Chapter-I

BASIC CONCEPTS OF FINSLER SPACES
1. COORDINATE, CURVE AND LINE ELEMENT:

Let \( R \) be a region of an \( n \)-dimensional space \( X_n \) covered by a coordinate system, such that any point \( P \) of \( R \) is represented by a set of \( n \)-independent variables \( x^i (i = 1, 2, 3, \ldots, n) \) called the coordinates of the point. The coordinates are assumed to be real. A transformation of coordinates is represented by a set of \( n \)-equations

\[
(1.1) \quad x^{i'} = x^i \left( x^1, x^2, \ldots, x^n \right) \quad (i', j' = 1, 2, \ldots, n)
\]

which shows that the coordinates \( x^i \) of a point \( P \) of \( X_n \) are represented in the new coordinate system by new variables \( x^{i'} \). We assume that the functions \( x^{i'} \) of (1.1) are at least of class \( C^2 \) and

\[
(1.2) \quad \det \left( \frac{\partial x^{i'}}{\partial x^i} \right) \neq 0.
\]

A set of points of \( R \) whose coordinates may be expressed as a function of single parameter \( t \) is regarded as a curve of \( X_n \).

Thus, the equations

\[
(1.3) \quad x^{i'} = x^i (t)
\]

define a curve \( C \) of \( X_n \). If the equations (1.3) are of class \( C^1 \), we shall regard the entity whose components are given by

\[
(1.4) \quad y^i = dx^i / dt
\]

as the tangent vector to the curve \( C \).

The combination \((x^i, y^i)\) is called a line element of the curve \( C \).

2. FINSLER SPACE:

Suppose that we are given a function \( F(x^i, y^i) \) of the line element \((x^i, y^i)\) of curves defined in \( R \). We shall assume \( F \) to be of class \( C^3 \) in all its \( 2n \) arguments. If we define the distance \( ds \) between the points \( P(x^i) \) and \( Q(x^i + dx^i) \) of \( C \) as
(2.1) \[ ds = F \left( x^i, dx^i \right) \]

then the space \( X_n \) equipped with the fundamental function defining the metric (2.1) is called a Finsler space \([22]\) if the function \( F(x^i, y^i) \) satisfies the following conditions:

**CONDITION(A):**

The function \( F(x^i, y^i) \) is positively homogeneous of degree one in \( y^i \), that is

\[ (2.2) \quad F \left( x^i, ky^i \right) = k \quad F \left( x^i, y^i \right) \quad \text{with} \quad k > 0 \]

for all the line elements \((x^i, y^i)\).

**CONDITION(B):**

The function \( F(x^i, y^i) \) is positive if all \( y^i \) do not vanish simultaneously, that is,

\[ (2.3) \quad F \left( x^i, y^i \right) > 0, \quad \text{with} \quad \sum_{i=1}^{n} (y^i)^2 \neq 0 \]

**CONDITION(C):**

The quadratic form

\[ \frac{\partial^2 F^2 \left( x^i, y^i \right)}{\partial y^i \partial y^j} \xi^i \xi^j \]

is assumed to be positive definite for all the variables \( \xi^i \).

From Euler's theorem on homogeneous functions, we have

\[ (2.4) \quad \frac{\partial F \left( x^i, y^i \right)}{\partial y^i} y^i = F \left( x^i, y^i \right) \]

and

\[ (2.5) \quad \frac{\partial^2 F \left( x^i, y^i \right)}{\partial y^i \partial y^j} y^j = 0. \]

We define a set of quantities \( g_{ij} (x^i, y^i) \) by
(2.6) \[ g_{ij} (x^i, y^j) = \frac{1}{2} \frac{\partial^2 F^2 (x^i, y^j)}{\partial y^i \partial y^j}. \]

Using the theory of quadratic forms and the condition C, we have

\[ g(x^i, y^j) = |g_{ij} (x^i, y^j)| > 0 \]

for all the line elements \((x^i, y^j)\).

If the function \( F \) is of the particular form

\[ (2.8) \quad F (x^i, dx^i) = \left[ g_{ij} (x^k) dx^i dx^j \right]^\frac{1}{2} \]

where the coefficients \( g_{ij} (x^k) \) are independent of the \( dx^i \), then the metric defined by such a function \( F \) is called Riemannian metric and the space \( X_n \) is called a Riemannian space \( V_n \) [8].

Throughout the entire discussion, the \( n \)-dimensional Finsler space will be denoted by \( F_n \).

3. **TANGENT SPACE, INDICATRIX, MINKOWSKIAN SPACE:**

**TANGENT SPACE:**

We consider a change of local coordinates as represented by equations (1.1). Along the curve (1.3), referred to an invariant parameter \( t \), the new components of the tangent vector

\[ y^i = \frac{dx^i}{dt} \]

are obtained by differentiating the relation

\[ (3.1) \quad x^i = x^i \left( x^j \left( t \right) \right) \]

with respect to \( t \), which gives,

\[ (3.2) \quad \frac{dx^i}{dt} = \frac{\partial x^i}{\partial x^j} \cdot \frac{dx^j}{dt} \]

or in terms of differentials,

\[ (3.2) \quad dx^i = \frac{\partial x^i}{\partial x^j} dx^j, \]
$dx^i$ are interpreted as the 'components' of a displacement in $X_n$ from a point $P(x^i)$ to a point $Q(x^i + dx^i)$.

If the point $P(x^i)$ is fixed, that is the coefficients $\frac{\partial x^j}{\partial x^i}$ of the transformation (3.2b) are fixed, the relation (3.2b) represents a linear transformation of the $dx^i$ onto the $dx^j$. The same is true for the variables $y^i$ and $y^j$ in the transformation (3.1). Therefore, the entities of this kind may be taken to define the elements of an n-dimensional linear vector space.

A system of 'n' quantities $X^i$ whose transformation law under (1.1) is equivalent to that of the $y^i$ is called a contravariant vector attached to the point $P(x^i)$ of $X_n$. Such contravariant vectors constitute the elements of our new vector space. Hence the totality of all contravariant vectors attached to $P(x^i)$ of $X_n$ is the tangent space denoted by $T_n(P)$ or $T_n(x^i)$ [22].

Further the transformation (3.2b) is homogeneous, we may regard the tangent space as 'centred' affine spaces, the centre or origin corresponding to the values $y^1 = 0, y^2 = 0, ..., y^n = 0$.

**INDICATRIX:**

We consider the function $L(x^i, y^i)$ defined for all line elements $(x^i, y^i)$ over the region $R$ of $X_n$. The equation

\[(3.3) \ L(x^i, y^i) = 1 \ (x^i \ \text{fixed}, \ y^i \ \text{variable})\]

represents an (n-1) dimensional locus in $T_n(P)$ that is a hypersurface. This hypersurface plays the role of the unit sphere in the theory of the vector space $T_n(P)$ and is called indicatrix [4][22].

A tensor $T$ of $F_n$ is called indicatory [18] if its components
$T_{ij\ldots k}$ satisfy

$$T_{ij\ldots k} = T_{i\ldots k} = T_{ij\ldots o} = 0$$

where and throughout the thesis $\circ$ denotes the contraction with $y^i$.

**MINKOWSKIAN SPACE:**

Minkowskian space is a vector space whose metric satisfies the conditions A to C of section 2[3,22]. Just as Riemannian space is locally Euclidean, Finsler space may be regarded as locally Minkowskian.

**4. METRIC TENSOR:**

With the help of the equations (3.2a), we can easily see that the set of quantities $g_{ij}(x^i, y^j)$ defined by the equation (2.6), form the components of a covariant tensor of rank 2. Also it is clear that $g_{ij}(x^i, y^j)$ are positively homogeneous of degree zero in $y^i$ and are symmetric in its indices $i$ and $j$. Because of the homogeneity condition (A) for the function $F(x^i, y^j)$, we have

$$\text{(4.1) } F^2(x, y) = g_{ij}(x, y) y^i y^j$$

The inverse of $g_{ij}$ denoted by $g^{ij}$ is defined as

$$\text{(4.2) } g_{ij}(x, y) g^{ik}(x, y) = \delta^k_j = \begin{cases} 
1 & \text{if } k = j \\
0 & \text{otherwise}
\end{cases}$$

where $\delta^k_j$ is the well known Kronecker delta.

**DEFINITION (4.1):**

The tensor with covariant components $g_{ij}$ and contravariant components $g^{ij}$ is called the metric tensor or the first fundamental tensor of the Finsler space $F_n$.

The tensor $C_{ijk}(x, y)$ defined by
(4.3) \( C_{ijk} (x, y)^{\text{def}} = \frac{1}{2} \frac{\partial g_{ij}(x, y)}{\partial y^k} = \frac{1}{4} \frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k} \)

is positively homogeneous of degree -1 in \( y^j \) and is symmetric in all three of its indices. This tensor is known as Cartan’s C-tensor and satisfies the following conditions

(4.4) \( C_{ijk}(x, y) y^i = C_{ijk}(x, y) y^j = C_{ijk}(x, y) y^k = 0 \)

and

(4.5) \( \frac{\partial C_{ijk}(x, y)}{\partial x^i} y^j = \frac{\partial C_{ijk}(x, y)}{\partial x^j} y^i = \frac{\partial C_{ijk}(x, y)}{\partial x^k} y^i = 0 \).

5. DUAL TANGENT SPACE:

Corresponding to every arbitrary contravariant vector \( y^i \) of \( T_n(P) \), we may associate a covariant vector \( y_i \) defined by the relation

(5.1) \( y_i = g_{ij}(x, y)y^j \)

where it may be noted that the directional argument in the \( g_{ij} \) coincides with the vector \( y^j \) under consideration.

The totality of all covariant vectors \( y_i \) given by the equation (5.1), associated to a point \( P \), is called the dual tangent space of \( X_n \) at \( P \) and is denoted by \( T_n^1(P) \).

6. MAGNITUDE OF A VECTOR, ANGLE AND ORTHOGONALITY:

Let \( X^i \) be a vector and \( y^i \) be an arbitrary fixed direction, then the scalar \( \mid X \mid \) given by

(6.1) \( \mid X \mid^2 = g_{ij}(x, y) X^i X^j \)
is called the square of the magnitude of the vector $X^i$ for the preassigned direction $y^i$. The Minkowskian magnitude of a vector $X^i$ is defined as

$$\langle X \rangle^2 = g_{ij} \left( x^i, x^j \right) X^i X^j.$$  

The Minkowskian Cosine corresponding to two arbitrary directions $\lambda^i$ and $\mu^i$ is defined by the ratio

$$\cos(\lambda, \mu) = \frac{g_{ij} \left( x^k, \lambda^k \right) \lambda^i \mu^j}{L(x^k, \lambda^k) L(x^k, \mu^k)}.$$  

From the equation (6.3), it is clear that the Minkowskian cosine is not symmetrical in its arguments $\lambda^i$ and $\mu^i$. However, for an arbitrary fixed direction $y^i$, the cosine for two arbitrary directions $\lambda^i$ and $\mu^i$ can be written as [22]

$$\cos(\lambda, \mu) = \frac{g_{ij} \left( x^k, y^k \right) \lambda^i \mu^j}{\left[ g_{ij} \left( x^k, y^k \right) \lambda^i \lambda^j \right]^{\frac{1}{2}} \left[ g_{ij} \left( x^k, y^k \right) \mu^i \mu^j \right]^{\frac{1}{2}}}.$$  

This expression is symmetrical in $\lambda$ and $\mu$ but it depends on the original choice of the direction $y^i$. Therefore, it is basically different from (6.3).

**DEFINITION (6.1):**

The vector $\mu^i$ is said to be orthogonal with respect to the vector $\lambda^i$ if

$$g_{ij} \left( x^k, \lambda^k \right) \lambda^i \mu^j = 0.$$  

Thus, we see that orthogonality is not a symmetrical relationship between the two vectors $\lambda^i$ and $\mu^i$. 


DEFINITION (6.2):

The vectors $\lambda^i$ and $\mu^i$ are called orthogonal (for a preassigned direction $y^i$) if

$$(6.6) \quad g_{ij} \left( x^k, y^k \right) \lambda^i \mu^j = 0.$$ 

This definition of orthogonality is symmetrical in $\lambda^i$ and $\mu^i$.

7. CONNECTIONS AND COVARIANT DIFFERENTIATION IN $F_n$:

The connection theories of Finsler space $F_n$ have been studied by many authors. These theories may be broadly divided into two types. In one $F_n$ is constructed of the line elements and is used by most of the researchers [14, 16, 18, 19, 22] and the other is derived from Minkowskian tangent spaces [1, 12].

FINSLER CONNECTION:

The Finsler connection $F\Gamma$ of a Finsler space $F_n$ is a triad $(F^i_{jk}, N^i_k, C^i_{jk})$ of a $V$-connection $F^i_{jk}$, a nonlinear connection $N^i_k$ and a vertical connection $C^i_{jk}$ [14,18]. In general, the vertical connection $C^i_{jk}$ is different from Cartan’s C-tensor obtained from $C_{ijk}$ given by the equation (4.3). However, there are certain Finsler connections to be discussed, in which the two quantities (vertical connection and Cartan’s C-tensor) are identical.

If a Finsler connection is given, the h- and v-covariant derivatives of any tensor field $T^i_j$ are defined as

$$(7.1) \quad T^i_j|_k = d_k T^i_j + T^m_j F^i_{mk} - T^i_m F^m_{jk}$$

and

$$(7.2) \quad T^i_j|_k = \hat{d}_k T^i_j + T^m_j C^i_{mk} - T^i_m C^m_{jk}$$

respectively, where
(7.3) \( d_{k} = \partial_{k} - N_{k}^{m} \hat{\partial}_{m}, \quad \partial_{k} = \partial/\partial x^{k}, \quad \hat{\partial}_{k} = \partial/\partial y^{k}, \)

\((\hat{\partial}_{k})\) and \((\partial_{k})\) denotes the h and v-covariant derivatives respectively.

For any Finsler connection \((F^{i}_{jk}, N^{i}_{k}, C^{i}_{jk})\) we have five torsion tensors which are expressed as follows:

(7.4) The (h) h-torsion tensor: \(T^{i}_{jk} = F^{i}_{jk} - F^{i}_{kj},\)

(7.5) The (v) v-torsion tensor: \(S^{i}_{jk} = C^{i}_{jk} - C^{i}_{kj},\)

(7.6) The (h) hv- torsion tensor: \(C^{i}_{jk} = \text{As the connection} C^{i}_{jk},\)

(7.7) The (v) h-torsion tensor: \(R^{i}_{jk} = d_{j} N^{i}_{j} - d_{j} N^{i}_{k},\)

(7.8) The (v) hv-torsion tensor: \(F^{i}_{jk} = \hat{\partial}_{k} N^{i}_{j} - F^{i}_{kj} \)

The deflection tensor field \(D^{i}_{j}\) of a Finsler connection is given by

(7.9) \(D^{i}_{j} = y^{k} N^{i}_{j} - F^{i}_{kj}.\)

When a Finsler metric is given, various Finsler connection may be defined from the metric. The well-known examples are the Rund’s connection, the Cartan’s connection and the Berwald’s connection which are given below.

**A) THE RUND CONNECTION:**

As in Riemannian geometry, the Christoffel’s symbols of first and second kinds have been defined as [22]

(7.10) \(\gamma_{hij} (x, y) = \frac{1}{2} \left( \partial_{j} g_{hi} + \hat{\partial}_{i} g_{hj} - \hat{\partial}_{i} g_{jh} \right)\)

and

(7.11) \(\gamma^{h}_{ij} (x, y) = g^{hk} (x, y) \gamma_{kij} (x, y).\)

From the definition it is clear that \(\gamma_{hij} (x, y)\) is symmetric in its extreme indices and \(\gamma^{h}_{ij} (x, y)\) is symmetric in its lower indices and satisfy the relation
(7.12) \[ \partial_k g_{ij}(x, y) = \gamma_{ijk}(x, y) + \gamma_{jik}(x, y). \]

The symbols \( \Gamma^h_{ij}(x, y) \) are defined as

(7.13) \[ \Gamma^h_{ij}(x, y) = \gamma^h_{ij}(x, y) - C^h_{im}(x, y)\gamma^m_{kj}(x, y)y^k \]

where

(7.14) \[ C^h_{ij}(x, y) = g^{hk}(x, y)C_{kj}(x, y) \]

and Cartan’s C-tensor \( C_{ij} \) is defined by (4.3)

For a vector \( X^i \), the components \( \frac{\delta X^i}{\delta t} \) defined by

(7.15) \[ \frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + \Gamma^i_{jk}(x, y)X^j \frac{dy^k}{dt} \]

for the covariant components of a vector the process of differentiation given by (7.15) is called the process of “\( \delta \)-differentiation.”

In particular, this process gives a well-defined parallel displacement. The vector \( X^i + dX^i \) of \( T_n(x^i + dx^i) \) is said to be obtained from the vector \( X^i \) of \( T_n(x^i) \) by parallel displacement if \( \delta X^i = 0 \). Hence, for such a displacement we have

(7.16) \[ dX^i = -\Gamma^i_{jk}(x, y)X^j dx^k \]

The partial \( \delta \)-derivative with respect to \( x^k \) in an arbitrary direction \( y^j \) of the arbitrary tensor \( T^i_j(x, y) \) is defined by the formula [22]

(7.17) \[ T^i_{j;k} = \partial_k T^i_j + \partial^h T^i_{j;k} \partial_h + T^m_{j;k} \Gamma^i_{mk}(x, y) - T^i_{m;k} \Gamma^m_{jk}(x, y) \]

where the coefficients \( \Gamma^i_{jk}(x, y) \) is given by

(7.18) \[ \Gamma^i_{jk}(x, y) = g^{ih}(x, y) \Gamma^i_{hjk}(x, y) \]

and
\[(7.19) \quad \Gamma^*_{jkl}(x, y) = \gamma^*_{jkl}(x, y) - [C_{kli}(x, y)\Gamma^i_{jm}(x, y) + \\
+C_{hjl}(x, y)\Gamma^i_{km}(x, y) - C_{jki}(x, y)\Gamma^i_{lm}(x, y)]y^m.\]

The symbol $\Gamma^*_{jkl}$ is symmetric in its lower indices $j$ and $k$, while $\Gamma^i_{jkl}$ is non-symmetric in $j$ and $k$. Also, we have
\[
(7.20) \quad \Gamma^*_{jkl}y^jy^k = \Gamma^i_{jkl}y^jy^k = \gamma^i_{jkl}y^jy^k, \\
(7.21) \quad \Gamma^i_{jkl}y^k = \Gamma^i_{jkl}y^k, \\
(7.22) \quad \Gamma^i_{jkl}y^j = \gamma^i_{jkl}y^j.
\]

The partial $\delta$-derivative of the metric tensor $g_{ij}(x, \xi)$ in the direction $y^i$ in view of (7.17) is
\[
(7.23) \quad g_{ij}(x, \xi)_{,k} = \partial_k g_{ij}(x, \xi) + 2C_{ijk}(x, \xi)\partial_k \xi^h - \\
-g_{ij}(x, \xi)\Gamma^*_{ik}(x, y) - g_{ih}(x, \xi)\Gamma^*_{jk}(x, y).
\]

If, in particular, $y^i = \xi^i$, the above equation reduces to
\[
(7.24) \quad g_{ij}(x, \xi)_{,k} = 2C_{ijk}(x, \xi)\xi^h_{,k}.
\]

We see that the partial $\delta$-derivative of the metric tensor $g_{ij}$ does not vanish in general. Therefore, further developments of the theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.

Further, it is to be noted that if the vector field $\xi^i$ is stationary, that is $\xi^i_{,ij} = 0$, then the partial $\delta$-differentiation of a tensor field is h-covariant derivative with respect to the Rund connection $(\Gamma^*_{jkl}, G^i_{j}, o)$, where $\Gamma^*_{jkl}$ is V-connection defined by the equation (7.19) and $G^i_{j}$ is defined by
(7.25) \[ G'^i_j(x,y) = \hat{\partial}_j G'^i, \ 2G'^i(x,y) = \gamma^i_{jk} y^j y^k \]

and the vertical connection \( C'_{jk} \) vanishes in this triad. Hence the \( v \)-covariant derivative of a tensor field is identical to the partial derivative with respect to the element of support \( y^i \) [10,18].

(B) THE CARTAN CONNECTION:

In 1934, E. Cartan [5] published his monograph “Les espaces de Finsler” and fixed his method to determine a notion of connection in the geometry of Finsler spaces. Although the aim of Cartan’s axioms is to determine both the fundamental tensor \( g \) and the connection from the Finsler metric, it seems that some of his axioms are rather artificial and are introduced after foreseeing the result. In 1966, his method was reconsidered by M. Matsumoto [13] and determined uniquely the Cartan connection by assuming the following axioms [17,18]:

(7.26) (a) The connection is \( h \)-metrical, i.e.

\[ g_{ijk} = 0 \]

(b) The connection is \( v \)-metrical, i.e.

\[ g_{y|k} = 0 \]

(c) The \( (h) \)-torsion tensor field \( T'^i_{jk} \) vanishes, i.e.

\[ T'^i_{jk} = F'^i_{jk} - F'^i_{kj} = 0 \]

(d) The \( (v) \) \( v \)-torsion tensor field \( S'^i_{jk} \) vanishes, i.e.

\[ S'^i_{jk} = C'^i_{jk} - C'^i_{kj} = 0 \]

(e) The deflection tensor field \( D'^i_j \) vanishes, i.e.

\[ D'^i_j = y^h F'^i_{hj} - N'^i_j = 0. \]
The components of the Cartan connection $\Gamma_{jk}^i$ are denoted by 
\[ (\Gamma_{jk}^i, G_j^i, C_{jk}^i) \]. The axioms (7.26b) and (7.26d) in view of the 
equation (7.2) give
\[ (7.27) \quad C_{jk}^i = \frac{1}{2} g^{ih} \tilde{\partial}_h g_{jk}, \]
this shows that the vertical connection and Cartan’s $C$-tensor are identical.

Further, axioms (7.26a) and (7.26c), view of relations (7.27) and 
(7.1), we get
\[ (7.28) \quad F_{ijk} = g_{jh} F_{ik}^h = \gamma_{ijk} - C_{ijm} N_k^m - C_{jkm} N_i^m + C_{kim} N_j^m. \]
Contracting the equation (7.28) with $y^i g^{jh}$ and then applying
the axiom (7.26c), we get
\[ (7.29) \quad N_k^h = \gamma_{ik}^h y^i - C_{km}^h N_i^m y^i. \]
Again contracting this equation with $y^k$, we have
\[ (7.30) \quad N_k^h y^k = \gamma_{ik}^h y^i y^k. \]
Substituting (7.29) and (7.30) in (7.28), we get
\[ F_{ijk} = \Gamma_{ijk}^* \]
where $\Gamma_{ijk}^*$ is defined by the equation (7.19). Thus, the Cartan $V$-connection and the Rund $V$-connection are identical. After
substituting from (7.30) in (7.29), the Cartan non-linear connection
$N_j^i$ is given by
\[ (7.31) \quad N_j^i = \gamma_{kj}^i y^k - C_{jm}^i \gamma_{hp}^m y^h y^p = G_j^i = \Gamma_{oj}^*. \]

The Cartan vertical connection $C_{jk}^i$ is given by (7.27). It is easy
to verify from the axioms (7.26a), (7.26e) and equation (4.1) that
\[ (7.32) \quad (a) \quad y_{|h}^i = 0 \quad (b) \quad L_{|h} = 0 \quad (c) \quad \Gamma_{|h}^i = 0, \]
where $\ell^i$ is a unit vector in the direction of the element of support $y_i$. That is

$$\ell^i = y^i | L(x, y).$$

Since $C_{jk}^i$ is an indicatory tensor, then from (7.2), we have

$$(7.33) \quad (a) \quad y^i | J = \delta^i_h \quad (b) \quad L^i_j = \frac{\partial L}{\partial y^j} = \ell^i_j \text{ where } \ell^i_j = g_{ij} \ell^j.$$  

It may also be verified that

$$(7.34) \quad (a) \quad \ell^i_j = \bar{L}^i_j \hat{h}^i_j \quad (b) \quad \ell^i_j = 0 \quad (c) \quad \ell^i_j = \bar{L}^i_j h_{ij}.$$  

$$(7.35) \quad (a) \quad h_{ijk} = 0 \quad (b) \quad h_{ij} | k = -\bar{L}^i_j \left( \ell^i_j h_{jk} + \ell^i_j h_{ki} \right),$$

where $h_{ij}$ are components of angular metric tensor defined by

$$(7.36) \quad h_{ij} = g_{ij} - \ell^i_j \ell_j$$

and

$$h^i_j = g^{ik} h_{kj}.$$

**(C) THE BERWALD CONNECTION**

L. Berwald defined a connection coefficient $G_{jk}^i$ by

$$(7.37) \quad G_{jk}^i(x, y) = \hat{\partial}_k \hat{\partial}_j G^i,$$

where $2G^i(x, y) = y^i_j(x, y) y^j y^k$.

He defined the covariant derivative in a manner analogous to that of Cartan, the only difference being that $\Gamma_{jk}^i$ are replaced by $G_{jk}^i$. Thus, the covariant derivative of a mixed tensor $T^i_j(x, y)$ in the sense of Berwald is defined by

$$(7.38) \quad T^i_{j(k)} = \hat{\partial}_k T^i_j - \hat{\partial}_m T^m_j G^i_k + T^m_j G^i_{mk} - T^i_j G^m_{jk}.$$  

The functions $G^i(x, y)$ are positively homogeneous of degree 2 in their directional arguments $y^i$ and $G^i_j$ are given by the equation (7.25).
Thus, the Berwald connection $\Gamma^i_{jk}$ of a Finsler space $F_n$ is a triad \( \left(G^i_{jk}, G^i_j, C^i_{jk} = 0\right) \) where $G^i_{jk}$ and $G^i_j$ are Berwald’s V-connection and non-linear connection respectively. The vertical connection vanishes in case of Berwald triad \([10,17]\).

The relation between Berwald’s and Cartan’s V-connection $G^i_{jk}$ and $\Gamma^i_{jk}$ is given by [22]

\[ (7.36) \ G^i_{jk} = \Gamma^i_{jk} + P^i_{jk} \]

where

\[ (7.40) \ P^i_{jk}(x, y) = C^i_{jk\mid o} = \dot{\partial}_k \Gamma^i_{jp}y^p = \dot{\partial}_j \Gamma^i_{kp}y^p. \]

Also, we can get

\[ (7.41) \ G^i_{jk}y^j = \Gamma^i_{jk}y^j. \]

Further, the Berwald’s covariant derivative of the metric tensor $g_{ij}$ is given by [22]

\[ (7.42) \ g_{ij(k)} = -2P^i_{jk} \] and therefore $g_{ij(k)}y^i = 0$,

where

\[ (7.43) \ P^i_{jk} = g^h_{jk}P^h_{ik} = C^i_{jk\mid o}. \]

This tensor $P^i_{jk}$ is a symmetric and indicatory tensor. Also we have the following relations

\[ (7.44) \ L_{(i)} = 0, \ \ell^i_{(j)} = 0, \ h^i_{(j)} = 0, \ h^h_{(k)} = 0, \ h^h_{(k)} = -2P^i_{jk}. \]

Taking $G^i_{ijk} = \dot{\partial}_h G^i_{jk}$, the following relations hold

\[ (7.45) \ (a) \ G^i_{jkh}y^j = 0, \ (b) \ g^h_{jk}G^h_{ik} = G^i_{jk} \] and \[ (c) \ \dot{\partial}_h G^i_{jk} = G^i_{jkh}. \]

Those Finsler spaces for which the functions $G^i_{jk}$ are independent of the directional argument $y^i$ are called ‘affinely
connected spaces. The affinely connected spaces are characterised by the condition $C_{ijk\nu} = 0$. It follows therefore that

$$(7.46) \ G_{jk}^i = \Gamma_{jk}^{*i}$$

for an affinely connected Finsler space.

8. GEODESICS IN A FINSLER SPACE:

The geodesics of a Finsler space are the curves of the extremum arc-length between any two points of the space. The differential equation of a geodesic in a Finsler space is given by [22]

$$(8.1) \ \frac{d^2x^i}{ds^2} + \gamma_{jk}^i \left( x, \frac{dx}{ds} \right) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

where $s$ is the arc length of the curve $x^i = x^i(s)$.

If we put $x'^{i} = \frac{dx^i}{ds}$ and $x''^{i} = \frac{d^2x^i}{ds^2}$ and use the relation (7.20), the equation (8.1) takes the form

$$(8.2) \ x''^{i} + \Gamma_{jk}^{*i}(x, y) y^j y^k = 0$$

or $$(8.3) \ \frac{\delta x'^{i}}{\delta s} = 0.$$

Thus, the geodesics of a Finsler space are autoparallel curves. In the case of Berwald connection, the differential equation of a geodesic is given by

$$(8.4) \ 2x''^{i} + 2G^i(x, x') = 0.$$  

9. CURVATURE TENSORS OF A FINSLER SPACE:

Several curvature tensors have been defined and studied in the literature of Finsler spaces with the help of different Finsler connections. We introduce some of them.

(a) The curvature tensor $K_{jhk}^i$ is defined as [22], p-97.

$$(9.1) \ (a) \ K_{jhk}^i = \partial_k \Gamma_{hj}^{*i} + \left( \partial_i \Gamma_{jk}^{*i} \right) \Gamma_{hl}^{*j} y^l + \Gamma_{mk}^{*i} \Gamma_{hj}^{*m} - \frac{k}{h},$$
\(-k/h\) means the subtraction from the former term by interchanging the indices \(k\) and \(h\).

This tensor is skew-symmetric in its last two lower indices, i.e.

\[
(9.1) \quad (b) \quad K_{jkh}^i = -K_{jkh}^i
\]

and is positively homogeneous of degree zero in \(y^i\). The curvature tensor \(K_{jkh}^i\) satisfies the following identities known as Bianchi identities

\[
(9.2) \quad (a) \quad K_{jkh}^i + K_{hjk}^i + K_{kij}^i = 0
\]

and

\[
(9.2) \quad (b) \quad K_{jklh}^r + K_{iklh}^r + K_{ihkl}^r + \left(\hat{\sigma}_S \Gamma^{sr}_{ij}\right) K_{jhlk}^S + \left(\hat{\sigma}_S \Gamma^{sr}_{ik}\right) K_{ijlh}^S + \left(\hat{\sigma}_S \Gamma^{sr}_{ih}\right) K_{ijkl}^S \cdot y' = 0.
\]

An obvious consequence of (9.2a) is

\[
(9.3) \quad (a) \quad K_{jkh} + K_{hjk} + K_{kij} = 0
\]

where the associate tensor \(K_{jrkh}\) of \(K_{jkh}^i\) is given by

\[
(9.3) \quad (b) \quad K_{jkh} = g_{ij} K_{ikh}^r,
\]

the tensor \(K_{jkh}\) also satisfy the condition

\[
(9.3) \quad (c) \quad K_{jkh} = -K_{jkh} - 2C_{ij} K_{rkh}^r y'
\]

**B. THE CURVATURE TENSORS OF CARTAN:**

The Ricci identities for a tensor \(T^i_j\) involving \(h\) and \(v\)-covariant derivatives with respect to the Cartan connection are given by [15]

\[
(9.4) \quad T^i_{jklh} - T^i_{jlk} = T^m_j R^i_{mhk} - T^i_{mjk} R^m_j - T^i_{jlm} R^m_l ,
\]

\[
(9.5) \quad T^i_{jklh} - T^i_{jlk} = T^m_j P^i_{mhk} - T^i_{mjk} P^m_j - T^i_{jlm} C^m_{hk} - T^i_{jlm} P^m_{jk} ,
\]

\[
(9.6) \quad T^i_{jklh} - T^i_{jlk} = T^m_j S^i_{mhk} - T^i_{mjk} S^m_j ,
\]

where
\begin{align}
(9.7) \quad R^i_{jk} &= \partial_h \Gamma^*_{jk} + \left( \partial_k \Gamma^*_{ij} \right) \Gamma^*_{ih} y^s + C^i_{jm} \left( \partial_k \Gamma^*_{sh} y^s - \Gamma^*_{ms} \Gamma^*_{sh} y^s \right) + \\
&+ \Gamma^*_{mk} \Gamma^*_{jh} - k j h,
\end{align}

\begin{align}
(9.8) \quad R^i_{hk} &= R^i_{hkm} y^m,
\end{align}

\begin{align}
(9.9) \quad P^i_{hkm} &= \frac{\partial \Gamma^*_{hk}}{\partial y^m} - C^i_{hk|m} + C^i_{hr} P^r_{km},
\end{align}

\begin{align}
(9.10) \quad S^i_{hkm} &= C^i_{rk} C^r_{hm} - C^i_{rm} C^r_{hk}.
\end{align}

The tensors defined by (9.7), (9.9), and (9.10) are called Cartan's curvature tensors. Also, it is known as h-curvature tensor, hv-curvature tensor and v-curvature tensor respectively. The v-curvature tensor \( S^i_{hkm} \) is an indicator tensor. These curvature tensors satisfy the following identities

\begin{align}
(9.11) \quad &\begin{align}
(a) \quad &R^i_{hijk} = -R^i_{hjk}, \quad (b) \quad &R^i_{hijk} = -R^i_{iijk}, \\
(9.12) \quad &\begin{align}
(a) \quad &P^i_{hijk} = -P^i_{ihjk}, \quad (b) \quad &P^i_{hijk} = -P^i_{hijk} = -S^i_{hijk|lo}, \\
&\quad (c) \quad P^i_{hijk} y^h = P^i_{ijk} = C^i_{ijk|lo}.
\end{align}
\end{align}

\begin{align}
(9.13) \quad &\begin{align}
(a) \quad &S^i_{hijk} = -S^i_{hijk}, \quad (b) \quad &S^i_{hijk} = -S^i_{ihjk}, \\
&\quad (c) \quad S^i_{hijk} = S^i_{jkh}. 
\end{align}
\end{align}

\begin{align}
(9.14) \quad &R^i_{ijk} = -R^i_{ikj}.
\end{align}

\begin{align}
(9.15) \quad &P^i_{ijk} = P^i_{ikj} = P^i_{jik} = P^i_{kji}.
\end{align}

where

\begin{align}
R^i_{hijk} &= g^m_{im} R^m_{hjk}, \quad P^i_{hijk} = g^m_{im} P^m_{hjk}, \\
S^i_{hijk} &= g^m_{im} S^m_{hjk}, \quad R^i_{ijk} = R^i_{hijk} y^h = g^m_{im} R^m_{ijk},
\end{align}

and \( P^i_{ijk} \) is given by the equation (7.43).
(C) CURVATURE TENSORS ARISING FROM BERWALD

CONNECTION:

The equations of geodesic deviation have been given by Berwald [2] in the form

\[ (9.16) \ \frac{\delta^2 Z^i}{\delta u^2} + H_k^j(x, y) Z^k = 0. \]

Where \( Z^i \) is called the variation vector. The tensor \( H_k^j(x, y) \) is called the ‘deviation tensor’ defined by

\[ (9.17) \ H_k^j(x, y) \overset{\text{def.}}{=} K_{hk}^i(x, y) y^i y^j. \]

It can also be written in the form

\[ (9.18) \ H_k^i(x, y) = 2\partial_k G^i - \partial_k \left( \partial^j G^i \right) y^h + 2G_{km}^i G^m - \partial_k G^m \partial^m G^i, \]

where we have used the fact that the function \( G^m(x, y) \) is positively homogeneous of degree 2 in \( y^i \).

The tensors defined by

\[ (9.19) \ H_{jk}^i(x, y) \overset{\text{def.}}{=} \frac{1}{2} \left( \partial_j H_k^i - \partial_k H_j^i \right) \]

and

\[ (9.20) \ H_{ijk}^i(x, y) \overset{\text{def.}}{=} \partial_{ij} H_{jk}^i \]

are explicitly written as

\[ (9.21) \ H_{jk}^i = \partial_k \partial_j G^i - \partial_j \partial_k G^i + G_{kr}^i \partial_j G^r - G_{rj}^i \partial_k G^r \]

and

\[ (9.22) \ H_{ijk}^i = \partial_k G_{ij}^i - \partial_j G_{hk}^i + G_{rj}^i G_{rk}^i - G_{rhk}^r G_{ij}^i + \]

\[ + G_{rjk}^i \partial_j G^r - G_{rjk}^i \partial_k G^r, \]

where \( G_{ijk}^i = \partial_k G_{ij}^i \) and \( G_{ijk}^i y^k = 0 \).

We also have the following relations:
(9.23) (a) $H^{i}_{jk}(x,y) = K^{i}_{hjk}(x,y) y^{h} = R^{i}_{hk}(x,y) y^{h}$

(b) $H^{i}_{hjk} = K^{i}_{hjk} + y^{r} \hat{\partial}_{r} K^{i}_{rhjk}.$

we can easily obtain the following

(9.24) (a) $H^{i}_{k}(x,y) y^{k} = 0$,  
(b) $H^{i}_{jk} y^{j} = H^{i}_{k}$,

(c) $H^{i}_{hk} y^{h} = H^{i}_{jk}$,

(d) $H^{h}_{ih} = H^{i}_{i}$,

(e) $H^{h}_{ijh} = H^{i}_{ij} = \hat{\partial}_{i} H^{j}_{j}$,

(f) $H^{i}_{ij} y^{j} = H^{i}_{j}$,

(g) $H^{i}_{i} y^{i} = H^{i}_{i} = (n-1) H$,

(h) $H^{i}_{hjk} + H^{i}_{jkh} + H^{i}_{kij} = 0$,

(i) $H^{i}_{jh} - H^{i}_{hj} = H^{k}_{kij}$.

10. THE PROJECTIVE CURVATURE TENSORS IN $F_{n}$:

The differential equation of the paths with respect to some special parameter $s$ are given by [22]

\[
(10.1) \quad \frac{d^2 x^{i}}{ds^2} + 2G^{i}(x, \frac{dx}{ds}) = \frac{dx^{ii}}{ds} + 2G^{i}(x, x') = 0.
\]

If we subject (10.1) to an arbitrary transformation of its parameters $t = t(s)$ where $\frac{dt}{ds} > 0$, the special form of (10.1) can not in general be preserved. In fact, we find that the differential equations of the path assume the form

\[
(10.2) \quad \frac{dy^{i} / dt + 2G^{i}(x,y)}{y^{i}} = \frac{dy^{k} / dt + 2G^{k}(x,y)}{y^{k}} (i, k = 1, 2, ..., n)
\]

It is obvious that these equations remain unchanged if we replace the functions $G^{i}(x,y)$ by new functions $\bar{G}^{i}(x,y)$, the latter being defined by

\[
(10.3) \quad \bar{G}^{i}(x,y) = G^{i}(x,y) + P(x,y) y^{i}
\]

where $P(x,y)$ is an arbitrary scalar function positively homogeneous of the first degree in $y^{i}$. Clearly this relation
represents the most general modification of the function $G^i$ which will leave equation (10.2) unchanged. We call (10.3) a projective change of the function $G^i$. The equation (10.3) may be regarded as representing a mapping of two path-spaces, characterised by the functions $G^i$ and $\tilde{G}^i$ respectively, onto each other (corresponding line elements being denoted by the same coordinates $(x, y)$). From equation (10.3), we find

\begin{equation}
(10.3a) \quad \tilde{G}^i_{jk} = G^i_{jk} + \dot{\gamma}_j P \delta^i_k + \ddot{\gamma}_k P \delta^i_j + \left( \dot{\gamma}_j \ddot{\gamma}_k P \right) y^i.
\end{equation}

The path space with entities $\tilde{G}^i, \tilde{G}^i_j \tilde{G}^i_{jk}$ will be denoted by $F_n$.

the tensor defined by

\begin{equation}
(10.4) \quad W^i_k \overset{\text{def.}}{=} H^i_k - H \delta^i_k - \frac{1}{n+1} \left( \dot{\gamma}_i H^i_k - \ddot{\gamma}_k H^i \right) y^j
\end{equation}

is invariant under the projective change (10.3) and is called the projective deviation tensor being the counterpart of the tensor $H^i_k$ of the equation (9.25) of geodesic deviation.

It satisfies the following identities:

\begin{equation}
(10.5) \quad (a) \quad W^i_i = 0, \quad \text{ (b) } \quad W^i_k y^k = 0, \quad \text{ (c) } \quad \dot{\gamma}_i W^i_k = 0.
\end{equation}

From the tensors defined by (10.4), we derive projective curvature tensors as follows,

\begin{equation}
(10.6) \quad W^j_{hk} \overset{\text{def.}}{=} \frac{1}{3} \left( \ddot{\gamma}_h W^j_k - \ddot{\gamma}_k W^j_h \right)
\end{equation}

\begin{align*}
&= H^j_{hk} + \frac{y^j}{n+1} \left( H^i_{hk} - H^i_{kh} \right) + \frac{\delta^j_h}{n^2 - 1} \left( n H_k + y^i H^i_k - \delta^j_k \right) - \frac{\delta^j_k}{n^2 - 1} \left( n H_k + y^i H^i_k \right)
\end{align*}

and
\[(10.7) \ W_{ihk}^{j \ \text{def.}} = \hat{\partial}_i W_{ihk}^{j} \]

\[= H_{ihk}^{j} + \frac{\delta_{k}^{j}}{n+1} (H_{hk} - H_{kh}) + \frac{\nu^{j}}{n+1} (\hat{\partial}_i H_{hk} - \hat{\partial}_j H_{kh}) + \]

\[+ \frac{\delta_{h}^{j}}{n^2 - 1} (nH_{ih} + H_{hi} + \nu^{r} \hat{\partial}_i H_{kr}) - \]

\[\frac{\delta_{k}^{j}}{n^2 - 1} (nH_{ih} + H_{hi} + \nu^{r} \hat{\partial}_i H_{kr}). \]

The tensor \( W_{ihk}^{j} \) represents the generalisation of a tensor introduced by Weyl in the restricted geometry of Paths [22].

Noting that \( W_{ij}^{j} \) is homogeneous of the second degree in its directional arguments, we deduce from \( (10.6) \) and \( (10.5b) \) that

\[(10.8) \ W_{hik}^{j} y^{h} = W_{h}^{j} \]

so that in view of \( (10.7) \),

\[(10.9) \ W_{ihk}^{j} y^{j} = W_{h}^{j} \]

11. SEMI SYMMETRIC CONNECTIONS IN A FINSLER MANIFOLD:

The concept of semi-symmetric metric connections in Riemannian manifold is introduced by Yano [25], subsequently Imai [11'] has expressed these connections in the local coordinate system of Riemannian space, Mehar and Patel [19'], introduced similar connections in \( F_{n} \), and they have noticed that the covariant derivative with respect to semi-symmetric connection of the fundamental function the unit vector \( \ell^{i} \) and the directional coordinates \( x^{i} \) vanish although that of the metric tensor does not. As such, they have called these connections as semi-symmetric connections in \( F_{n} \). Similar to definition of Imai [11'], Mehar and Patel [19'] call
\( (11.1) \ \Pi_{jk}^i = G_{jk}^i + U_{jk}^i \)

with

\( (11.2) \ U_{jk}^i = \delta_j^i v_k - g_{jk} v^i, \)

as semi-symmetric connections in \( F_n \). The covariant vector field \( v_j \) is related with the contravariant vector field \( v^i \) by \( v_j = g_{ij} v^i \) and \( v_j \) is not homogeneous in general in the directional arguments.

Thus, the connection parameters \( \Pi_{jk}^i \) are not homogeneous functions in \( \dot{x}^i \)'s as well as not-symmetric in their covariant indices. The covariant derivative of a vector field \( X^i \) with respect to \( \Pi_{jk}^i \) denoted by \( \zeta_j X^i \) is defined as

\( (11.3) \ \zeta_j X^i = \partial_j X^i - \left( \dot{\Pi}_{jk}^m X^r \right) \Pi_{jm}^i \dot{x}^r + X^m \Pi_{jm}^i, \)

which can be deduced, by the help of (7.38) and (11.1) in the form

\( (11.4) \ \zeta_j X^i = X^i_{(jk)} + X^m U_{jm}^i - \left( \dot{\Pi}_m^i X^i \right) U_{jr}^m \dot{x}^r. \)

It can easily be verified that

\( (11.5) \quad (a) \ \zeta_j \ell^i = 0, \quad (b) \ \zeta_j \dot{x}^i = 0 \quad (c) \ \zeta_j F = 0. \)

For a vector field \( X^i(x, \dot{x}) \) the operators \( \dot{\partial}_j \) and \( \zeta_k \) commute according to

\( (11.6) \ \left( \dot{\partial}_j \zeta_k - \zeta_k \dot{\partial}_j \right) X^i = X^s \Pi_{jks}^i \left( \dot{\partial}_s X^i \right) \Pi_{jkr}^i \dot{x}^r. \)

where \( \Pi_{jkr}^i = \dot{\partial}_j \Pi_{kr}^i \), the term \( \Pi_{jkr}^i \dot{x}^r \) evolving in (11.6) does not vanish as \( \Pi_{jkr}^i \) are not homogeneous functions in \( \dot{x}^i \)'s.

For \( \zeta_j \) the commutation formula of a vector field \( X^i \) is expressed by

\( (11.7) \ 2 \zeta_j \zeta_{k1} X^i = X^h R_{jkh}^i - \left( \dot{\partial}_m X^i \right) R_{jkh}^i \dot{x}^h - 2 \left( \partial_h X^i \right) \Pi_{[jkh]}^i \),

where the curvature type quantities \( R_{jkh}^i \) introduced here are defined by
(11.8) \[ R_{jkh}^i = 2 \left\{ \partial_{[j} \Pi_{k]}^i_{m} - \Pi_{m[k}^i \Pi_{j]}^r \dot{x}^r + \Pi_{m[j}^i \Pi_{k]}^i_{m} + \Pi_{m[k}^i \Pi_{j]}^r \right\} \]

and are different than the curvature defined by the same notation of Rund [22].

Like the Berwald's contracted curvature, the contracted curvature type quantities are defined by

(11.9) \[ R_{kh}^i = R_{ikh}^i. \]

\( R_{jkh}^i \) on contraction on the indices \( i \) and \( h \) satisfies

(11.10) \[ R_{khi}^i - 2R_{[kk]}^i = 2nL_{k}^{j}u_{k}\]

where the symbol \( L_{j}^{i} \) over a tensor field \( T_{k}^{i}(x, \dot{x}) \) is defined by

(11.11) \[ L_{j}^{i}T_{k}^{i} = T_{k(j)}^{i} - \left( \hat{\partial}_{j}T_{k}^{i} \right) u_{j}\dot{x}^{r} + \left( \hat{\partial}_{m}T_{k}^{i} \right) g_{jr}u^{m}\dot{x}^{r}. \]

Under the restriction \( \zeta_{[k}u_{h]} = 0 \), the contracted curvature type quantities \( R_{kh}^i \) are expressed by

(11.12) \[ R_{ihi}^i = 2R_{[kk]}^i, \]

The curvature type quantities \( R_{jik}^i \) satisfy the identity

(11.13) \[ R_{jkh}^i + R_{kkj}^i + R_{jik}^i = 2 \left( \delta_{[j}^{l}L_{k]}^{i}u_{k} + \delta_{k}^{l}L_{j}^{i}u_{k} + \delta_{i}^{l}L_{j}^{i}u_{j} \right) \]

whereas in a Finsler space \( F_{n}^{*} \) admitting semi-symmetric connection \( R_{jkh}^i \) satisfies

(11.14) \[ R_{jkh}^i + R_{kkj}^i + R_{jik}^i = 0 \]

under the condition \( \zeta_{[k}u_{h]} = 0 \).

12. LIE-DIFFERENTIATION:

Let \( v^{i}(x^{i}) \) be a contravariant vector field independent of directional arguments defined over the differentiable space \( X_{n} \) of a Finsler space \( F_{n} \). With the help of this field we consider the transformation
(12.1) (a) $\bar{x}^i = x^i + \varepsilon i ^i (x^i)$,

where $\varepsilon$ is an infinitesimal constant. The corresponding variation of $y^i$ is represented by

(12.1) (b) $\bar{y}^i = y^i + \varepsilon i ^i (\partial_j y^j) y^j$.

The transformation represented by (11.1) is called an infinitesimal transformation because of the infinitesimal constant $\varepsilon$ appearing in (11.1). Also this transformation gives rise to a process of differentiation called Lie-differentiation. Interpreting (11.1a) as a general shift as well as an infinitesimal coordinate transformation, we may find the changes in an arbitrary vector field $X^i$ over $X_n$. If $d^i X^i$ and $d^m X^i$ are these changes, then the Lie-derivative of $X^i$ with respect to (11.1a) is defined by

(12.2) $\mathcal{L}_X X^i = \lim_{\varepsilon \to 0} \frac{d^i X^i - d^m X^i}{\varepsilon}$

where the symbol $\mathcal{L}_X$ stands for Lie-differentiation.

Let $T_{kh}^i$ be an arbitrary tensor field, its Lie-derivative with respect to the above infinitesimal transformation and also with respect to Cartan’s and Berwald’s connections respectively are given by

(12.3) (a) $\mathcal{L}_X T_{kh}^i = T_{kh}^i v^r - T_{kh}^i v^r_k + T_{rh}^i v^r_k + T_{kr}^i v^r_h + (\partial_{,} T_{kh}^i) v^r_s y^s$

and

(b) $\mathcal{L}_X T_{kh}^i = v^r T_{kh(r)}^i - T_{kh}^i v_{(r)}^i + T_{rh}^i v_{(r)}^i + T_{kr}^i v_{(h)} + (\partial_{,} T_{kh}^i) v_{(s)}^r y^s$.

obviously the Lie-derivatives of $y^i$ and $v^i$ with respect to above infinitesimal transformation vanish, i.e.

(12.4) (a) $\mathcal{L}_X y^i = 0$ (b) $\mathcal{L}_X v^i = 0$. 


The Lie-derivative of the connection parameters $\Gamma^i_{kh}$ [22, 24] and $G^i_{kh}$ [7, 22, 25] are given by

\begin{equation}
(12.5) \quad \mathcal{L}_{\Gamma^i_{kh}} = v^i_{jkh} + K^i_{hkr} v^r + \left( \hat{\partial}_r \Gamma^i_{kh} \right) v^r_s y^s
\end{equation}

and

\begin{equation}
(12.5) \quad \mathcal{L}_{G^i_{kh}} = v^i_{(h)(k)} + H^i_{hkr} v^r + \left( \hat{\partial}_r G^i_{kh} \right) v^r_s y^s.
\end{equation}

The process of Lie-differentiation commutes with those of partial and covariant differentiation for both connections according to

\begin{equation}
(12.6) \quad (\hat{\partial}_j \mathcal{L}_\Omega - \mathcal{L}_j \hat{\partial}_\Omega) \Omega = 0,
\end{equation}

\begin{equation}
(12.6) \quad \mathcal{L} \left( X^i_{(j)} \right) - \left( \mathcal{L} X^i \right)_{(j)} = X^r \mathcal{L} \Gamma^i_{rk} - \left( \hat{\partial}_r X^i \right) \mathcal{L} \Gamma^s_{sk} y^s
\end{equation}

and

\begin{equation}
(12.6) \quad \mathcal{L} \left( X^i_{(k)} \right) - \left( \mathcal{L} X^i \right)_{(k)} = X^r \mathcal{L} G^i_{rk} - \left( \hat{\partial}_r X^i \right) \mathcal{L} G^s_{sk} y^s
\end{equation}

where $\Omega$ is any geometric object such as scalar, vector or connection coefficient.

The infinitesimal transformation (11.1a) defines a motion, affine motion, projective motion or conformal motion (transformation) if it preserves the distance between two points, parallelism of vectors, the geodesic or the angle between pair of vectors respectively. Necessary and sufficient conditions for the transformation (11.1a) to be a motion, affine motion, projective motion and conformal motion are respectively given by

\begin{equation}
(12.7) \quad (a) \quad \mathcal{L} g_{kh} = 0,
\end{equation}

\begin{equation}
(12.7) \quad (b) \quad (i) \quad \mathcal{L} \Gamma^i_{kh} = 0 \quad \text{(Cartan’s connection)},
\end{equation}

\begin{equation}
(12.7) \quad \text{(ii)} \quad \mathcal{L} G^i_{kh} = 0 \quad \text{(Berwald’s connection)},
\end{equation}

\begin{equation}
(12.7) \quad (c) \quad \mathcal{L} G^i_{kh} = \delta^i_k p_h + \delta^i_h p_k + y^i p_{kh}
\end{equation}

and

\begin{equation}
(12.7) \quad (d) \quad \mathcal{L} g_{kh} = \phi g_{kh},
\end{equation}
\( \phi \) is a scalar point function where \( p \) is a scalar positively homogeneous of degree one in \( y^j \) and \( \phi \) as mentioned above is a function of \( x^j \), i.e. \( \phi = \phi(x^j) \), a function of positional coordinates only.

It is well known that even motion is an affine motion and every affine motion is a projective motion. A projective motion need not be an affine motion. A projective motion, which is not an affine motion, will be called as non-affine projective motion.
REFERENCES


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