Chapter-II

LIE DERIVATIVES AND AFFINE MOTION IN A FINSLER SPACE EQUIPPED WITH SEMI SYMMETRIC CONNECTION
1. INTRODUCTION:

Davies, E. T.[2] has considered the generalisation of the Lie-derivatives to Finsler space $F_n$ and its application to the theory of subspaces. By considering the infinitesimal transformation [I-(11.1a)] Rund, H. [11] and Yano, K. [16] have defined the Lie-derivatives of an arbitrary vector $X^i(x, \dot{x})$ and the symmetric connection parameter $\Gamma^{ij}_{jk}(x, \dot{x})$ as given in [I-(11.3a) and [I-(11.5)].

In this chapter, we have studied the Lie-derivatives of covariant and contravariant vectors in a Finsler space equipped with semi-symmetric connection as has been introduced by Mehar and Patel [10] and have generalised the results thus obtained for tensor fields of various orders. The Lie-derivatives of semi-symmetric connection parameter have also been discussed. In concluding part of the chapter an attempt has been made to obtain the commutation formulae involving the Lie-derivatives.

2. LIE DERIVATIVES OF TENSOR FIELDS IN A FINSLER SPACE EQUIPPED WITH SEMI-SYMMETRIC CONNECTION:

Let $v^i = v^i(x)$ be a vector field of class $C^2$ defined over a region $R$ of the Finsler space equipped with semi-symmetric connection, we may always associate an infinitesimal transformation of the type [I-(11.1a)] with such a vector field, i.e.

\[(2.1) \quad \bar{x}^i = x^i + v^i(x) \, dt,\]
where $dt$ is an infinitesimal point constant. Under the above transformation considered at each point of the space, we may interpret a shift or displacement $dx^i$ as given below

$$\tag{2.2} dx^i = v^i(x) \, dt,$$

and the corresponding variation of the component $\dot{x}^i$ of the element of support is given by

$$\tag{2.3} \ddot{x}^i = \dot{x}^i + (\partial_h v^i) \dot{x}^h \, dt.$$

Let $X^i(x, \dot{x})$ be a vector field defined over the region R of the Finsler space equipped with semi-symmetric connection. If we assume that $X^i(x, \dot{x})$ is homogenous of degree zero in its directional arguments $x^k$ then this vector will be affected by variations given in (2.1) and (2.3). Following the notations of Rund, H. [11], if we denote the variation arising from (2.1) and (2.3) by $d^r X^i$, we shall have

$$\tag{2.4} d^r X^i = (\partial_k X^i) v^k(x) dt + \left\{ \dot{\partial}_k X^i \left( \partial_h v^k \right) \dot{x}^h \right\} dt.$$

On adding and subtracting the same term (2.4) may be written as

$$\tag{2.5} d^r X^i = \left\{ \partial_k X^i - \left( \dot{\partial}_m X^i \right) \Pi_{kh}^{m} \dot{x}^h \right\} v^k(x) dt + \left( \dot{\partial}_k X^i \right) \left( \partial_h v^k \right) \dot{x}^h dt.$$

By interpreting (2.1) merely as an infinitesimal coordinate transformation and denoting by $\ddot{X}^i$ as the components of $X^i$ in new coordinate system $\left( \ddot{x}^i, \ddot{\dot{x}}^i \right)$, we have

$$\tag{2.6} \ddot{X}^i = (\partial_j \ddot{x}^i) X^j = X^j \left\{ \delta^j_i + \left( \partial_j v^i \right) dt \right\}. $$
We say that the vector $X^i$ is displaced from $(x^i, \dot{x}^i)$ to $(\bar{x}^i, \ddot{x}^i)$ and write

\begin{equation}
(2.7) \quad d^m X^i = \bar{X}^i - X^i = \left( \partial_j v^j \right) X^j dt .
\end{equation}

Rund, H. [11] has defined the Lie-derivative of the vector field $X^i$ in the Finsler space $F_n$ by

\begin{equation}
(2.8) \quad \mathcal{L}_\nu X^i \overset{\text{def.}}{=} \frac{d^v X^i - d^m X^i}{dt} ,
\end{equation}

where $\mathcal{L}_\nu$ denotes the Lie-derivative with respect to the infinitesimal transformation (2.1). We now apply the definition given by (2.8) for the vector field $X^i(x, \dot{x})$ in the Finsler space $F_n^*$ equipped with semi-symmetric connection. Using (2.5), (2.7), (2.8) and adding subtracting the same term, in view of [I-(12.3)], we have

\begin{equation}
(2.9) \quad \mathcal{L}_\nu X^i = \left( \zeta_k X^i \right) v^k - \left( \zeta_j v^j \right) X^j + \left( \dot{\partial}_k X^i \right) \left( \zeta_k v^k \right) \dot{x}^h .
\end{equation}

In view of [I-(12.3)] and (2.9) it can be easily verified that the Lie-derivative of the directional argument $\dot{x}^i$ vanishes identically, i.e.

\begin{equation}
(2.10) \quad \mathcal{L}_\nu \dot{x}^i = 0 .
\end{equation}

Differentiating (2.1) partially, with respect to $\bar{x}^j$ and using the properties of infinitesimal constants of first order, we have

\begin{equation}
(2.11) \quad \frac{\partial x^j}{\partial \bar{x}^i} = \delta^j_i - \left( \partial_j v^j \right) dt .
\end{equation}

For the co-variant vector field $X^i(x, \dot{x})$, we have the following transformation, law, Rund, H. [11].

\begin{equation}
(2.12) \quad \bar{X}_i = X_j \frac{\partial x^j}{\partial \bar{x}^i} = X_j \left\{ \delta^j_i - \left( \partial_i v^j \right) dt \right\} .
\end{equation}
Hence, we shall have

\[(2.13) \ d^m X_i = \ddot{X}_i - X_i = -X_j \left( \partial_j v^i \right) dt \]

and

\[(2.14) \ \mathcal{L}_v X_i = \left( \xi_k X_i \right) v^k + \left( \dot{\partial}_k X_i \right) \xi_h v^k \dot{x}^h + \left( \xi_i v^j \right) X_j \]

In view of (2.9) and (2.14), the Lie-derivative for the mixed tensor \( T^i_j(x,\dot{x}) \) can be given as follows:

\[(2.15) \ \mathcal{L}_v T^i_j = \left( \xi_k T^i_j \right) v^k + \left( \dot{\partial}_k T^i_j \right) \xi_h v^k \dot{x}^h - T^k_j \left( \xi_k v^i \right) + T^k_j \left( \xi_j v^k \right). \]

We can extend these formulae for finding the Lie-derivative of an arbitrary tensor \( T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \) as follows:

\[(2.16) \ \mathcal{L}_v T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \left( \xi_k T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \right) v^k + \left( \dot{\partial}_k T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \right) \xi_h v^k \dot{x}^h - \sum_n T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \xi_{p_{i_r}} v^p + \sum m T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \xi_{j_q} v^k. \]

3. **LIE DERIVATIVE OF TENSOR FIELDS IN THE FINSLER SPACE \( F^*_n \)** **EQUIPPED WITH SEMI-SYMMETRIC CONNECTION:**

Let us consider an infinitesimal transformation with a vector field \( v^i(x,\dot{x}) \) defined over a region \( R \) of \( F^*_n \) as follows:

\[(3.1) \ \ddot{x}^i = x^i + v^i(x,\dot{x}) dt, \]

where \( dt \) is an infinitesimal constant. If we interpret each point of a displacement \( dx^i = v^i(x,\dot{x}) dt \), then the corresponding variation of \( \dot{x}^i \) is given by
\( (3.2) \quad \ddot{x}^i = \dot{x}^i + \left( \partial_h v^i \dot{x}^h + \dot{\partial}_h v^i \ddot{x}^h \right) dt. \)

The variation \( d^\nu X^i \) in \( X^i \), arising from (3.1) and (3.2) can be given by the following relation

\( (3.3) \quad d^\nu X^i = \left\{ \left( \partial_k X^i \right) v^k + \hat{\partial}_k X^i \left( \partial_h v^k \dot{x}^h + \dot{\partial}_h v^k \ddot{x}^h \right) \right\} dt. \)

Using [(I-(12.5a)] and [(I-(12.5b)], equation (3.3) can be rewritten as

\( (3.4) \quad d^\nu X^i = \left( \partial_k X^i - \hat{\partial}_m X^i \Pi_{kh}^m \dot{x}^h \right) v^k dt + \)
\[ + \left( \hat{\partial}_k X^i \right) \left( \zeta_h v^k \right) \dot{x}^h dt + \]
\[ + \left( \hat{\partial}_m X^i \right) \left( \hat{\partial}_m v^k \right) \left( \ddot{x}^m + \Pi_{ph}^m \dot{x}^p \dot{x}^h \right) dt, \]

and \( d^m X^i \) is given by (2.7). Using equations (3.4), (2.7) and [(I-(11.3)], we can get

\( (3.5) \quad \mathcal{L}_v X^i = \left( \hat{\partial}_k X^i \right) \left( \zeta_h v^k \right) \dot{x}^h - X^j \left( \zeta_j v^i \right) + \left( \zeta_k X^i \right) v^k + \)
\[ + \left( \hat{\partial}_k X^i \right) \left( \hat{\partial}_m v^k \right) \left( \ddot{x}^m + \Pi_{ph}^m \dot{x}^p \dot{x}^h \right). \]

Proceeding as above, the Lie-derivative of a covariant vector field \( X_i \) and a mixed tensor \( T^j_i \) can be obtained as under:

\( (3.6) \quad \mathcal{L}_v X_i = \left( \hat{\partial}_k X_i \right) \left( \zeta_h v^k \right) \dot{x}^h + X_j \left( \zeta_i v^j \right) + \left( \zeta_k X_i \right) v^k + \)
\[ + \left( \hat{\partial}_k X_i \right) \left( \hat{\partial}_m v^k \right) \left( \ddot{x}^m + \Pi_{ph}^m \dot{x}^p \dot{x}^h \right) + \left( \hat{\partial}_m v^k \right) X_k \Pi_{pi}^m \dot{x}^p, \]

and

\( (3.7) \quad \mathcal{L}_v T^j_i = \left( \zeta_k T^j_i v^k \right) + \left( \hat{\partial}_k T^j_i \right) \left( \zeta_h v^k \right) \dot{x}^h + T^j_i \left( \zeta_j v^k \right) - \)
\[ - T^j_i \left( \zeta_k v^i \right) + \left( \hat{\partial}_k T^j_i \right) \left( \hat{\partial}_m v^k \right) \left( \ddot{x}^m + \Pi_{ph}^m \dot{x}^p \dot{x}^h \right) + \]
\[ + \left( \hat{\partial}_m v^k \right) T^j_i \Pi_{pj}^m \dot{x}^p - \left( \hat{\partial}_m v^i \right) T^j_i \Pi_{ph}^m \dot{x}^p. \]
4. **LIE DERIVATIVE OF THE CONNECTION PARAMETER $\Pi^i_{jk}$:**

Let the law of transformation for $\Pi^i_{jk}$ be analogous to that of the connection parameter $\Gamma^i_{hk}$ as has been given in Rund [11] and as such, we may write

$$
\tag{4.1} \Pi^i_{jk} = \frac{\partial x^i}{\partial \bar{x}^r} \left( \frac{\partial^2 \bar{x}^r}{\partial x^j \partial \bar{x}^k} + \bar{\Pi}^r_{st} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial x^t}{\partial \bar{x}^k} \right).
$$

As $\Pi^i_{jk}$ do not form the components of a tensor, hence we are helpless to use formulae (2.15) or (2.16) for obtaining the Lie-derivative of $\Pi^i_{jk}$, we therefore revert to the definition given by (2.8)

$$
\tag{4.2} d^r \Pi^i_{jk} = \left( \partial_h \Gamma^i_{jk} \right) v^h dt + \left( \partial_r \Gamma^i_{jk} \right) (\partial_h v^r) \dot{x}^h dt.
$$

Using [I-(11.3)] and [I-(11.4)], the above relation can be rewritten as under

$$
\tag{4.3} d^r \Pi^i_{jk} = \left\{ \partial_h \Pi^i_{jk} - \left( \partial_r \Pi^i_{jk} \right) \Pi^r_{hm} \dot{x}^m \right\} v^h dt + \left( \partial_r \Pi^i_{jk} \right) (\xi_h v^r) \dot{x}^h dt.
$$

In view of (2.1), (2.11) and (4.1), we have

$$
\tag{4.4} d^{m} \Pi^i_{jk} = \bar{\Pi}^i_{jk} - \Pi^i_{jk} = - \left( \partial_j^2 v^i + \Pi^i_{jk} \partial_j v^s + \Pi^i_{jt} \partial_k v^t - \Pi^r_{jk} \partial_r v^i \right) dt.
$$

Using (2.8), (4.3) and (4.4), we get

$$
\tag{4.5} \mathcal{L}_v \Pi^i_{jk} = \left\{ \partial_h \Pi^i_{jk} - \left( \partial_r \Pi^i_{jk} \right) \Pi^m_{rh} \dot{x}^m \right\} v^h + \left( \partial_r \Pi^i_{jk} \right) (\xi_h v^r) \dot{x}^h + \partial_j^2 v^i + \Pi^i_{sk} \partial_j v^s + \Pi^i_{j} \partial^i_k - \Pi^i_{jk} \partial_r v^i.
$$

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Let $\xi_k \left( \xi_j v^j \right)$ denote the semi-symmetric derivative of $\left( \xi_j v^j \right)$ with to $x^k$, hence we have

\[
(4.6) \xi_k \left( \xi_j v^j \right) = \partial^2_{jk} v^j + \Pi^i_{jm} \partial_k v^m + \Pi^i_{mk} \partial_j v^m - \Pi^m_{jk} \partial_m v^i + \\
+ v^m \left\{ \partial_k \Pi^i_{jm} - \left( \partial_r \Pi^i_{jm} \right) \Pi^r_{pk} \dot{x}^p + \Pi^p_{jm} \Pi^i_{pk} - \Pi^i_{pm} \Pi^p_{jk} \right\}.
\]

Substituting the value of $\partial^2_{jk} v^j$ from (4.6) in (4.5) and thereafter using [I-(11.3)], we get

\[
(4.7) \mathcal{L}_v \Pi^i_{jk} = \xi_k \left( \xi_j v^j \right) + \left( \partial_r \Pi^i_{jk} \right) \left( \xi_h v^r \right) \dot{x}^h + v^h R^i_{jkh}.
\]

Again, proceeding as above, it can easily be verified that

\[
(4.8) \mathcal{L}_v \Pi^i_{jk} = \xi_{jk} v^j + \left( \partial_r \Pi^i_{jk} \right) \left( \xi_h v^r \right) \dot{x}^h + \\
+ v^h \left( \xi_h \Pi^i_{jk} - \xi_k \Pi^i_{jh} \right).
\]

5. COMMUTATION FORMULAE:

In view of (2.16), we have

\[
(5.1) \mathcal{L}_v \left( \partial_k T^i_j \right) = \xi_h \left( \partial_k T^i_j \right) v^h + \left( \partial_h T^i_j \right) \xi_k v^h + \\
+ \left( \partial_k T^i_h \right) \xi_j v^h - \left( \partial_k T^i_j \right) \xi_h v^j + \\
+ \left( \partial^2_{hk} T^i_j \right) \left( \xi_p v^h \right) \dot{x}^p.
\]

and

\[
(5.2) \mathcal{L}_v T^i_j = \left( \xi_h T^i_j \right) v^h + T^i_h \xi_j v^h - T^h_j \xi_h v^i + \\
+ \left( \partial_h T^i_j \right) \left( \xi_p v^h \right) \dot{x}^p.
\]

In view of the covariant derivative [I-(11.3)], the commutation formula [I-(11.6)] takes the following form
(5.3) \( \dot{\sigma}_k \left( \zeta_h T^i_j \right) - \zeta_k \left( \dot{\sigma}_h T^i_j \right) = T^m_j \Pi^i_{kmj} - T^i_j \Pi^m_{kmj} - \\
- \left( \dot{\sigma}_m T^i_j \right) \Pi^m_{knj} \dot{x}^p. \)

Differentiating (5.2) partially with respect to \( \dot{x}^k \) and subtracting the equation thus obtained from (5.1), we have the following commutation formula, in view of the fact that \( \Pi^i_{jk} = \dot{\sigma}_j \Pi^i_{kr} \) and (5.3) and the fact that \( v^i \) is independent of directional arguments

(5.4) \( \mathcal{L}_v \left( \dot{\sigma}_k T^i_j \right) - \dot{\sigma}_k \left( \mathcal{L}_v T^i_j \right) = 0. \)

The relation (5.4) asserts that Lie-derivative and partial derivative with respect to directional argument \( \dot{x}^k \) commute with each other in the Finsler space \( F^* \) equipped with semi-symmetric connection.

using [I-(11.3)] and (2.16), we have

(5.5) \( \mathcal{L}_v \left( \zeta_k T^i_j \right) - \zeta_k \left( \mathcal{L}_v T^i_j \right) = \\
= \left( \zeta_{kh} T^i_j - \zeta_{kh} T^i_j \right) v^h + \zeta_h T^i_j \left( \zeta_k v^h \right) + \\
+ \left( \dot{\sigma}_h \left( \zeta_k T^i_j \right) - \dot{\sigma}_k \left( \zeta_h T^i_j \right) \right) v^h \dot{x}^p - T^i_j \zeta_k \left( \zeta_j v^h \right) + \\
+ T^h_j \zeta_k \left( \zeta_h v^i \right) - \left( \dot{\sigma}_h T^i_j \right) \zeta_k \left( \zeta_v v^h \right) \dot{x}^p. \)

In view of [I-(11.7)] and (4.7), the commutation formula (5.5) reduces to its simplest form as

(5.6) \( \mathcal{L}_v \left( \zeta_k T^i_j \right) - \zeta_k \left( \mathcal{L}_v T^i_j \right) = T^h_j \mathcal{L}_v \Pi^i_{hk} - T^i_j \mathcal{L}_v \Pi^h_{jk} - \\
- \left( \dot{\sigma}_h T^i_j \right) \left( \mathcal{L}_v \Pi^h_{pk} \right) \dot{x}^p. \)

Taking the Lie-derivative of [I-(11.3)], we obtain
\[ (5.7) \quad \mathcal{L}_v (\xi_k T^i_j) = \mathcal{L}_v (\partial_k T^i_j) - (\partial_m T^i_j) (\mathcal{L}_v \Pi^{m}_{pk}) \dot{x}^p - \]

\[ - \mathcal{L}_v (\partial_m T^i_j) \Pi^{m}_{pk} \dot{x}^p + (\mathcal{L}_v T^p_j) \Pi^i_{jk} + \]

\[ + T^p_j \mathcal{L}_v \Pi^i_{pk} - (\mathcal{L}_v T^i_p) \Pi^p_{jk} - T^i_p \mathcal{L}_v \Pi^p_{jk}. \]

In view of [I-(11.3)], we also have

\[ (5.8) \quad \xi_k (\mathcal{L}_v T^i_j) = \partial_k (\mathcal{L}_v T^i_j) - (\partial_m T^i_j) \Pi^{m}_{pk} \dot{x}^p + \]

\[ + (\mathcal{L}_v T^p_j) \Pi^i_{pk} - (\mathcal{L}_v T^i_p) \Pi^p_{jk}. \]

Subtracting (5.8) from (5.7) and thereafter using (5.4) and (5.6), we get

\[ (5.9) \quad \mathcal{L}_v (\partial_k T^i_j) - \partial_k (\mathcal{L}_v T^i_j) = 0. \]

Differentiating (4.7) covariantly in the manner as has been given in [I-(11.3)] and then using [I-(11.7)], we have

\[ (5.10) \quad \xi_k (\mathcal{L}_v \Pi^i_{lj}) - \xi_j (\mathcal{L}_v \Pi^i_{hk}) = \]

\[ = \xi_k \Pi^i_{rhj} (\xi_r v^r) \dot{x}^j + \Pi^i_{rlj} \xi_k (\xi_r v^r) \dot{x}^j + (\xi_k v^r) R^i_{lj} + \]

\[ + v^j \xi_k R^i_{ljl} - \xi_j \Pi^i_{rhl} (\xi_r v^r) \dot{x}^j - \Pi^i_{rhl} \xi_j (\xi_r v^r) \dot{x}^j - \]

\[ + \xi_j v^j R^i_{hjl} - v^j \xi_j R^i_{hjl} - \partial_r (\xi_h v^r) R^i_{jk} + (\xi_h v^r) R^i_{rjk} - \]

\[ - (\xi_r v^r) R^i_{lj} \Pi^i_{rlm} v^m. \]

Using (2.16) in (5.10), we get

\[ (5.11) \quad \mathcal{L}_v R^i_{lj} = (\xi^j R^i_{lj}) v^j - R^i_{hjk} (\xi_r v^r) + R^i_{rjk} (\xi_r v^r) + \]

\[ + R^i_{hrl} (\xi_r v^r) + R^i_{rjr} (\xi_k v^r) + (\partial_r R^i_{lj}) (\xi_r v^r) \dot{x}^p. \]

It can also be easily verified that

\[ (5.12) \quad \partial_r (\xi_h v^r) = \Pi^i_{rlm} v^m. \]

Using (5.11), (5.13), [I-(11.7)], (4.7) and (5.10), we get
\[(5.13) \quad \varsigma_k \left( \mathcal{L}_v \Pi_{ij}^l \right) - \varsigma_j \left( \mathcal{L}_v \Pi_{hk}^i \right) = \mathcal{L}_v R_{ijk}^l + \dot{x}^i \Pi_{r lj}^i \mathcal{L}_v \Pi_{rk}^i - \dot{x}^i \Pi_{r hi}^l \mathcal{L}_v \Pi_{lj}^i + \dot{x}^i \Pi_{r hj}^l \mathcal{L}_v \Pi_{rk}^i + N_{kj}^r \mathcal{L}_v \Pi_{hr}^i + P_{p lj}^i \left( \varsigma_q \mathcal{V}^p \right) \dot{x}^q , \]

where \[(5.14) \quad P_{p lj}^i = -\dot{\varsigma}_p R_{ijk}^l + \varsigma_k \Pi_{plj}^i - \left( \varsigma_j \Pi_{plh}^i \right) \Pi_{p lj}^i \Pi_{rlh}^i \dot{x}^f + \Pi_{phr}^i \Pi_{r hj}^l \mathcal{L}_v \Pi_{rk}^i - \Pi_{phr}^i N_{kj}^r . \]

In view of [I-(11-8)] and the fact that \(\Pi_{jkr}^i = \dot{\varsigma}_j \Pi_{kr}^i\), it can easily be verified that the right hand member of (5.14) vanishes identically thus, we have

\[(5.15) \quad P_{p lj}^i = 0 . \]

Using (5.15), the commutation formula (5.13) takes the form

\[(5.16) \quad \varsigma_k \left( \mathcal{L}_v \Pi_{ij}^l \right) - \varsigma_j \left( \mathcal{L}_v \Pi_{hk}^i \right) = \mathcal{L}_v R_{ijk}^l + \dot{x}^i \Pi_{r lj}^i \mathcal{L}_v \Pi_{rk}^i - \dot{x}^i \Pi_{r hi}^l \mathcal{L}_v \Pi_{lj}^i . \]

6. AFFINE MOTION IN A FINSLER SPACE EQUIPPED WITH SEMI-SYMMETRIC CONNECTION:

In a series of papers [12] to [15] Takano, K. studied the existence of affine motion in non-Riemannian spaces of recurrent curvature. He studied in [15] a special type of an affine motion which he calls it as a recurrent affine motion in a \(K^*_r\) space. Under this heading we are going to discuss the existence of affine motion in a recurrent Finsler space \(F^*_n\) equipped with semi-symmetric connection. For this purpose, we consider an infinitesimal point transformation \(\mathcal{X}^i = x^i + v^i(x)dt\), where \(v^i(x)\) is a contravariant vector field and \(dt\) is an infinitesimal point constant, such a point transformation considered at each point
of recurrent $F_n^*$ equipped with semi-symmetric connection is called $\zeta$-affine motion if and only if $\mathcal{L}_\nu \Pi^i_{jk} = 0$. With such as our basic assumptions we have also derived the complete condition for the vanishing of $\mathcal{L}_\nu R^i_{jkh}$ where $R^i_{jkh}$ is the curvature tensor type entity with respect to semi-symmetric connection $\Pi^i_{jk}$. By virtue of the above point transformation the Lie-derivative of any tensor field $T^i_j(x, \dot{x})$ and the semi-symmetric connection $\Pi^i_{jk}(x, \dot{x})$ are respectively given by (3.7) and (4.7). The commutation formulae involving the operators $\mathcal{L}_\nu, \partial$ and $\zeta_k$ are respectively given by (5.4), (5.6) and (5.16), here $\zeta_k$ stands for the covariant derivative of an object tensorial in nature with respect to the semi-symmetric connection.

We now give the following definitions which we shall use in the later discussions.

**DEFINITION (6.1):**

A Finsler space $F_n^*$ equipped with semi-symmetric connection $\Pi^i_{kj}$ is said to be $\zeta$-recurrent $F_n^*$ if

\[ \zeta_t R^i_{jkh} = \lambda_t R^i_{jkh} \quad (6.1) \]

where $\lambda_t(x)$ is a non-null recurrence vector.

**DEFINITION (6.2):**

An n-dimensional $F_n^*$ equipped with semi-symmetric connection is said to be an affinely connected $\zeta$-space if

\[ \partial_t \Pi^i_{jk} = 0 \quad (6.2) \]
If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ considered at each point of $F^*_n$ defines an affine motion then form (5.16), we immediately get

$$\mathcal{L}_v R^i_{jkh} = 0.$$ (6.3)

Taking the Lie-derivative of both sides of (2.8) and thereafter using (5.16), (5.6) and (6.3), we get

$$\mathcal{L}_v \Lambda^i = 0.$$ (6.4)

Consequently, if the $\zeta$-recurrent $F^*_n$ is considered to be non-flat (a non-flat $F^*_n$ is characterised by $R^i_{jkh} \neq 0$) then we have

$$\mathcal{L}_v \Lambda^i = 0.$$ (6.5)

Thus, with the help of (6.5), we can say that the recurrence vector of a non-flat $\zeta$-recurrent $F^*_n$ must be a Lie-invariant.

Next, we propose to study a $\zeta$-recurrent $F^*_n$ admitting and infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ satisfying (6.5), for brevity, we shall onwards call such a restricted space as $\zeta$-recurrent space.

**THE VANISHING OF $\mathcal{L}_v R^i_{jkh}(x, \dot{x})$:**

First of all we prove the following Lemma.

**Lemma (6.1):**

In a $\zeta$-recurrent $F^*_n$ equipped with semi-symmetric connection if the recurrence vector $\Lambda_s(x)$ is a gradient one, then we should have $\Lambda_s v^s = \text{constant.}$
PROOF:

For brevity, we write $\alpha = \lambda_m v^m$ then with help of equations (2.15) and (6.5), we get

\[
(6.6) \quad \mathcal{L}_v \lambda_m = (\zeta_p \lambda_m) v^p + \lambda_p (\zeta_m v^p).
\]

If we now assume that $\zeta_p \lambda_m = \zeta_m \lambda_p$, then it is obvious that $\zeta_p \alpha = 0$ which will mean that $\alpha = \lambda_m v^m$ is a constant. By virtue of (2.15), the Lie derivative of $R^i_{jkh}(x, \dot{x})$ in view of (6.1) can be written as

\[
(6.7) \quad \mathcal{L}_v R^i_{jkh} = \alpha R^i_{jkh} - R^p_{jkh} (\zeta_p v^i) + \]
\[
+ R^i_{phk} (\zeta_j v^p) + R^i_{ijh} (\zeta_k v^p) + \]
\[
+ R^i_{jhp} (\zeta_h v^p) + (\hat{\partial}_p R^i_{jkh})(\zeta_m v^p) \dot{x}^m.
\]

The commutation formula [(1-(11.7)] when used for $R^i_{jkh}(x, \dot{x})$ gives

\[
(6.8) \quad (\zeta_m R^i_{jkh} - \zeta_m R^i_{jkh}) = - (\hat{\partial}_p R^i_{jkh}) R^{pq}_{jkl} \dot{x}^q + R^p_{jkh} R^i_{p\ell} - \]
\[
- R^i_{phk} R^p_{j\ell m} - R^i_{jpih} R^p_{k\ell m} - R^i_{jhp} R^p_{k\ell m}.
\]

Using (6.1) in (6.8) we get

\[
(6.9) \quad (\zeta_m \lambda_\ell - \zeta_\ell \lambda_m) R^i_{jkh} = - (\hat{\partial}_p R^i_{jkh}) R^{pq}_{jkl} \dot{x}^q + R^p_{jkh} R^i_{p\ell} - \]
\[
- R^i_{phk} R^p_{j\ell m} - R^i_{jpih} R^p_{k\ell m} - R^i_{jhp} R^p_{k\ell m}.
\]

Now, we assume that $\alpha = \lambda_m v^m$ is not a constant, then from the stated lemma, we can find the following:

\[
(6.10) \quad \beta^i_{\ell m}(x) = (\zeta_m \lambda_\ell - \zeta_\ell \lambda_m) \neq 0.
\]
Now, we consider a suitable non-symmetric tensor $\psi^{jk}$ which satisfies the relation

\begin{equation}
R^i_{jkh} \psi^{kh} = \xi^j v^i.
\end{equation}

We now multiply (6.9) by $\psi^{\ell m}$ and sum it over $\ell$ and $m$ and get

\begin{equation}
\beta_{\ell m} \psi^{\ell m} R^i_{jkh} = -\left( \hat{\partial}_p R^i_{jkh} \right) \left( \xi^q v^p \right) \hat{x}^q + R^p_{jkh} \left( \xi^q v^p \right) - \left( \xi^j v^p \right) - R^i_{jkh} \left( \xi^q v^p \right) - R^i_{jkh} \left( \xi^q v^p \right).
\end{equation}

Comparing (6.12) with (6.7), we get

\begin{equation}
\ell_v R^i_{jkh} = \left( \alpha - \beta_{\ell m} \psi^{\ell m} \right) R^i_{jkh}.
\end{equation}

From (6.13) it is almost obvious that $\ell_v R^i_{jkh}$ vanishes only when $\alpha = \beta_{\ell m} \psi^{\ell m}$.

With the help of (6.7) and (6.9), we can construct following identity

\begin{equation}
\beta_{\ell m} \ell_v R^i_{jkh} = R^i_{jkh} \left( \alpha R^i_{\ell m} - \beta_{\ell m} \xi^p \right) - R^i_{jkh} \left( \alpha R^p_{jkh} - \beta_{\ell m} \xi^p v^p \right) - R^i_{jkh} \left( \alpha R^p_{k\ell m} - \beta_{\ell m} \xi^p v^p \right) - R^i_{jkh} \left( \alpha R^p_{k\ell m} - \beta_{\ell m} \xi^p v^p \right) - \left( \hat{\partial}_p R^i_{jkh} \right) \left( \alpha R^p_{q\ell m} - \beta_{\ell m} \xi^p v^p \right) \hat{x}^q.
\end{equation}

From (6.14), we conclude that $\ell_v R^i_{jkh} = 0$ if

\begin{equation}
\alpha R^i_{jkh} = \beta_{kh} \left( \xi^j v^i \right).
\end{equation}

We now given the following definition.
DEFINITION (6.3):

A $\zeta$-recurrent $F_n^*$ equipped with semi-symmetric connection satisfying $\lambda_m v^m \neq \text{constant}$ is called a special one of the first kind.

Now, we again comeback to the case $\lambda_m v^m = \text{constant}$ of the foregoing lemma then (6.9) is replaced by

\begin{equation}
(6.16) -\left( \hat{\partial}_p R_{jkh}^i \right) R_{qlm}^p \hat{x}^q + R_{jkh}^p R_{pim}^i - R_{jkh}^i R_{jkm}^p - R_{jkh}^p R_{kkm}^i - R_{jkp}^i R_{nkm}^p = 0.
\end{equation}

In view of the equation (6.11), multiplying (6.16) by $\psi^{lm}$ and summing over the indices $\ell$ and $m$, we get

\begin{equation}
(6.17) -\left( \hat{\partial}_p R_{jkh}^i \right) \left( \zeta_q v^p \right) \hat{x}^q + R_{jkh}^p \left( \zeta_p v^i \right) - R_{pkh}^i \left( \zeta_j v^p \right) - R_{jkh}^i \left( \zeta_k v^p \right) - R_{jkp}^i \left( \zeta_h v^p \right) = 0.
\end{equation}

We now introduce (6.17) into the right hand side of (6.7) and get

\begin{equation}
(6.18) \mathfrak{L}_v R_{jkh}^i = \alpha R_{jkh}^i \quad \text{where} \quad \alpha = \lambda_m v^m.
\end{equation}

With the help of (6.18) we conclude that if $\alpha = 0$ then $\mathfrak{L}_v R_{jkh}^i = 0$ hold. We now give the following definition:

DEFINITION (6.4):

In a $\zeta$-recurrent $F_n^*$ equipped with semi-symmetric connection when $\alpha = \lambda_m v^m = \text{constant}$ holds then such a $F_n^*$ is said to be special one of the second kind.

We now summarise all these results in the form of following theorems.
THEOREM (6.1):

In a \( \zeta \)-recurrent \( F_n^* \) of the first kind, \( \mathcal{L}_\zeta R^i_{jkh} = 0 \) hold
good only when (6.15) holds.

THEOREM (6.2):

In a \( \zeta \)-recurrent \( F_n^* \) of the second kind, \( \mathcal{L}_\zeta R^i_{jkh} = 0 \)
hold good if \( \alpha = 0 \) where \( \alpha = \lambda_m v^m \).

7. COMPLETE CONDITION:

In this section we propose to establish the necessary and
sufficient condition in order that (6.15) may hold good. In view
of the assumption (6.5), we have

\[
(7.1) \quad \mathcal{L}_\zeta \lambda_m = \left( \xi_p \lambda_m \right) v^p + \xi_m \left( \lambda_p v^p \right) - \left( \xi_m \lambda_s \right) v^s = 0.
\]

(7.1) by virtue of (6.10) reduces to

\[
(7.2) \quad \xi_m \alpha + \beta_{ms} v^s = 0.
\]

In view of (2.15), the Lie-derivative of \( \beta_{lm}(x) \) is given by

\[
(7.3) \quad \mathcal{L}_\zeta \beta_{lm} = \left( \xi_p \beta_{lm} \right) v^p + \beta_{pm} \left( \xi_p v^p \right) + \beta_{ip} \left( \xi_m v^p \right).
\]

(7.3) by virtue of (5.6), (6.5) and (6.10) reduces into the following form

\[
(7.4) \quad \mathcal{L}_\zeta \beta_{lm} = 0.
\]

Differentiating (6.9) \( \zeta \)-covariantly (with respect to semi-

symmetric connection) with respect to \( x^n \) and using (6.1) and

(6.9), we get

\[
(7.5) \quad \left( \xi_n \beta_{lm} \right) R^i_{jkh} = \lambda_n \left[ R^p_{jkh} R^i_{p\ell m} - R^i_{pjh} R^p_{j\ell m} - R^i_{pjh} R^p_{k\ell m} - R^i_{jkp} R^p_{h\ell m} - \left( \partial_p R^i_{jkh} \right) R^p_{q\ell m} \dot{\chi}^q \right]
\]

We now use (6.16) in (7.5) and get

\[
(7.6) \quad \left( \xi_n \beta_{lm} \right) = 0.
\]
Now after making use of (7.3), (7.4) and (7.6), we get

\[(7.7) \quad \beta_{pm} (\xi^p_j v^j) + \beta_{tp} (\xi^p_m v^m) = 0.\]

From (7.2), we have

\[(7.8) \quad \xi^m_n \alpha = -\xi^m_n \left( \beta_{ns} v^s \right).\]

Interchanging the indices \(m\) and \(n\) in (7.8) and subtracting the equation thus obtained from (7.8) itself, we get

\[(7.9) \quad (\xi^m_n \alpha - \xi^m_n \alpha) = \xi^m_n \left( \beta_{ns} v^s \right) - \xi^m_n \left( \beta_{ns} v^s \right).

\(\alpha\)-being a non-constant scalar function, therefore under this assumption (7.9) gives

\[(7.10) \quad \left( \xi^m_n \beta_{ns} \right) v^s + \left( \xi^m_n \beta_{ns} \right) v^s = \beta_{ns} \left( \xi^m_n v^s \right) - \beta_{ns} \left( \xi^m_n v^s \right).\]

Therefore, we can state:

**THEOREM (7.1):**

In a \(\xi\)-recurrent \(F_n^*\) equipped with semi-symmetric connection, the necessary and sufficient condition for the existence of (6.15) is given by (7.10).

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REFERENCE:


