Chapter-III

CURVATURE COLLINEATION AND CONFORMAL MOTION IN A FINSLER SPACE EQUIPPED WITH NON-SYMMETRIC AND SEMI-SYMMETRIC CONNECTION COEFFICIENTS
1. INTRODUCTION:

Katzin, Levine and Davis[6] have defined curvature collineation in a Riemannian space and studied its properties. They also studied the properties of curvature collineations in conformally flat spaces. In a Finsler space the theory of curvature collineation have been studied by Singh and Prasad [13], Pande and Kumar[11], while carrying out such studies in a Finsler space they have taken into account Cartan’s and Berwald’s curvature tensors and have obtained the relations between curvature collineation and other symmetries admitted by the Finsler space. We have discussed the theory of curvature collineation and also the theory of conformal motion with the help of curvature tensor type entities $R_{jkh}^{i}(x,\dot{x})$ depending on the non-symmetric connection coefficient $\Gamma_{hk}^{i}(x,\dot{x})$ and $R_{jkh}^{i}(x,\dot{x})$ depending on the semi-symmetric connection coefficient $\Pi_{hk}^{i}(x,\dot{x})$. In different sections of this chapter, several relations between the curvature collineations, conformal motion and other symmetries admitted by such a Finsler space have been obtained.

The $\Theta$-covariant derivative of an arbitrary tensor field

\begin{equation}
(1.1) \quad T_{j}^{\ i} \quad \quad \quad (+) = \partial_{k} T_{j}^{\ i} - \left( \partial_{m} T_{j}^{\ i} \right) \Gamma_{pk}^{m} \dot{x}^{p} + T_{j}^{\ p} \Gamma_{pk}^{i} - T_{p}^{\ i} \Gamma_{jk}^{p}
\end{equation}

where $\Gamma_{jk}^{i}$ is the non-symmetric connection coefficient written in the form

\begin{equation}
(1.2) \quad \Gamma_{jk}^{i} = M_{jk}^{i} + N_{jk}^{i}.
\end{equation}

Here $M_{jk}^{i}$ and $\frac{1}{2} N_{jk}^{i}$ are respectively the symmetric and skew-symmetric parts of $\Gamma_{jk}^{i}$. In view of the $\Theta$-covariant derivative, the
Lie derivative of $T^i_j(x, \dot{x})$ and non-symmetric connection $\Gamma^i_{jk}$ are given by Gupta[2] as:

\begin{equation}
\mathcal{L}_v T^i_j(x, \dot{x}) = T^i_j + \left( \partial^i_k \right) T^i_j \left|_k \right. v^k + \left( \partial^i_k T^i_j \right) \left|_h \right. \dot{x}^h - T^i_j v^i + \left|_k \right. + T^i_k v^k \left|_j \right.
\end{equation}

and

\begin{equation}
\mathcal{L}_v \Gamma^i_{jk}(x, \dot{x}) = \left( v^i + \right|_j \left. \right|_k - \left( \partial^i_s \Gamma^i_{sk} \right) \left|_h \right. \dot{x}^h + v^h R^i_{jkh},
\end{equation}

where $R^i_{jkh}(x, \dot{x})$ is given as

\begin{equation}
R^i_{jkh} = \partial^i_k \Gamma^h_{ij} - \partial^i_j \Gamma^h_{ik} + \partial^i_m \Gamma^h_{ik} \Gamma^m_{sj} \dot{x}^s + \Gamma^r_{ij} \Gamma^h_{rk} \Gamma^h_{pk} - \Gamma^p_{ik} \Gamma^h_{pj}.
\end{equation}

In between the operators $\mathcal{L}_v$, $\partial^i_k$ and $v^i + |_k$, we have the following commutation formulae

\begin{equation}
\mathcal{L}_v \left( \partial^i_k T^i_j \right) - \partial^i_k \left( \mathcal{L}_v T^i_j \right) = 0,
\end{equation}

\begin{equation}
\mathcal{L}_v \left( T^i_j + \right|_k \left. \right|_l - \left( \mathcal{L}_v T^i_j \right) + \left|_k \right. = T^h_j \mathcal{L}_v \Gamma^i_{hk} - T^i_l \mathcal{L}_v \Gamma^h_{jk} + \left( \partial^i_k T^i_j \right) \left( \mathcal{L}_v \Gamma^h_{sk} \right) \dot{x}^s
\end{equation}

and

\begin{equation}
\mathcal{L}_v \Gamma^i_{jk}(x, \dot{x}) + \left|_k \right. - \left( \mathcal{L}_v \Gamma^i_{hk} \right) + \left|_j \right. = \mathcal{L}_v R^i_{jkl} + \dot{x}^j \Gamma^r_{rlj} \mathcal{L}_v \Gamma^r_{ik} - \dot{x}^j \Gamma^r_{rlk} \Gamma^r_{lj} + N^r_{kj} \mathcal{L}_v \Gamma^i_{hr}.
\end{equation}

We give the following definitions, which shall be used in the later discussions.

**DEFINITION (1.1):** Affine motion, Yano [14]

A Finsler space $F_n$ is said to admit an affine motion provided that there exists a vector $v^i(x)$ such that

\begin{equation}
\mathcal{L}_v G^i_{jk}(x, \dot{x}) = 0,
\end{equation}

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where $G^i_{jk}$ is the Berwald's connection coefficient.

**DEFINITION (1.2):**

In Finsler space $F_n$, if the Berwald's curvature tensor field $H^i_{hjk}(x, \dot{x})$ satisfies the relation

$$ (1.10) \quad H^i_{hjk(m)} = 0, $$

then such a Finsler space is called a symmetric Finsler space, Pande and Kumar[11]. In view of (1.10), the following relations can also be obtained

$$ (1.11) \quad a) \quad H^i_{jk(m)} = 0, \quad b) \quad H^i_{j(m)} = 0, \quad c) \quad H_{(m)} = 0. $$

2. **PROJECTIVE CURVATURE COLLINEATION IN A FINSLER SPACE EQUIPPED WITH NON-SYMMMETRIC CONNECTION:**

Let us consider an infinitesimal point transformation

$$ (2.1) \quad \bar{x}^i = x^i + v^i(x)d\zeta $$

where $v^i(x)$ determines a non-zero contravariant vector field defined over the domain of the space under consideration and $d\zeta$ is an infinitesimal constant.

**DEFINITION (2.1):**

The infinitesimal point transformation (2.1) defines a projective curvature collineation in a Finsler space equipped with non-symmetric connection provided that the space under consideration admits a vector field $v(x)$ such that

$$ (2.2) \quad \mathcal{L}_v R^i_{jkh} = 0. $$
DEFINITION (2.2):
The Finsler space equipped with non-symmetric connection is said to admit a Ricci type projective curvature collineation if there exists a vector field $v^i(x)$ such that
\[
(2.3) \quad \mathcal{L}_v R^i_{kh} = 0 \quad \text{where we have written} \quad R^i_{kh} = R^i_{ikh}.
\]

DEFINITION (2.3):
The infinitesimal point transformation (2.1) defines a special projective motion in a Finsler space equipped with non-symmetric connection, if the Lie-derivative of $\Gamma^i_{jk}(x,\dot{x})$ with (2.1) itself has the form
\[
(2.4) \quad \mathcal{L}_v \Gamma^i_{jk}(x,\dot{x}) = \delta^i_j \varepsilon^k_k + \delta^i_k \varepsilon^j_j
\]
where $\varepsilon^k_k$ is any non-zero covariant vector and satisfies the relation
\[
(2.5) \quad \varepsilon^k_k \dot{x}^k = 0.
\]

We now propose to investigate the conditions under which a special projective motion becomes a projective curvature collineation. The Lie-derivative of $R^i_{ijk}(x,\dot{x})$ in view of (1.8) assumes the following form
\[
(2.6) \quad \mathcal{L}_v R^i_{ijk} = \left( \mathcal{L}_v \Gamma^i_{hk} \right)^i_j - \left( \mathcal{L}_v \Gamma^i_{hk} \right)^i_j - \dot{x}^i \Gamma^i_{rhj} \mathcal{L}_v \Gamma^r_{ik} + \dot{x}^i \Gamma^i_{rhk} \mathcal{L}_v \Gamma^r_{ij} - N^r_{kj} \mathcal{L}_v \Gamma^i_{jk}.
\]
We use (2.4) in (2.6) and get
\[
(2.7) \quad \mathcal{L}_v R^i_{ijk} = \delta^i_h \left( \varepsilon^j_k \varepsilon^i_j - \varepsilon^i_k \varepsilon^j_j \right) + \delta^i_j \left( \varepsilon^i_h \varepsilon^j_k \right) - \delta^i_k \varepsilon^j_h \varepsilon^i_j - \Gamma^i_{rkh} \varepsilon^j_k \dot{x}^i - \Gamma^i_{rjk} \varepsilon^j_r \varepsilon^i_h - N^r_{kj} \varepsilon^j_r \varepsilon^i_h - N^r_{kj} \varepsilon^i_h.
\]
Using the equation (2.5) and the fact that $\varepsilon^i_k = \dot{\varepsilon}_k \varepsilon^i$ in (2.7), we get...
$$\text{(2.8)} \quad \ell^i \epsilon^j_{\ell k} \delta^i_j \left( \epsilon^i_k - \epsilon^i_k \right) + \delta^i_j \epsilon^i_j - \delta^i_k \epsilon^i_j - N_i^j \delta^i_j \in_r - N_i^j \in_h. $$

If we now suppose that the special projective motion in the space under consideration is also a projective curvature collineation, then we get the following with the help of (2.2) and (2.8)

$$\text{(2.9)} \quad \delta^i_j \left( \epsilon^i_k - \epsilon^i_k \right) + \delta^i_j \epsilon^i_j - \delta^i_k \epsilon^i_j - N_i^j \delta^i_j \in_r - N_i^j \epsilon^i_j = 0. $$

Contracting (2.9) with respect to the indices $i$ and $j$, we get

$$\text{(2.10)} \quad n \epsilon^i_k + \epsilon^i_k \in_j - N_i^j \epsilon^i_j - N_i^j \epsilon^i_j = 0. $$

Therefore, we can state the following:

**THEOREM(2.1):**

In a Finsler space equipped with non-symmetric connection if a special projective motion becomes a projective curvature collineation then (2.11) necessarily holds.

At this stage, if we now assume that the non-zero covariant vector $\epsilon^i_k$ appearing in (2.4) is a covariant constant then under this assumption (2.10) can alternatively be written as (the Finsler space having this property shall be termed as special Finsler space)

$$\text{(2.11)} \quad N^r_i \delta^i_j \in_r + N^r_i \epsilon^i_j = 0. $$

Hence, we can state:

**THEOREM(2.2):**

In a special Finsler spaces equipped with non-symmetric connection, if a special projective motion becomes a projective collineation then (2.11) necessarily holds.
We now contract (2.8) with respect to the indices $i$ and $h$ and therefore use (2.3) to get

\[(2.12) \; \mathcal{E}_{\alpha} R^\alpha_{jk} = (n+1) \left( \varepsilon_j^i \varepsilon_k^i - \varepsilon_k^i \varepsilon_j^i - N^i_{kj} \right). \]

\[(2.12) \text{ after making use of (2.3) can alternatively be written as} \]

\[(2.13) \; \varepsilon_j^i \varepsilon_k^i - \varepsilon_k^i \varepsilon_j^i - N^i_{kj} \varepsilon_i = 0. \]

Thus, we can state:

**THEOREM (2.3):**

In a Finsler space equipped with non-symmetric connection, if the special projective motion becomes a Ricci type projective curvature collineation then the equation \((2.13)\) necessarily holds.

At this stage, if we again suppose that the non-zero covariant vector appearing in (2.3) is a covariant constant then under this assumption \((2.13)\) can alternatively be written as

\[(2.14) \; N^i_{kj} \varepsilon_i = 0. \]

With the help of \((2.14)\), we can state the following.

**THEOREM (2.4):**

In a special Finsler space equipped with non-symmetric connection, if a special projective motion becomes a projective collineation then either of the following two holds always (i) Either $\varepsilon_i = \partial_i \varepsilon$ is a constant or (ii) The skew-symmetric part of $\Gamma^i_{jk}$ i.e. $N^i_{jk}$ vanishes.

Now, we apply the commutation formula (1.7) to $R^i_j(x, \dot{x})$ and thereafter use the equation \((1.11^b)\) and \((2.4)\) to get
(2.15) \( \left( \mathcal{L}_x R^i_j \right)^+ \big|_k = R^i_k \in_j + \left( \hat{\partial}_s R^i_j \right) \in_k \dot{x}^s - R^h_j \delta^i_k \in_h \). 

Contracting (2.15) with respect to the indices i and j, we get

(2.16) \( \left( \mathcal{L}_x R^i_j \right)^+ \big|_k = \left( \hat{\partial}_s R^i_j \right) \in_k \dot{x}^s \).

Thus, we can state:

**THEOREM (2.5):**

*If a projective symmetric Finsler space equipped with non-symmetric connection defined in analogy to (1.11b) admits a special projective motion then (2.15) always holds.*

3. **CONFORMAL MOTION IN A FINSLER SPACE EQUIPPED WITH NON-SYMMETRIC CONNECTION:**

If the infinitesimal point change (2.1) implies that the magnitude of the vectors defined in the same tangent space are proportional and the angle between the two directions is also the same with respect to the metrics then it is called a conformal motion in Fn. We adopt this concept in the Finsler space equipped with non-symmetric connection as well. The variation of \( G^i_{jk} (x, \dot{x}) \) under the infinitesimal point transformation is \( \mathcal{L}_x G^i_{jk} (x, \dot{x}) \) and that under the conformal change is \( \widetilde{G}^i_{jk} (x, \dot{x}) \). The two transformations shall coincide if the corresponding variations are the same.

The necessary and sufficient condition in order that the infinitesimal point change (2.1) may define a conformal motion in the Finsler space equipped with non-symmetric connection is that the Lie-derivative of connection coefficient \( \Gamma^i_{jk} (x, \dot{x}) \) satisfies the relation, Izumi [5]
(3.1) \( f^i_v \Gamma^j_{jk} = \delta_j^i \epsilon_k + \delta^i_k \epsilon_j - \epsilon^i g_{jk}, \)

where \( \epsilon^i (x) \) is a non-zero vector field and satisfies the relation

(3.2) \( \epsilon^i = g^{ik} \epsilon_k. \)

In the light of facts mentioned in the foregoing lines, we can also conclude the following

(3.3) \( f^i_v \Gamma^i_{shk} = -2 \epsilon^i C^*_x \)

and

(3.4) \( f^i_v \Gamma^i_{jk} \dot{x}^j = \epsilon_k \dot{x}^i - \epsilon^i g_{jk} \dot{x}^j, \)

where

(3.5) a) \( C^*_x = \frac{1}{2} \dot{\partial}_s g_{shk} \) b) \( \epsilon_k = \partial_k \epsilon. \)

We shall now investigate the circumstances under which a conformal motion given by (3.1) becomes a curvature collineation.

In view of (3.1), the commutation formula (1.8) gives

(3.6) \( f^i_v R^i_{hjk} = \epsilon^r \left( g^{rk} \Gamma^i_{rkj} - g_{ir} \Gamma^r_{rkj} \right) \dot{x}^r + \Gamma^r_{rhk} \left( \epsilon^r g_{rk} \dot{x}^r - \epsilon_k \dot{x}^r \right) + \)

\( + \delta^r_h \left( \epsilon^r_k - \epsilon^r_k \right) + \delta^r_j \epsilon^r_k \epsilon^r_k - \epsilon^r_k g_{hk} - \epsilon^r_k g_{bk} \)

\( - \delta^r_k \epsilon^r_k \epsilon^r_k - \epsilon^r_k g_{hk} - \epsilon^r_k g_{bk} \)

We now make two different assumptions one that the covariant vector \( \epsilon_i \) is a covariant constant and the other that the covariant vector \( \epsilon_i \) is a constant as well as the fundamental tensor is metrical, then under these two different assumptions we get two more equations from (3.6) as

(3.7) \( f^i_v R^i_{hjk} = \epsilon^r g_{rk} + \epsilon^r \epsilon^r g_{hk} - \epsilon^r g_{rk} \Gamma^r_{rkj} \dot{x}^r + \)

\( + N^r_{kj} \left( \delta^r_h \epsilon_k + \delta^r_i \epsilon_k - \epsilon^r g_{hr} \right), \)
and

\begin{align*}
(3.8) \quad \mathcal{F}_i^\ell R^i_{hjk} &= \varepsilon^r g_{(k} \Gamma^i_{lj} \chi^\ell + \Gamma^i_{rbk} \varepsilon^r g_{(i} \chi^\ell - \varepsilon^r \Gamma^i_{rhk} g_{ij} \chi^\ell - \\
& \quad - N^r_{kj} \left( \delta^i_h \varepsilon^r + \delta^i_r \varepsilon^h - \varepsilon^i g_{hr} \right). 
\end{align*}

Introducing (2.2) in (3.6), (3.7) and (3.8), we get

\begin{align*}
(3.9) \quad \delta^i_h \varepsilon^i_j |_{k} - \varepsilon^i_k |_{j} + \delta^i_j \varepsilon^i_h |_{k} - \varepsilon^i_k |_{j} g_{hk} - \varepsilon^i g_{hk} + |_{k} - \delta^i_h \varepsilon^i_j |_{k} - \\
& \quad - \varepsilon^i_j g_{hk} - \varepsilon^i g_{hk} + \Gamma^i_{rhj} \varepsilon^r + \varepsilon^r g_{(k} \Gamma^i_{lj} \chi^\ell + \varepsilon^i j \Gamma^i_{rhk} \chi^r - \\
& \quad - \varepsilon^r \Gamma^i_{rhk} g_{ij} \chi^\ell - N^r_{kj} \left( \delta^i_h \varepsilon^r + \delta^i_r \varepsilon^h - \varepsilon^i g_{hr} \right) = 0, \\
(3.10) \quad \varepsilon^i g_{hk} |_{k} + \varepsilon^i g_{hk} + |_{j} + \Gamma^i_{rhj} \varepsilon^r - \varepsilon^r g_{(k} \Gamma^i_{lj} \chi^\ell + \\
& \quad + N^r_{kj} \left( \delta^i_h \varepsilon^r + \delta^i_r \varepsilon^h - \varepsilon^i g_{hr} \right) = 0, \\
\text{and}
\end{align*}

\begin{align*}
(3.11) \quad \varepsilon^r g_{(k} \Gamma^i_{lj} \chi^\ell - \Gamma^i_{rhj} \varepsilon^r _{k} \chi^r - \varepsilon^r \Gamma^i_{rhk} g_{ij} \chi^\ell - \\
& \quad - N^r_{kj} \left( \delta^i_h \varepsilon^r + \delta^i_r \varepsilon^h - \varepsilon^i g_{hr} \right) = 0. 
\end{align*}

Contracting (3.9), (3.10) and (3.11) with respect to the indices $i$ and $j$, we get

\begin{align*}
(3.12) \quad (n-2) \varepsilon^i_j |_{k} - \varepsilon^i_k |_{j} - \varepsilon^i |_{k} - \varepsilon^i k |_{j} g_{hk} - \varepsilon^i g_{hk} + |_{i} - \Gamma^i_{rhj} \varepsilon^r + \\
& \quad + \varepsilon^r g_{(k} \Gamma^i_{lj} \chi^\ell + \Gamma^i_{rhj} \varepsilon^r _{k} \chi^r - \varepsilon^r \Gamma^i_{rhk} g_{ij} \chi^\ell - N^r_{kj} \varepsilon^r - \\
& \quad - N^r_{ki} \varepsilon^r + \varepsilon^r N^r_{ki} g_{hr} = 0, \\
(3.13) \quad \varepsilon^r g_{(k} \Gamma^i_{rhj} \chi^\ell - \Gamma^i_{rhj} \varepsilon^r _{k} \chi^r - \varepsilon^i g_{hk} + |_{i} + \Gamma^i_{rhj} \varepsilon^r - \\
& \quad - \varepsilon^r \Gamma^i_{rhk} g_{ij} \chi^\ell - N^r_{kh} \varepsilon^r _{i} - N^r_{ki} \varepsilon^i + \varepsilon^i N^r_{ki} g_{hr} = 0, \\
\text{and}
\end{align*}

\begin{align*}
(3.14) \quad \varepsilon^r g_{(k} \Gamma^i_{rhj} \chi^\ell - \Gamma^i_{rhj} \varepsilon^r _{k} \chi^r - \varepsilon^r \Gamma^i_{rhk} g_{ij} \chi^\ell - N^r_{kh} \varepsilon^r - \\
& \quad - N^r_{kr} \varepsilon^r + \varepsilon^r N^r_{ki} g_{hr} = 0.
\end{align*}
Thus, we can state the following:

**THEOREM (3.1):**

If a Finsler space equipped with non-symmetric connection admits a conformal motion characterised by (3.1) and a curvature collineation given by (2.2) then the equation (3.12) necessarily holds.

**THEOREM (3.2):**

In a Finsler space (equipped with non-symmetric connection) admitting a conformal motion characterised by (3.1) and a curvature collineation given by (2.2) if the non-zero vector $\varepsilon_k$ appearing in (3.1) be assumed to be a covariant constant then equation (3.13) necessarily holds.

**THEOREM (3.3):**

In a Finsler space admitting a conformal motion and curvature collineation respectively characterised by (3.1) and (2.2) if the non-zero vector $\varepsilon_k$ appearing in (3.1) be a covariant constant and the fundamental tensor be assumed to be metrical then (3.14) necessarily holds.

Contracting (3.6) with respect to the indices $i$ and $k$, we get

\[
(3.15) \quad \ell^i_{,k} R^i_{jkl} = \varepsilon^r \left( g_{ij} \Gamma^i_{rj} - g_{ij} \Gamma^i_{ri} \right) \hat{x}^r + \Gamma^i_{rj} \left( \varepsilon^r g_{ij} \hat{x}^r - \varepsilon_i \hat{x}^r \right) - \\
- \delta^i_k \varepsilon_j^+ |^j + \varepsilon_j^+ |^h - (n-1) \varepsilon^+ |^j - \varepsilon^+ |^r g_{ij} - \varepsilon^r g_{ij} |^r \hat{x}^r - \\
- \varepsilon^i g_{hi} - \varepsilon^i g_{hi} |^j - N^r_{ij} \varepsilon_r - N^r_{ij} \varepsilon_h + N^r_{ij} \varepsilon^r g_{hr}.
\]

At this stage, we now suppose that the covariant vector $\varepsilon_i$ appearing in (3.1) is a covariant constant as well as the fundamental tensor is metrical, then under this assumption we get the following from (3.15)

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(3.16) \( \mathcal{L}_v R^i_{hji} = \varepsilon^r \left( g_{\ell i} \Gamma^i_{r hj} - g_{ij} \Gamma^i_{r hi} \right) \dot{x}^\ell + \Gamma^i_{r hj} \left( \varepsilon^r g_{\ell i} \dot{x}^\ell - \varepsilon_i \dot{x}^r \right) - N^r_{bj} \varepsilon_r - N^i_{ij} \varepsilon_i + N^r_{ij} g_{hr} \varepsilon^i. \)

Thus, we can state the following:

**THEOREM (3.4):**

If an Fn equipped with non-symmetric connection admits a conformal motion and a Ricci type curvature collineation respectively characterised by (3.1) and (2.3) then we can always have.

\[
(3.17) \varepsilon^r \left( g_{\ell i} \Gamma^i_{r hj} - g_{ij} \Gamma^i_{r hi} \right) \dot{x}^\ell + \Gamma^i_{r hj} \left( \varepsilon^r g_{\ell i} \dot{x}^\ell - \varepsilon_i \dot{x}^r \right) - \delta^i_h \varepsilon_i |^j + \\
+ \varepsilon_j |^h - (n-1) \varepsilon_h |^j - \varepsilon^i |^j, g_{bj} - \varepsilon^i g_{bj} |^i - \varepsilon^i g_{hi} - \\
- \varepsilon^i g_{hi} |^j - N^r_{bj} \varepsilon_r - N^i_{ij} \varepsilon_i + N^r_{ij} \varepsilon^i g_{hr} = 0.
\]

**THEOREM (3.5):**

In a Finsler space admitting a conformal motion and Ricci type curvature collineation respectively characterised by (3.1) and (2.3) if the non-zero vector \( \varepsilon_k \) appearing in (3.1) be a covariant constant and the fundamental tensor be assumed to be metrical then we shall always have

\[
(3.18) \varepsilon^r \left( g_{\ell i} \Gamma^i_{r hj} - g_{ij} \Gamma^i_{r hi} \right) \dot{x}^\ell + \Gamma^i_{r hj} \left( \varepsilon^r g_{\ell i} \dot{x}^\ell - \varepsilon_i \dot{x}^r \right) - \\
- N^r_{bj} \varepsilon_r - N^i_{ij} \varepsilon_i + N^r_{ij} g_{hr} \varepsilon^i = 0.
\]

By applying the commutation formula (1.7) to the tensor field \( R^i_j (x, \dot{x}) \) and thereafter using (3.1), we get

\[
(3.19) \left( \mathcal{L}_v R^i_j \right)^+ |^k_k = \mathcal{L}_v \left( R^i_j \right)^+ |^k_k + \delta^i_k R^i_j \varepsilon_i - \varepsilon^i R^\ell_j g_{k\ell} - R^i_j \varepsilon_j + \\
+ R^i_j \varepsilon_i g_{bj} - \left( \hat{o}_m R^i_j \right) \varepsilon_k \dot{x}^m + \left( \hat{o}_j R^i_j \right) \varepsilon^\ell g_{km} \dot{x}^m.
\]
If we now assume that $R^i_j$ is independent of directional arguments, then from (3.19) we get

$$\left(\mathcal{L}_v R^i_j\right)^{\perp}_k = \partial^i_k R^i_j \in \varepsilon - \varepsilon^i R^i_{kj} - R^i_j \in_j + R^i_j \in^\ell g_{kj},$$

where we have written $R^\ell_j g_{kj} = R^\ell_j$ and have also assumed that $R^i_j$ satisfies an equation of the type (1.11b).

Contracting (3.19) with respect to the indices $i$ and $j$, we get

$$\left(\mathcal{L}_v R^i_i\right)^{\perp}_k = R^i_t \in^\ell g_{ki} - \varepsilon^i R^i_{ki} - \left(\partial^\ell_m R^i_i\right) \in_k \dot{x}^m +$$

$$\left(\partial^\ell_t R^i_i\right) \in^\ell g_{km} \dot{x}^m.$$

If we now assume that $R^i_i$ too is independent of directional arguments then from (3.21), we get

$$\left(\mathcal{L}_v R^i_i\right)^{\perp}_k = R^i_t \in^\ell g_{ki} - \varepsilon^i R^i_{ki}.$$

Thus, we can state the following:

**THEOREM (3.6):**

If a symmetric Finsler space equipped with non-symmetric connection admits a conformal motion characterised by (3.1) then (3.21) always holds.

**THEOREM (3.7):**

If in a symmetric Finsler space (equipped with non-symmetric connection) admitting conformal motion the contracted tensor $R^i_t$ be supposed to be independent of direction then (3.22) always holds.
4. PROJECTIVE CURVATURE COLLINEATION IN A FINSLER SPACE EQUIPPED WITH SEMI-SYMMETRIC CONNECTION:

We give the following definitions which shall extensively be used in carrying out the studies under this heading.

**DEFINITION (4.1):**

The infinitesimal point transformation (2.1) defines a projective curvature collineation in a Finsler space equipped with semi-symmetric connection provided the space under consideration admits a vector field $v^i(x)$ such that

$$\mathcal{L}_v R^{i}_{jkh} = 0,$$

where $R^{i}_{jkh}$ is a curvature type quantity as has been introduced in [I-(11.8)]. The curvature type quantities $R^{i}_{jkh}$ introduced here are quite different than the curvature defined by the same notation of Rund[12].

**DEFINITION (4.2):**

The Finsler space equipped with semi-symmetric connection is said to admit a Ricci type projective curvature collineation if there exists a vector field $v^i(x)$ such that

$$\mathcal{L}_v R_{kh} = 0$$

where $R_{kh} = R^{i}_{jkh}$.

**DEFINITION (4.3):**

The infinitesimal point transformation (2.1) defines a special projective motion in a Finsler space equipped with semi-symmetric connection (to be denoted by $F^+_n$) if the Lie derivative of the semi-symmetric connection $\Pi^{i}_{jk}(x, \dot{x})$ with (2.1) itself has the form
(4.3) \( \mathcal{L}_v \Pi^i_{jk}(x, \dot{x}) = \delta^i_j \epsilon_k + \delta^i_k \epsilon_j, \)

where \( \epsilon_k \) is any non-zero covariant vector and satisfies the relation

(4.4) \( \epsilon_k \dot{x}^k = 0. \)

Like the studies carried out in section two of this chapter, here too we shall derive the conditions under which a special projective motion becomes a projective curvature collineation in a Finsler space \( F^+_n \) equipped with semi-symmetric connection. The Lie-derivative of \( R^i_{hk}(x, \dot{x}) \) in view of [I-(5.16)] assumes the following form

(4.5) \( \mathcal{L}_v R^i_{hk} = \zeta^i_k \left( \mathcal{L}_v \Pi^i_{kj} \right) - \zeta^i_j \left( \mathcal{L}_v \Pi^i_{hk} \right) - \Pi^i_{rbj} \left( \mathcal{L}_v \Pi^r_{lk} \right) \dot{x}^r + \\
+ \Pi^i_{rhk} \left( \mathcal{L}_v \Pi^r_{lj} \right) \dot{x}^l, \)

where \( \zeta^i_k \) stands for differentiation with respect to semi-symmetric connection as has been mentioned in [I-(11.3)]. We use (4.3) in (4.5) and get

(4.6) \( \mathcal{L}_v R^i_{hk} = \delta^i_j \left( \zeta^i_k \epsilon_j - \zeta^i_j \epsilon_k \right) + \delta^i_j \left( \zeta^i_k \epsilon_h \right) - \delta^i_k \left( \zeta^i_j \epsilon_h \right) - \\
- \epsilon_k \Pi^i_{hj} + \epsilon_j \Pi^i_{hk}, \)

where we have written \( \Pi^i_{rhk} \dot{x}^r = \Pi^i_{hk} \) and have taken into account (4.4). If we now suppose that the special projective motion described in the space \( F^+_n \) is also a special projective collineation then with the help of (4.1) and (4.6), we get

(4.7) \( \delta^i_h \left( \zeta^i_k \epsilon_j - \zeta^i_j \epsilon_k \right) + \delta^i_j \left( \zeta^i_k \epsilon_h \right) - \delta^i_k \left( \zeta^i_j \epsilon_h \right) - \\
- \epsilon_k \Pi^i_{hj} + \epsilon_j \Pi^i_{hk} = 0. \)

We now contract (4.7) with respect to the indices \( i \) and \( k \) and get
(4.8) \( n\xi_j \xi_h - \xi_h \xi_j + \varepsilon_i \Pi_{bj}^i - \varepsilon_j \Pi_{hi}^i = 0 \).

Therefore, we can state the following:

**THEOREM (4.1):**

In a Finsler space \( F_n^+ \) equipped with semi-symmetric connection if a special projective motion becomes a special projective collineation then (4.8) necessarily holds.

Like the assumptions made in section 2 of this chapter we again assume that the non-zero covariant vector appearing in (4.3) is a covariant constant then under this assumption (4.8) can alternatively be written as

\[
(4.9) \varepsilon_i \Pi_{bj}^i - \varepsilon_j \Pi_{hi}^i = 0.
\]

Hence, we can state:

**LEMMA (4.1):**

In the Finsler space \( F_n^+ \) equipped with semi-symmetric connection if the covariant vector appearing in (4.3) be assumed to be a covariant constant then (4.9) necessarily holds.

We now contract (4.6) with respect to the indices \( i \) and \( h \) and get

\[
(4.10) \mathcal{L}_v R_{jk}^i = \mathcal{L}_v R_{jk} = (n+1)\xi_k \xi_j + 2n \xi_j \xi_k + \varepsilon_j \Pi_{ik}^i - \varepsilon_k \Pi_{ij}^i.
\]

If we now assume that the space under consideration admits a Ricci type collineations then from (4.10), we get

\[
(4.11) (n+1)\xi_k \xi_j - 2n \xi_j \xi_k + \varepsilon_j \Pi_{ik}^i - \varepsilon_k \Pi_{ij}^i = 0.
\]

Therefore, we can state:
THEOREM (4.2): 

In a Finsler space $F_n^+$ equipped with semi-symmetric connection if the special projective motion becomes a Ricci type projective curvature collineation then (4.11) necessarily holds.

At this stage, if we again suppose that non-zero vector $\xi_k$ appearing in (4.3) is a covariant constant then under this assumption (4.11) can alternatively be written as

(4.12). $\varepsilon_j \Pi^i_{jk} = \varepsilon_k \Pi^i_{ij} = 0$.

With the help of (4.12), we can therefore state the following:

LEMMA (4.2): 

In a Finsler space $F_n^+$ equipped with semi-symmetric connection if a special projective motion becomes a special projective curvature collineation then (4.12) always holds provided the covariant vector appearing in (4.3) be assumed to be a covariant content.

5. CONFORMAL MOTION IN A FINSLER SPACE $F_n^+$ EQUIPPED WITH SEMI-SYMMETRIC CONNECTION: 

Keeping in mind the discussions made in early section of section 3, we say that the Finsler space $F_n^+$ equipped with semi-symmetric connection defines a conformal motion if

(5.1) $\pounds_x \Pi^i_{jk} = \delta^i_j = \xi_k + \delta^i_j \xi_j - \varepsilon^i g_{jk}$,

where $\varepsilon^i (x)$ is a non-zero vector field and satisfies the relation

(5.2) a) $\varepsilon^i = g^{ik} \xi_k$, b) $\xi_k = \delta^i_k \varepsilon^i$, c) $\pounds_x \Pi^i_{jk} = \xi_k \dot{x}^i = \varepsilon^i g_{jk} \dot{x}^i$.

We shall now investigate the circumstances under which a conformal motion given by (5.1) becomes a curvature collineation.
In view of (5.1), the commutation form [Π-(5.16)] assumes the following form

\begin{equation}
L_v R^i_{hk} = \zeta^i_k \left( \delta^i_h \epsilon_j + \delta^i_j \epsilon_h - \epsilon^i g_{hj} \right) - \zeta^i_j \left( \delta^i_h \epsilon_k + \delta^i_k \epsilon_h - \epsilon^i g_{hk} \right) - \Pi^i_{lj} \epsilon_k + \Pi^i_{hk} \epsilon_j + \epsilon^r g_{lk} \Pi^i_{rl} \dot{x}^\ell - \epsilon^r g_{lj} \Pi^i_{rh} \dot{x}^\ell.
\end{equation}

We shall now make two different assumptions, one that the covariant vector \( \epsilon_k \) appearing in (5.1) is a covariant constant and the other that the covariant vector \( \epsilon_k \) of (5.1) is a covariant constant and also that the fundamental tensor is metrical, then under these two assumptions, we have the following:

\begin{equation}
L_v R^i_{hk} = \epsilon^i \left( \zeta^i_j g_{hk} - \zeta^i_k g_{kj} \right) + g_{hk} \left( \zeta^i_j \epsilon^i \right) - g_{hk} \left( \zeta^i_j \epsilon^i \right) - \Pi^i_{lj} \epsilon_k + \Pi^i_{hk} \epsilon_j + \epsilon^r g_{lk} \Pi^i_{rl} \dot{x}^\ell - \epsilon^r g_{lj} \Pi^i_{rh} \dot{x}^\ell.
\end{equation}

and

\begin{equation}
L_v R^i_{hk} = g_{hk} \left( \zeta^i_j \epsilon^i \right) - g_{hk} \left( \zeta^i_j \epsilon^i \right) - \Pi^i_{lj} \epsilon_k + \Pi^i_{hk} \epsilon_j + \epsilon^r g_{lk} \Pi^i_{rl} \dot{x}^\ell - \epsilon^r g_{lj} \Pi^i_{rh} \dot{x}^\ell.
\end{equation}

Introducing (4.1) in (5.3), (5.4) and (5.5), we respectively get

\begin{equation}
\zeta^i_k \left( \delta^i_h \epsilon_j + \delta^i_j \epsilon_h - \epsilon^i g_{hj} \right) - \zeta^i_j \left( \delta^i_h \epsilon_k + \delta^i_k \epsilon_h - \epsilon^i g_{hk} \right) - \Pi^i_{lj} \epsilon_k + \Pi^i_{hk} \epsilon_j + \epsilon^r g_{lk} \Pi^i_{rl} \dot{x}^\ell - \epsilon^r g_{lj} \Pi^i_{rh} \dot{x}^\ell = 0,
\end{equation}

\begin{equation}
\epsilon^i \left( \zeta^i_j g_{hk} - \zeta^i_k g_{kj} \right) + g_{hk} \left( \zeta^i_j \epsilon^i \right) - g_{hk} \left( \zeta^i_j \epsilon^i \right) - \Pi^i_{lj} \epsilon_k + \Pi^i_{hk} \epsilon_j + \epsilon^r g_{lk} \Pi^i_{rl} \dot{x}^\ell - \epsilon^r g_{lj} \Pi^i_{rh} \dot{x}^\ell = 0,
\end{equation}

and

\begin{equation}
g_{hk} \left( \zeta^i_j \epsilon^i \right) - g_{hk} \left( \zeta^i_j \epsilon^i \right) - \Pi^i_{lj} \epsilon_k + \Pi^i_{hk} \epsilon_j + \epsilon^r g_{lk} \Pi^i_{rl} \dot{x}^\ell - \epsilon^r g_{lj} \Pi^i_{rh} \dot{x}^\ell = 0.
\end{equation}
We now contract (5.6), (5.7) and (5.8) with respect to the indices $i$ and $j$ and get

\begin{equation}
(5.9) \quad (n+1)\left(\xi_k e_h - \xi_k e_k \right) - \xi_k \left(\epsilon^i g_{hi} \right) + \xi_i \left(\epsilon^i g_{hk} \right) - \\
-\Pi_{hi}^i e_k + \Pi_{hk}^i e_i + \epsilon^r g_{tk} \Pi_{ri}^i \dot{x}^i - \epsilon^r g_{ti} \Pi_{rk}^i \dot{x}^i = 0,
\end{equation}

\begin{equation}
(5.10) \quad \xi_i \left(\epsilon^i g_{hk} \right) - \xi_k \left(\epsilon^i g_{hi} \right) - \Pi_{hi}^i e_k + \Pi_{hk}^i e_i + \\
+ \epsilon^r g_{tk} \Pi_{ri}^i \dot{x}^i - \epsilon^r g_{ti} \Pi_{rk}^i \dot{x}^i = 0,
\end{equation}

and

\begin{equation}
(5.11) \quad g_{hk} \left(\xi_k \epsilon^i \right) g_{hi} - \left(\xi_k \epsilon^i \right) - \Pi_{hi}^i e_k + \Pi_{hk}^i e_i + \\
+ \epsilon^r g_{tk} \Pi_{ri}^i \dot{x}^i - \epsilon^r g_{ti} \Pi_{rk}^i \dot{x}^i = 0.
\end{equation}

In the light of these observations, we can therefore state the following:

**THEOREM (5.1):**

If a Finsler space $F^+_n$ equipped with semi-symmetric connection admits a conformal motion and curvature collineation respectively characterised by (5.1) and (4.1) then the equation (5.9) necessarily holds.

**THEOREM (5.2):**

If a Finsler space $F^+_n$ (equipped with semi-symmetric connection) admitting a conformal motion and a curvature collineation respectively characterised by (5.1) and (4.1), if the non-zero vector $\xi_i$ appearing in (5.1) be assumed to be a covariant constant then equation (5.10) necessarily holds.
THEOREM (5.3): 

If a Finsler space \( F_n^+ \) equipped with semi-symmetric connection admitting a conformal motion and a curvature collineation respectively characterised by (5.1) and (4.1), if the non-zero vector \( e_i \) appearing in (5.1) be a covariant constant and the fundamental tensor be assumed to be metrical then (5.11) holds necessarily.

We now contract (5.3) with respect to the indices \( i \) and \( h \) and get

\[
(5.12) \; f_{\nu} R^i_{jk} = f_{\nu} R_{jk} = (n+1) \zeta_i^{[k} \epsilon_j^{l]} - \zeta^{i}_{[k} \epsilon^{j]} g_{\epsilon_l^{\nu},j} - \\
- \Pi_{ij}^{i} \epsilon_k^{j} + \Pi_{ik}^{i} \epsilon_j^{j} + \epsilon_{r} g_{\epsilon_l^{i} \Pi_{rj}^{l}} \dot{x}^{l} - \epsilon_{r} g_{\epsilon_l^{i} \Pi_{rk}^{l}} \dot{x}^{l}.
\]

Therefore, we can state the following:

THEOREM (5.4): 

If an \( F_n^+ \) equipped with semi-symmetric connection admits a conformal motion and a Ricci type curvature collineation respectively characterised by (5.1) and (4.2) then we shall always have

\[
(5.13). \; (n+1) \zeta_i^{[k} \epsilon_j^{l]} - \zeta^{i}_{[k} \epsilon^{j]} g_{\epsilon_l^{\nu},j} - \Pi_{ij}^{i} \epsilon_k^{j} + \Pi_{ik}^{i} \epsilon_j^{j} + \\
+ \epsilon_{r} g_{\epsilon_l^{i} \Pi_{rj}^{l}} \dot{x}^{l} - \epsilon_{r} g_{\epsilon_l^{i} \Pi_{rk}^{l}} \dot{x}^{l} = 0.
\]

At this stage, we make the supposition that the covariant vector appearing in (5.1) is a covariant constant and also that the fundamental tensor is metrical, then under these two suppositions we can get the following from (5.13).

\[
(5.14) \; g_{ik} \left( \zeta_j^{i} \epsilon_l^{j} \right) - g_{ij} \left( \zeta_k^{l} \epsilon_l^{k} \right) - \Pi_{ij}^{i} \epsilon_k^{j} + \Pi_{ik}^{i} \epsilon_j^{j} + \\
+ \epsilon_{r} g_{\epsilon_l^{i} \Pi_{rj}^{l}} \dot{x}^{l} - \epsilon_{r} g_{\epsilon_l^{i} \Pi_{rk}^{l}} \dot{x}^{l} = 0.
\]

Thus, we can state:
THEOREM (5.5):

In a Finsler space $F^+_n$ (equipped with semi-symmetric connection) admitting a conformal motion and a Ricci type curvature collineation respectively characterised by (5.1) and (4.2) if the non-zero vector $e_k$ appearing in (5.1) be a covariant constant and the fundamental tensor be assumed to be metrical then we shall always have (5.14).

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REFERENCES:


