CHAPTER - 3
DISCRETE TIME TRANSIENT ANALYSIS OF A SYSTEM WITH QUEUE DEPENDENT SERVERS
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QUEUE DEPENDENT SERVERS

This chapter of thesis deals with discrete time analysis of an infinite capacity Markovian Queueing model with number of heterogeneous channels depending on system size. The model consists of one regular channel and a maximum of C-additional channel by defining probability generating functions.

Introduction:

Most of the Queueing theory literature concentrates on finding the steady state solutions or approximations. Very little seems to have been done to evaluate the transient solutions. Even at time steady state solutions are difficult to compute.

Chaudhry, Agarwal & Templeton (1992) have mostly concentrated on this using the techniques of roots. Earlier attempts at finding transient solutions can be attributed to Tacka’s (1962) & Morse (1959). However there are computations difficulties with their methods. Now with the increased skill available in computations with the use of computers researches especially in computer science have started looking for transient solutions and easy to compute closed form solutions. Sharma & Dass (1988) have provided transient solutions to a class of Markovian Queueing models in Queueing Theory. However they did not concentrate on the computational difficulties of finding roots or eigen values if the matrices involved are large.

Most application of Queueing theory involves queues which are emptied and restarted periodically. These queues never reach equilibrium state. Hence steady state solutions are not always adequate and it is desirable to have time dependent solutions.
Sharma and Shobha (1988) have considered multisyver Markovian Queueing model with finite waiting space and derived transient results.

Sharma and Tarabia (2000) have obtained the transient state probabilities of a Singh server Markovian Queue with finite source in a closed form using Laplace transform technique. Tarabia (2001) has analysed the transient behaviour of a non-empty $M/M/1/N$ queues model using Laplace transform technique.

In real situation it is common that the executive concerned is obliged to offer special service facility to his customers to gain their good will. Systems with additional servers may be required in places like banks, tax offices, transport facilities etc. hence a system with queue depended servers is proposed, Gross and Harries (1985) have mentioned that the transient solution of single channel queue becomes more complicated with the restriction on waiting room capacity is relaxed. However in this chapter we presents the discrete time transient solution of an finite capacity queuing model with additional servers. Here we assume that the system capacity is $N$.

Moreover, no attempt seems to have been made to obtain similar results in discrete time for finite waiting space problems in Queueing theory.

As the transient solution is not independent of the initial state of the system’s behaviors, further, some systems may not exist long enough, to reach their steady state.

There are several systems which operate at discrete times see Kobayashi (1983). As a result it becomes important to study them, in such cases events are clock controlled.

In this chapter we analyze a discrete time transient analysis of a Markovian Queueing model of a system with queue dependent servers. Such problems occur not only in Queueing theory but also in bio- science and now in computer science we given closed
form solution to this class of problems in terms of the roots of a polynomial in Z-transform and results are computed even where the matrices involved are large. It is also shown how the results for the continuous can be obtained. Interesting analogy exists between the discrete time models and their continuous time counter parts.

Results presented in this chapter further unify the treatment given by Chaudhry, Kapur, Templeton (1991). It is worth noting though, continuous time models are particular cases of discrete time models, yet this area of research has remain neglected. Assume that the customers arrive individually at a service facility in accordance with a Poisson process having rate \( \lambda \). The queue discipline is FIFO. The service facility consists of one regular service channels. The system starts with a regular service channels. The system starts with a regular service channel and \( c \)- additional service channel having service rate \( \mu \).

The regular channel is always open irrespective of queue length. If the system is empty then the regular channel is idle. If the number of customers in the system increases to \( r (0) = 1 \) then the regular channel will become busy. If the number of customer in the system is \( r (1) \) then one additional service channel with service rate \( \mu^{(1)} \) is started, which will be dropped at the termination of a service, if the system size becomes less than \( r (1) \) at that epoch.

When two channels are operating, if the number of units in the system increases to \( r (2) ( > r (1)) \), the second additional channels is limited to a maximum of \( c \) and when all the \( c \)-channels are operating together with the regular channels.

The queue is allowed to grow without limit initially system is assumed to have \( c \)-units. Let \( p_m(n) \) be the probability of \( n \) customers in the system at \( m^\text{th} \) epoch or time slot.
Let $\mu^i = \mu + \sum_{n=1}^{\infty} \mu^{(n)}$

The following difference equation may easily be written as

$$p_{m+1}(0) - p_m(0) = -\lambda p_m(0) + \mu p_m(1) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.1)$$

$$p_{m+1}(n) - p_m(n) = -(\lambda + \mu_k) p_m(n) + p_m(n-1) \lambda + \mu_k p_m(n+1) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.2)$$

$$; r(k) \leq n \leq r(k+1)$$

$$p_{m+1}(r(j)-1) - p_m(r(j)-1) = -(\lambda + \mu_j - 1) p_m(r(j)-1) + \mu_j p_m(r(j)) + \lambda p_m(r(j) - 2)$$

Where $j = 1, 2, 3 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.3)$

$$p_{m+1}(n) - p_m(n) = -\lambda p_m(n-1) + \mu_c p_m(n+1) - (\lambda + \mu_c) p_m(n) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.4)$$

$$; n \geq r(c)$$

Defining the probability generating function as

$$P(z,m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_m(n) z^m + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p_m(n+r(c)) z^m \quad |z| \leq 1$$

$$= \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} z^m R(m) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} p_m(n+r-1)$$

using the initial condition, we have

$$P(z,0) = p_0(n) = z^{k(i)} \text{ or } z^{s(i)}$$

now taking the probability generating function of equation (3.1), (3.2), (3.3), (3.4)

$$R(m) = \sum_{n=0}^{\infty} p_m(n)$$

$$P(z,m,n) = R(m) + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p_m(n+r(c)) z^n$$

$$P(z,m,n) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} p_m(n+r(c)) z^n$$

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\[ P(z, m, n) = \sum_{m=0}^{\infty} \sum_{n=1}^{N} z^m p_m(n) \]
\[ = \sum_{n=1}^{N} z^0 p_0(n) + z^1 p_1(n) + z^2 p_2(n) + \ldots \]

\[ p_m(n) = \text{Coeff of } z^m \text{ in the expansion of } P(z, m, n). \]

Multiplying eq. (3.1) by \( z^n \) and summing from 0 to \( \infty \), we have

\[ z^m p_{m+1}(0) - p_m(0) z^m = -\lambda P_m(0) z^m + \mu P_m(1) \cdot z^m \]

\[ \sum_{m=0}^{\infty} z^m p_{m+1}(0) - p_m(0) z^m = -\lambda \sum_{m=0}^{\infty} z^m + \mu \sum_{m=0}^{\infty} z^m \cdot z^m \]

\[ (1/z) \sum_{m=0}^{\infty} z^{m+1} p_{m+1}(0) - p(z, m, o) = -\lambda p(z, m, o) + \mu p(z, m, 1) \]

\[ (1/z) [p(z, m, o) - p_0(0)] - P(z, m, 0) = -\lambda P(z, m, o) + \mu P(z, m, 1) \]

\[ (1 - z)/z P(z, m, o) - 1/2 p_0(0) = -\lambda P(z, m, o) + \mu P(z, m, 1) \]

Denoting \( (1 - z)/z = s \), we have

\[ s P(z, m, o) - 1/z p_0(0) = -\lambda P(z, m, o) + \mu P(z, m, 1) \]

\[ (s + \lambda) P(z, m, o) - \mu P(z, m, 1) = 1/z p_0(0) \quad \text{(3.5)} \]

Multiplying eqn (3.2) by \( z^m \) and summing over 0 to \( \infty \), we have

\[ \sum_{m=0}^{\infty} p_{m+1}(n) z^m - \sum_{m=0}^{\infty} p_m(n) z^m = - (\lambda + \mu_k) \sum_{m=0}^{\infty} p_m(n) z^m + 1 \sum_{m=0}^{\infty} p_{m-1}(n) z^m + \mu R p_m(n+1) z^m \]

\[ (1/z) \sum_{m=0}^{\infty} p_{m+1}(n) z^m = 1 - \sum_{m=0}^{\infty} p_m(n) z^m = - (\lambda + \mu_k) \sum_{m=0}^{\infty} p_m(n) z^m + \lambda \sum_{m=0}^{\infty} p_{m-1}(n) z^m + \mu_k \]

\[ (1/z) [P(z, m, o) - P_0(n)] - P(z, m, n) = - (\lambda + \mu_k) P(z, m, n) + P(z, m, n-1) + \mu_k P(z, m, n+1) \]

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\[(1-z)/z \ P(z, m, n) - 1/z \ p_0(n) = -(\lambda + \mu_k) \ P(z, m, n) + \lambda \ P(z, m, n-1) + \mu_k \ P(z, m, n+1)\]

\[s \ P(z, m, n) + P(z, m, n) - \lambda \ P(z, m, n-1) - \mu_k \ P(z, m, n+1) = 1/z \ p_0(n)\]

\[(s + \lambda + \mu_j) \ P(z, m, n) - \lambda \ P(z, m, n-1) - \mu_j \ P(z, m, n+1) = 1/z \ p_0(n) ; \ r(r) \leq n \leq r(r+1)\]...........(3.6)

Multiplying eqn (3.3) by \(z^m\) and summing from 0 to \(\infty\), we have

\[\sum_{m=0}^{\infty} p_{m+1} (r(j) - 1) z^m - \sum_{m=0}^{\infty} p_m (r(j) - 1) z^m = - (\lambda + \mu_j) \sum_{m=0}^{\infty} p(r(j) - 1) z^m + \mu_j \sum_{m=0}^{\infty} p_m(r(j)) z^m\]

\[+ \lambda \sum_{m=0}^{\infty} p_m (r(j) - 2) z^m\]

\[1/z \sum_{m=0}^{\infty} p_{m+1}(r(j) - 1)z^{m+1} - \sum_{m=0}^{\infty} p_m (r(j) - 1) z^m = - (\lambda + \mu_j) \sum_{m=0}^{\infty} p_m(r(j) - 1) z^m + \mu_j \sum_{m=0}^{\infty} p_m (r(j)) z^{m+1} + \sum_{m=0}^{\infty} p_m (r(j) - 2) z^m\]

\[1/z [P(z, m, (r(j) - 1)) \ P_0(r(j) - 1)] - P(z, m, (r(j) - 1)) = - (\lambda + \mu_j) P(z, m, r(j) - 1) + \mu_j P(z, m, r(j)) + \lambda P(z, m, r(j) - 2)\]

\[(1 - z)/z \ P(z, m, r(j) - 1) - 1/z \ p_0 (r(j) - 1)\]

\[= - (\lambda + \mu_j) P(z, m, r(j) - 1) + \mu_j P(z, m, r(j)) + \lambda P(z, m, r(j) - 2)\]

\[s \ P(z, m, r(j) - 1) + - (\lambda + \mu_j) P(z, m, r(j) - 1) - \mu_j P(z, m, r(j)) - \lambda P(z, m, r(j) - 2)\]

\[= 1/z \ p_0(r(j)-1)\]

\[(s+ \lambda + \mu_j) P(z, m; r(j) - 1) + \mu_j P(z, m, r(j)) + \lambda P(z, m, r(j) - 2) = 1/z \ p_0(r(j)-1)\]...........(3.7)

Multiplying eqn. (3.4) by \(z^m\) and summing over 0 to \(\infty\), we get
\[
\sum_{m=0}^{\infty} p_{m+1}(n) z^m - \sum_{m=0}^{\infty} p_m(n) z^n = \lambda \sum_{m=0}^{\infty} p_m(n-1) z^m + \mu_c \sum_{m=0}^{\infty} p_m(n+1) z^m
\]

\[
- (\mu_k \lambda) + \sum_{m=0}^{\infty} P_m(n) Z^m
\]

\[
1/z \sum_{m=0}^{\infty} p_{m+1}(n) z^{m+1} - \sum_{m=0}^{\infty} p_m(n) z^n = \lambda \sum_{m=0}^{\infty} p_m(n-1) z^m + \mu_c \sum_{m=0}^{\infty} p_m(n+1) z^m
\]

\[
- (\mu_k \lambda) + \sum_{m=0}^{\infty} p_m(n) z^m
\]

\[
1/z [P(z, m, n) - p_0(n)] - P(z, m, n) = - \lambda P(z, m, n-1) + \mu_c P(z, m, n+1) - (\lambda + \mu_c) P(z, m, n)
\]

\[
(1-z)/z P(z, m, n) - p_0(n) = - \lambda P(z, m, n-1) + \mu_c P(z, m, n+1) - (\lambda + \mu_c) P(z, m, n)
\]

\[
s P (z, m, n) + (\lambda + \mu_c) P(z, m, n) + \lambda P(z, m, n-1) - \mu_c P(z, m, n+1) = 1/z p_0(n)
\]

\[
(s + \lambda + \mu_c) p(z, m, n) - \lambda p(z, m, n-1) + \mu_c p(z, m, n+1) = 1/z p_0(n)
\]

\[
- \lambda p(z, m, n-1) + (s + \lambda + \mu_c) p(z, m, n) + \mu_c p(z, m, n+1) = 1/z p_0(n)
\]

\[
\text{AP} = [\delta_{ij} \delta_{i1} \cdots \delta_{iN}]'
\]

Where A is a real tri-diagonal matrix of order (N+1)x (N+1) matrix.

P is a column vector.

\[
\delta_{ij} \text{ is the Kronecker delta defined as}
\]

\[
\delta_{ij} = \begin{cases} 
\frac{1}{2} & \text{if } j=i \\
0 & \text{if } j \neq i 
\end{cases}
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