, 3), (10, 3), (12, 3), (14, 3) \} \]
Therefore, we easily verify that

\[ \int_{a}^{b} f \oplus \int_{a}^{b} g \supseteq \int_{a}^{b} (f + g) \]

6.6 Fuzzy Differentiation:

Introduction:

In analogy to fuzzy integration, Fuzzy differentiation is defined.

The results will depend on the type of function considered.

Differentiation of fuzzy functions is considered by Dubois and Prade [1980a, p. 116 and 1982b, p. 227]

Here we consider only fuzzy differentiation of a differentiable function.

\[ f : \mathbb{R} \supseteq [a, b] \to \mathbb{R} \text{ at a "fuzzy point"}. \]

"A fuzzy point" \( \tilde{X}_0 \) [Dubois and Prade, 1982b, p. 225]

is a convex fuzzy subset of the real line \( \mathbb{R} \).
In present case, fuzzy point is considered for which the support is contained in the interval \([ a, b ]\), that is \(S(\tilde{x}) \subseteq [a, b]\).

Such a fuzzy point can be interpreted as the possibility distribution of a point \(x\) whose precise location is only approximately known.

The uncertainty of the knowledge about the precise location of the point induces an uncertainty about the derivative \(f'(x)\) of a function \(f(x)\) at this point. The derivative might be the same for several \(x\) belonging to \([a, b]\).

The possibility of \(f'(\tilde{X}_0)\) is therefore defined [Zadeh 1978] to be the supremum of the values of the possibilities of \(f'(x) = t, x \in [a, b]\).

The "Derivative" of a real valued function at a fuzzy point can be interpreted as the fuzzy set \(f'(\tilde{x}_0)\), the member-
ship function of which expresses the degree to which a specific 

\( f'(x) \) is the first derivative of a function \( f \) at point \( x_0 \).

6.7 Definition:

The membership function of the fuzzy set "derivative of a real valued function at a fuzzy point \( \tilde{x}_0 \)" is defined by the extension principle as

\[
\mu_{f'}(\tilde{x}_0)(y) = \operatorname{Sup}_{x \in f^{-1}(y)} \mu_{\tilde{x}_0}(x)
\]

Where \( \tilde{x}_0 \) is the fuzzy number that characterizes the fuzzy location.

6.3 Ex. Let \( f(x) = x^3 \)

and \( \tilde{x}_0 = \{ (-1, .4), (0, 1), (1, .6) \} \)

be a fuzzy location.

Since \( f'(x) = 3x^2 \)

We obtain

\( f'(\tilde{x}_0) = \{ (0, 1), (3, .6) \} \)
as derivative of a real valued function at the fuzzy point \( \tilde{x}_0 \).

For fuzzy differentiation, following propositions hold:

Proposition (1): For the extended sum \( \oplus \) of the derivative of two real valued functions \( f \) and \( g \)

\[
f' (\tilde{x}_0) \oplus g' (\tilde{x}_0) \supseteq (f' + g') \tilde{x}_0.
\]

Proposition (2): If \( f \) and \( g \) are continuous and both non-decreasing or non-increasing then

\[
f' (\tilde{x}_0) \oplus g' (\tilde{x}_0) = (f' + g') \tilde{x}_0.
\]
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