Chapter 3. Inventory Model for non-linear price-dependent demand and constant inflation rate
3.1 Introduction

This chapter is concerned primarily with non-linear price dependent demand, which is generally observed for finished goods. Ordering a large quantity reduces the ordering cost since orders are less frequent. It also reduces the ordering cost divided over the units ordered and the shipping expenses per unit. However, ordering a small quantity will reduce the holding cost (due to the tied-up capital in the items) and the storage space required since there will be less inventory on hand. Economic order quantity (EOQ) models have been developed to balance these different costs to obtain a minimum total cost or maximum total profit.

As indicated earlier, in classical inventory models it is generally assumed that the demand rate remains constant over time. However, this assumption is usually valid only in the mature stage of the life cycle of a product. In real-world applications, the demand of a product may depend on a number of factors, like price of the item, rate of deterioration, stock height, etc. A wide variety of items like fashion goods, medicines, etc. demand is likely to be linear in price and time, and may also depend on the stock level. Studies along this line has been conducted by many authors, see, for example, Hariga and Benkherouf (1994), Hariga (1996), Jalan and Choudhuri (1999), Mandal et al. (2003), You (2005). Low selling price can result in an increase in demand rate, while high selling price may cause a decline in demand. Thus, the demand function shows a downward slope with respect to price. The most commonly used mathematical functions to describe the price-demand relationship are (i) linear \(D = a - bp, a,b>0\), and (ii) iso-elastic \(D = Kp^{-a}, K>0\). A mixture of linear and iso-elastic demand gives hybrid demand. To the best of our knowledge, iso-elastic and hybrid demand functions have not been used to find the EOQ policy. In this chapter we, therefore, aim at studying an integrated inventory model with such demand rates that considers operations and pricing decisions, and investigate the effect of this coordination on the system. For items like fruit juice, soft drinks, detergents, paper towels, etc. the demand functions are often found to be iso-elastic, and seldom linear in price. Iso-elastic demand is widely used in production economics as seen in the studies by Petruzzi and Dada (1999), Tramontana (2010), Wang et al. (2012), and Nilsen (2013). On the other hand, commodities like coffee, cotton, tin, copper, wristwatch, automobile man’s suit, snow skis,, sailboat, mattress, floor lamp, refrigerator, PC game, trash compactor, ice maker, etc exhibit...
hybrid demand. Studies using hybrid demand have been carried out by Bearden and Etzel (1982), Deaton and Laroque (1992), Lau and Lau (2003), Chen and Chen (2007), Lau et al. (2008).

While classical inventory models assume the cost parameters to remain unchanged all through the planning horizon, from financial point of view, it is important to investigate how time value of money influences an inventory policy. Buzacott (1975) was the first to consider EOQ model with inflation, subject to different types of pricing policies. Misra (1979) developed a discounted-cost model and included internal (company) and external (general economy) inflation rates for various costs associated with an inventory system. Sarker and Pan (1994) surveyed the effects of inflation and the time value of money on order quantity with finite replenishment rate. Other papers which also consider the time value of money and inflation are by Uthayakumar and Geetha (2009), Maity (2010), Vrat and Padmanabhan (1990), Dutta and Pal (1991), Hariga (1995), Kuo-Lung Hou (2006), to name a few. All these papers, however, assume equal replenishment cycles. Hariga and Ben-Daya (1996) developed an inventory model with unequal replenishment cycles taking into account the effects of inflation and time value of money. Sabahno (2008) studied an inventory model with time value of money and inflation, finite replenishment rate and unequal replenishment cycle lengths.

Though classical models are based on the assumption that the inventory manager pays his dues to the supplier immediately on receiving the items in actual financial transactions it is observed that the supplier often offers the manager a delay period, known as the trade credit period, to pay his dues. Offering such credit period encourages the supplier’s selling and reduces his on-hand stock level. At the same time, this arrangement comes out to be very advantageous to the manager as he may delay the payment till the end of the permissible delay period. During the credit period the manager can start to accumulate revenues on the sales and earn interest on that revenue. Thus, the delay in the payment offered by the supplier is a kind of price discount to the manager since paying later indirectly reduces the cost of holding and it encourages the inventory manager to increase his order quantity. Moreover, paying later indirectly reduces the cost of holding stock. Hence, trade credit can play an important role in inventory model for both the supplier as well as the inventory manager in integrated
inventory model. The first study along this line was carried out by Goyal (1985). Thereafter many authors investigated inventory models. See, for example, Shinn et al. (1996), Hwang and Shinn (1997), Jamal et al. (1997), Pal and Ghosh (2006, 2007), Shah and Shah (1998), Ghosh (2007). Inventory models allowing permissible delay in payment and also taking into account inflation and time value of money have been considered by Shah (2006), Lio et al. (2006), Mishra et al. (2011). They, however, considered equal replenishment cycles.

Inventory models allowing shortages have been studied by many authors, for example Shinn et al. (1996), Ouyang et al. (1999), Chang and Dye (2001), Pal and Ghosh (2006), to name a few. However, in most studies it is assumed that during stock-out, demand is either completely backlogged or it is lost. In reality, often some customers are willing to wait until replenishment, especially if the wait is short, while others may be impatient and go elsewhere. The willingness of a customer to wait during a shortage period is likely to decline with the length of waiting time. To capture this situation, the backlogging rate should be taken as a function of time. If \( \delta(t) \) denotes the proportion of backlogged demand at time \( t \), \( 1 - \delta(t) \) will be the proportion of lost sale at \( t \). Demands for foods, medicines, etc. are usually lost during the shortage period, so that \( \delta(t) \) is very small for such items even for small \( t \). On the other hand, for fashionable commodities, like jewellery and high-tech products, demands are usually backlogged so that \( 1 - \delta(t) \) is very small for moderate to large \( t \).

In this chapter we follow the discounted cash flow approach to develop an inventory model with varying lengths of replenishment cycles, taking into account the effects of inflation and time value of money. Demand is assumed to be price dependent iso-elastic and hybrid. Price inflation is taken into account and the inventory manager is allowed a constant grace period to repay his dues.
3.2 An Ordering policy for iso-elastic demand under permissible delay in payments and price inflation

3.2.1 Notations and Assumptions

The notations used in the sub-section are as follows:

\( H = \) Planning horizon
\( n = \) number of replenishment periods during the planning horizon
\( [t_{j-1}, t_j) = j\)-th reorder interval, \( 1 \leq j \leq n, \ t_0 = 0, t_n = H \)
\( T_j = t_j - t_{j-1}, \ 1 \leq j \leq n \)
\( p_i = \) nominal selling price per item in inventory at time \( t \)
\( f = \) discount rate
\( r = \) inflation rate
\( R = f - r = \) net discount rate of inflation
\( \bar{p}_t = e^{rt} = \) general price level at time \( t (\bar{p}_0 = 1) \).
\( z_t = \frac{p_t}{\bar{p}_t} = \) real selling price for every item in inventory at time \( t \)
\( D(t) = a z_t^b = a(p_t e^{-rt})^b = a p_t^b e^{br} = \) iso-elastic price dependent demand rate at time \( t \), \( b > 1 \)
\( A_0 = \) ordering cost per order at time \( t = 0 \)
\( h_0 = \) unit inventory holding cost per unit time at time \( t = 0 \)
\( c_0 = \) purchasing cost per unit of item at time \( t = 0 \)
\( M = \) permissible delay period for settling account.
\( \theta = \) constant fraction of the on-hand inventory deteriorating per unit time
\( I_e = \) interest that can be earned per rupee during the planning horizon
\( I_p = \) interest paid per rupee investment in stocks during the planning horizon
The policy is to place orders at time points \(0 = t_0 < t_1 < t_2 < \ldots < t_{n-1}(< t_n = H)\), and the nominal, unadjusted selling price remains fixed at \(p_j\) in the \(j^{th}\) reorder interval \([t_{j-1}, t_j]\). It is assumed that \(pe^{rt_{j-1}} \leq p_j \leq pe^{rt_j}\), where \(pe^{rt}\) is the inflation adjusted initial price at \(t\). The real price per item at time point \(t\) in the \(j^{th}\) interval is \(z_j = p_je^{-rt}\), and the demand rate is \(D_j(t) = ap_j^{-b}e^{b\cdot rt}\). It is assumed that the demand during a reorder cycle is completely met, and there is no stock on hand or shortage at the end of the cycle.

**Figure 3.2.1:** Inventory level over the time horizon

3.2.2 Mathematical Model

Let \(I_j(t)\) denote the inventory level at time \(t \in [t_{j-1}, t_j)\).

Since depletion from stock occurs simultaneously due to demand and deterioration, the differential equation that describes the instantaneous state of \(I_j(t)\) is given by

\[
\frac{dI_j(t)}{dt} + \theta I_j(t) = -D_j(t), \quad t_{j-1} < t < t_j, \quad j = 1, 2, \ldots, n
\]

with boundary condition \(I_j(t_j) = 0\).

The solution to the above differential equation is given by
\[ I_j(t) = \frac{ap_j \t b}{\theta + br} \left[ e^{(\theta+br)t_j - \theta} - e^{brt} \right], \quad t_{j-1} < t < t_j, \quad j = 1, 2, \ldots, n. \]

The decision variables are \( n \) and \( t_j, p_j, j = 1, 2, \ldots, n-1 \), which are determined so as to maximize the present value of the total profit over the planning horizon.

Let \( Z(n, t, p) \) denote the present value of the total profit over \([0, H]\), where \( t = (t_1, t_2, \ldots, t_{n-1}) \) and \( p = (p_1, p_2, \ldots, p_{n-1}, p_n) \). The different components of \( Z(n, t, p) \) are as follows:

(i) Ordering Cost: There are \( n \) replenishment cycles in \([0, H]\). Hence the present value of the total ordering cost is

\[ C_R = A_0 \sum_{j=1}^{n} e^{-Rt_{j-1}}. \]

(ii) Holding Cost: The present value of the inventory holding cost in the \( n \) cycles is

\[
C_h = h_0 \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} I_j(t)e^{-Rt} dt
\]

\[
= h_0 \frac{1}{\theta + br} \sum_{j=1}^{n} p_j^{-b} \left[ e^{(\theta+R)t_{j-1}} - e^{(\theta+R)t_j} \right] - \frac{1}{R - br} \left[ e^{(br-R)t_{j-1}} - e^{(br-R)t_j} \right].
\]

(iii) Deteriorating Cost: Since a fraction \( \theta \) of the stock on hand deteriorates per unit time, the present value of inventory deteriorating cost is

\[ C_\theta = \theta c_0 C_h. \]

(iv) Purchase Cost: The present value of total purchasing cost is given by

\[
C_p = c_0 \sum_{j=1}^{n} I_j(t_{j-1})e^{-Rt_{j-1}}
\]

\[
= c_0 \frac{1}{br + \theta} \sum_{j=1}^{n} p_j^{-b} \left[ e^{(\theta+br)t_{j-1}} - e^{brt_{j-1}} \right] e^{-Rt_{j-1}}.
\]

(v) Sales Revenue: The present value of total sales revenue over \([0, H]\) is
\[ S_R = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} p_j D_j(t) e^{-rt} dt \]
\[ = \frac{a}{R - br} \sum_{j=1}^{n} p_j^{1-b} \left( e^{(br-R)t_j} - e^{(br-R)t_{j-1}} \right) \]

(vi) Interest paid and interest earned: In any replenishment cycle, say the \( j \)-th cycle, the interest earned or paid by the inventory manager depends on whether \( T_j \geq M \) or \( T_j < M \).

Case 1: \( T_j \geq M \), i.e. \( t_{j-1} + M \leq t_j \).

In this case, the inventory manager uses his sales revenue to earn interest throughout the cycle.

However, the unsold stock remaining after the end of the grace period has to be financed at the specified rate of interest.

Thus, the present value of the total interest earned in the \( j \)-th cycle is given by

\[ I_{E1}(j) = I_e \int_{t_{j-1}}^{t_j} p_j D_j(t) e^{-rt} dt \]
\[ = I_e \frac{a}{(R - br)^2} p_j^{1-b} \left[ e^{(br-R)t_j} \left((R-br)t_{j-1} + 1\right) - e^{(br-R)t_j} \left((R-br)t_j + 1\right) \right] \]

And, the total interest payable is

\[ I_{P1}(j) = c_0 I_p \int_{M+t_{j-1}}^{t_j} I_j(t) e^{-rt} dt \]
\[ = c_0 I_p \frac{a}{\theta + br} p_j^{1-b} \left[ e^{(\theta+br)t_j} \left(e^{-\theta t_j} - e^{-\theta t_{j-1}}\right) - e^{(\theta+br)t_j} \left(e^{(br-R)(M+t_{j-1})} - e^{(br-R)t_j}\right) \right] \]

Case 2: \( T_j \leq M \), i.e. \( t_{j-1} + M \geq t_j \).

In this case, the inventory manager earns interest on his sales revenue till the end of the permissible period and does not have to pay any interest.

Thus, the interest earned is given by
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\[ I_{E2}(j) = I_{e}p_{j}^{t_{j}} \int_{t_{j-1}}^{t_{j}} D_{j}(t)e^{-Rt} dt + \left(M - T_{j}\right) \int_{t_{j-1}}^{t_{j}} D_{j}(t)e^{-Rt} dt \]

\[ = aI_{e}p_{j}^{1-b} \left[ \frac{1}{(R - br)^{2}} e^{(br-R)t_{j-1}} \left( (R - br)t_{j-1} + 1 \right) - e^{(br-R)t_{j}} \left( (R - br)t_{j} + 1 \right) \right] + \frac{M - T_{j}}{R - br} \left( e^{(br-R)t_{j-1}} - e^{(br-R)t_{j}} \right) \]

Therefore, the present value of the total profit over the planning horizon is

\[ Z(n,t,p) = S_{R} + \sum_{j=1}^{n} \{ \delta_{j}I_{E1}(j) + (1 - \delta_{j})I_{E2}(j) \} - \left( C_{R} + C_{h} + C_{0} + C_{p} + \sum_{j=1}^{n} \delta_{j}I_{p1}(j) \right), \quad (3.2.1) \]

where

\[ \delta_{j} = 1, \text{ if } t_{j-1} + M \leq t_{j} \]
\[ = 0, \text{ if } t_{j-1} + M \geq t_{j}. \]

Then, the optimal values of \( n, t \) and \( p \) satisfy the following:

\[ Z(n+1,t,p) - Z(n,t,p) \leq 0 \leq Z(n,t,p) - Z(n-1,t,p) \]

\[ \frac{\partial Z(n,t,p)}{\partial t_{j}} = 0 \quad \text{for } j = 1,2,\ldots,n-1, \]

\[ \frac{\partial Z(n,t,p)}{\partial p_{j}} = 0 \quad \text{for } j = 1,2,\ldots,n. \]

We consider the following two situations.

**Situation1:** \( \delta_{j} = 1 \) for all \( j = 1, 2, \ldots, n. \)

In this case, \( \frac{\partial Z(n,t,p)}{\partial p_{j}} = 0 \) gives
\[
p_j = \frac{b}{b-1} \frac{F_2(t_{j-1}, t_j)}{F_1(t_{j-1}, t_j)},
\]

where
\[
F_1(t_{j-1}, t_j) = \begin{cases} 
\frac{R - br}{e^{(br-R)t_{j-1}} - e^{-(br-R)t_j}} + I e^{(br-R)t_{j-1}} 
\end{cases} \frac{1}{(R - br)^2}
\]
\[
F_2(t_{j-1}, t_j) = \frac{(\theta_0 + h_0)}{\theta + br} \left( e^{(\theta + br)t_j} \left( e^{-(\theta + R)t_{j-1}} - e^{-(\theta + R)t_j} \right) \right) + I e^{(\theta + br)t_j} \left( e^{-(\theta + R)(M + t_{j-1})} - e^{-(\theta + R)t_j} \right)
\]

Expressing \( p_j \) as (3.2.2) in \( Z(n, t, p) \), we have
\[
Z(n,t) = a \left( \frac{b}{b-1} \right)^{1-b} - \left( \frac{b}{b-1} \right)^{-b} \sum_{j=1}^n \left( F_2(t_{j-1}, t_j) \right)^{1-b} \left( F_1(t_{j-1}, t_j) \right)^b - \sum_{j=1}^n e^{-Rt_{j-1}}.
\]

Thus, for fixed \( n \), the problem becomes:

Maximize \( Z(n, t) \)

subject to \( t_j > M + t_{j-1}, 1 \leq t_j \leq n \), \( t_0 = 0 \), \( t_n = H \),

and \( \frac{b}{b-1} \frac{F_2(t_{j-1}, t_j)}{F_1(t_{j-1}, t_j)} \leq p e^{rt_{j-1}} \).

**Situation2**: \( \delta_j = 0 \) for all \( j = 1, 2, \ldots, n \).

In this case, \( \frac{\partial Z(n,t,p)}{\partial p_j} = 0 \) gives
\( p_j = \frac{b}{b-1} - \frac{G_2(t_{j-1}, t_j)}{G_1(t_{j-1}, t_j)} \) \( (3.2.3) \)

where

\[
G_1(t_{j-1}, t_j) = \left( e^{(b-R)y_{j-1}} - e^{(b-R)y_j} \right) + I e \left( \frac{e^{(b-R)y_{j-1}} - e^{(b-R)y_j}}{R - br} \right) + \frac{(M - T_j)}{R - br} \left( e^{(b-R)y_{j-1}} - e^{(b-R)y_j} \right),
\]

\[
G_2(t_{j-1}, t_j) = \frac{(\theta + b)^j}{\theta + br} \left( e^{(\theta+br)y_{j-1}} - e^{(\theta+br)y_j} \right) - \frac{1}{R - br} \left( e^{(b-R)y_{j-1}} - e^{(b-R)y_j} \right) + \frac{c_0}{\theta + br} \left( e^{(\theta+br)y_{j-1}} - e^{(b-R)y_{j-1}} \right)
\]

Hence, expressing \( p_j \) as \( (3.2.3) \) in \( Z(n,t,p) \), we can write the profit function as

\[
Z(n,t) = a \left( \frac{b}{b-1} \right)^{-b} - \left( \frac{b}{b-1} \right)^{-b} \sum_{j=1}^{n} \left( \frac{G_2(t_{j-1}, t_j)}{G_1(t_{j-1}, t_j)} \right)^{-b} \left( G_1(t_{j-1}, t_j) \right)^b - A_0 \sum_{j=1}^{n} e^{-R_{t_{j-1}}}. \]

So, for fixed \( n \), our problem becomes:

Maximize \( Z(n,t) \)

subject to \( M + t_{j-1} > t_j, \ \ t_0 = 0, \ \ t_n = H \) and \( \frac{b}{b-1} \frac{G_2(t_{j-1}, t_j)}{G_1(t_{j-1}, t_j)} \leq pe^{rt_j}, \ 1 \leq j \leq n, \)

where \( p \) is the selling price in the first cycle.

### 3.2.3 Some Results

**Theorem 3.2.1:** \( Z(n) \) is concave in \( n \).

**Proof:** When \( t_{j-1} + M \leq t_j \) for all \( j = 1, 2, \ldots, n \), i.e. \( \delta_j = 1 \) for all \( j = 1, 2, \ldots, n \), we can write

\[
Z(n,t) = Q(n) + T(n,0,H),
\]

where

\[
Q(n) = -A_0 \sum_{j=1}^{n} e^{-R_{t_{j-1}}} \quad \text{total ordering cost in } [0, H],
\]

\[
T(n,0,H) = \sum_{j=1}^{n} e^{-R_{t_{j-1}}} \quad \text{total holding cost in } [0, H].
\]
\[ T(n,0,H) = a \left( \left( \frac{b}{b-1} \right)^{1-b} - \left( \frac{b}{b-1} \right)^{b} \right) \sum_{j=1}^{n} (F_2(t_{j-1}, t_j))^{1-b} (F_1(t_{j-1}, t_j))^b \]

= total profit in [0, H], excluding the ordering cost, when there are \( n \) cycles in [0, H].

Clearly, \( Q(n) \) is an integer concave and decreasing function of \( n \).

Now by Bellman’s principle of optimality (1957) the maximum value \( T^*(n,0,H) \) of \( T(n,0,H) \) with respect to \( t \) is

\[ T^*(n,0,H) = \text{Max}_{t=0}^{H} [T^*(n-1,0,t) + T(1,t,H)] \quad (3.2.4) \]

where \( T(1,t,H) \) denotes the total profit in the cycle \([t, H]\), excluding the ordering cost.

Let \( t = H \). Then, we have \( T^*(n,0,H) > T^*(n-1,0,H) \), since the maximum in (3.2.4) occurs at an interior point. Hence, \( T^*(n,0,H) \) is strictly increasing in \( n \).

Recursive application of (3.2.4) yields the following relations:

\[ t_i^*(n,0,H) = t_i^*(n-j,0,t_{n-j}^*(n,0,H)), \quad i = 1,2, ..., n-j-1. \]

In order to prove that \( T^*(n,0,H) \) is strictly concave in \( n \), let us choose \( H_1, H_2 > H \) such that

\[ t_n^*(n+1,0,H_1) = t_n^*(n+2,0,H_2) = H. \]

Then, using Bellman’s principle of optimality, we have

\[ T^*(n+1,0,H_1) = \text{Max}_{t=0}^{H_1} [T^*(n,0,t) + T(1,t,H_1)] \]

\[ = T^*(n,0,H) + T(1,H_1), \]

and

\[ T^*(n+2,0,H_2) = \text{Max}_{t=0}^{H_2} [T^*(n+1,0,t) + T(1,t,H_2)] \]

\[ = T^*(n+1,0,H) + T(1,H_2). \]

Since \( H \) is an optimal interior point in \( T^*(n+1,0,H_1) \) and \( T^*(n+2,0,H_2) \), we get
\[
\frac{\partial T^*(n,0,t)}{\partial t} + \frac{\partial T(1,t,H_1)}{\partial t} \bigg|_{t=H} = 0 \quad \text{and} \quad \frac{\partial T^*(n+1,0,t)}{\partial t} + \frac{\partial T(1,t,H_2)}{\partial t} \bigg|_{t=H} = 0.
\]

Now,
\[
T(1,x,y) = ap^{-b} \left[ \left( e^{(br-R)x} - e^{(br-R)y} \right) \frac{R - br}{R - br} \right] + \frac{e^{(br-R)x} ((R - br)x + 1) - e^{(br-R)y} ((R - br)y + 1)}{(R - br)^2}
\]
\[
- \frac{p}{\theta + br} a \left( \frac{\theta}{\theta + br} \right) e^{-\frac{\theta}{\theta + br}} \left( e^{-\theta y} - e^{-\theta x} \right) - \frac{1}{R - br} \left( e^{(br-R)x} - e^{(br-R)y} \right)
\]
\[
+ \frac{ac_0}{\theta + br} \left[ e^{-\frac{\theta}{\theta + br}} \left( e^{-\theta y(M + x)} - e^{-\theta x} \right) \right]
\]
\[
\frac{\partial T(1,x,y)}{\partial x} = e^{my-x} \left\{ ap^{-b} \frac{\theta c_0 + h_0}{m} + ap^{-b} \frac{c_0 w}{m} + ap^{-b} I_p e^{-w} \right\}
\]
\[
- e^{ly} \left[ ap^{-b} \left( 1 + I_c \right) + ap^{-b} \frac{\theta c_0 + h_0}{m} + ap^{-b} \frac{c_0 l}{m} - ap^{-b} I_p e^{-l} \right].
\]

Hence,
\[
\frac{\partial T^*(n,0,t)}{\partial t} \bigg|_{t=H} = - \frac{\partial T(1,t,H_1)}{\partial t} \bigg|_{t=H}
\]
\[
e^{-h_{(x,0,H)}} \left\{ ap^{-b} \frac{\theta c_0 + h_0}{m} + ap^{-b} \frac{c_0 w}{m} + ap^{-b} I_p e^{-w} \right\}
\]
\[
- e^{-h_{(x,0,H)}} \left[ ap^{-b} \left( 1 + I_c \right) + ap^{-b} \frac{\theta c_0 + h_0}{m} + ap^{-b} \frac{c_0 l}{m} - ap^{-b} I_p e^{-l} \right].
\]

And
\[
\frac{\partial T^*(n+1,0,t)}{\partial t} \bigg|_{t=H} = - \frac{\partial T(1,t,H_2)}{\partial t} \bigg|_{t=H}
\]
\[
e^{-h_{(x,0,H)}} \left\{ ap^{-b} \frac{\theta c_0 + h_0}{m} + ap^{-b} \frac{c_0 w}{m} + ap^{-b} I_p e^{-w} \right\}
\]
\[
- e^{-h_{(x,0,H)}} \left[ ap^{-b} \left( 1 + I_c \right) + ap^{-b} \frac{\theta c_0 + h_0}{m} + ap^{-b} \frac{c_0 l}{m} - ap^{-b} I_p e^{-l} \right].
\]
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\[
= e^{aH - w/(n+1,0,H)} \left\{ ap^{-1b} \frac{\partial e_0 + h_0}{m} + ap^{-1b} \frac{c_0 w}{m} + ap^{-1b} I_p e^{-m M} \right\} \\
- e^{b/(n+1,0,H)} \left\{ ap^{-1b} (1 + I e_l) + ap^{-1b} \frac{\partial e_0 + h_0}{m} + ap^{-1b} \frac{c_0 l}{m} - ap^{-1b} I_p e^{m M} \right\},
\]

(3.2.6)

Subtracting (3.2.6) from (3.2.5) we get

\[
\frac{\partial}{\partial t} \left( T^*(n,0,t) - T^*(n+1,0,t) \right)_{t = H} = D \left( T^*(n+1,0,H), H \right) - D \left( T^*(n,0,H), H \right) < 0,
\]

where

\[
D(t, H) = e^{aH - wt} \left\{ ap^{-1b} \frac{\partial e_0 + h_0}{m} + ap^{-1b} \frac{c_0 w}{m} + ap^{-1b} I_p e^{-m M} \right\} \\
- e^{b/(n+1,0,H)} \left\{ ap^{-1b} (1 + I e_l) + ap^{-1b} \frac{\partial e_0 + h_0}{m} + ap^{-1b} \frac{c_0 l}{m} - ap^{-1b} I_p e^{m M} \right\}.
\]

Clearly, \( D(t, H) \) is a decreasing function of \( t \) for \( t < H \), since

\[
\frac{\partial D(t, H)}{\partial t} = -we^{aH - wt} \left\{ ap^{-1b} \frac{\partial e_0 + h_0}{m} + ap^{-1b} \frac{c_0 w}{m} + ap^{-1b} I_p e^{-m M} \right\} \\
- le^{b} \left\{ ap^{-1b} (1 + I e_l) + ap^{-1b} \frac{\partial e_0 + h_0}{m} + ap^{-1b} \frac{c_0 l}{m} - ap^{-1b} I_p e^{m M} \right\} < 0.
\]

Hence, \( T^*(n,0,H) - T^*(n+1,0,H) \) is a strictly decreasing function in \( H \) so that

\[
T^*(n,0,H) - T^*(n+1,0,H) > T^*(n,0,H_1) - T^*(n+1,0,H_1)
\]

(3.2.7)

Again, from (3.2.4),

\[
T^*(n,0,H_1) - T^*(n+1,0,H_1) = \max_{t \in [0,H]} \left\{ T^*(n-1,0,t) + T^*(1,t,H_1) \right\} - T^*(n,0,H) - T^*(1,H,H_1).
\]

Let \( t = H \). Then,

\[
T^*(n,0,H_1) - T^*(n+1,0,H_1) > T^*(n-1,0,t) - T^*(n,0,H)
\]

(3.2.8)

From (3.2.7) and (3.2.8)
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\( T^*(n,0,H) - T^*(n+1,0,H) > T^*(n-1,0,H) - T^*(n,0,H) \), which implies that \( T^*(n,0,H) \) is integer concave in \( n \), and hence \( Z(n) \) is also integer concave in \( n \).

For the case \( t_{j-1} + M \geq t_j \) for all \( j = 1, 2, \ldots, n \), i.e. \( \delta = 0 \) for all \( j = 1, 2, \ldots, n \), it can be similarly shown that \( Z(n) \) is integer concave in \( n \).

**Theorem 3.2.2:** For \( br - R \geq 0 \), and \( e^{(br-R)M} \leq \theta + br \), optimal reorder intervals satisfy \( T_1 > T_2 > \ldots > T_n \).

**Proof:** We have,

\[
Z(n, \{ \{ p_j \} \}) = a \sum_{j=1}^{n} p_j^{1-b} \left[ e^{(br-R)k_{j-1}} - e^{(br-R)k_j} \right] \left( \frac{1}{R - br} \right) + I_e \left( e^{(br-R)k_{j-1}} \left( (R - br)k_{j-1} + 1 \right) - e^{(br-R)k_j} \left( (R - br)k_j + 1 \right) \right) \left( \frac{1}{(R - br)^2} \right) \\
- A_0 \sum_{j=1}^{n} e^{-RT_j} \left[ \sum_{j=1}^{n} p_j \left( \frac{e^{(\theta+br)k_j}}{\theta + br} \left( e^{(\theta+R)k_{j-1}} - e^{(\theta+R)k_j} \right) - \frac{1}{R - br} \left( e^{(br-R)k_{j-1}} - e^{(br-R)k_j} \right) \right) \right] \\
+ \frac{ac_0}{\theta + br} \left( e^{(\theta+br)k_j} - 1 \right) \left( e^{-RT_j} + I_p \left( e^{(\theta+br)k_j} \left[ e^{(\theta+R)(M+t_{j-1})} - e^{(\theta+R)k_j} \right) \right) \right) \\
- \frac{1}{R - br} \left( e^{(br-R)(M+t_{j-1})} - e^{(br-R)k_j} \right) \right].
\]

Let, \( m = \theta + br \) \( w = R + \theta \) \( l = br - R \).

Then,
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\[ Z_1(n, \{t_j\}, \{p_j\}) = d \sum_{j=1}^{n} p_j^{1-b} \left[ - \left( \frac{e^{l_{t_j} - l_{t_j - 1}}}{l} - I_e \frac{e^{l_{t_j}} (l_{t_j - 1} - 1) - e^{l_{t_j}} (l_{t_j - 1})}{l^2} \right) - A_0 \sum_{j=1}^{n} e^{-R_{t_j}} \right] \]

\[ -a \sum_{j=1}^{n} p_j^{1-b} \left[ \frac{(h_0 + h_0)}{m} \left( \frac{e^{m_{t_j}}}{w} (e^{-w_{t_j - 1} - e^{-w_{t_j}}}) + \frac{1}{l} (e^{l_{t_j} - e^{l_{t_j}}}) \right) \right] \]

\[ + c_0 \left\{ e^{m_{t_j} - w_{t_j - 1} - e^{l_{t_j}}} + I_p \left[ \frac{e^{m_{t_j}}}{w} (e^{-w_{t_j} + e^{l_{t_j}}}) + \frac{1}{l} (e^{l_{t_j} - e^{l_{t_j}}}) \right] \right\} \]

\[ \frac{\partial Z_1(n, \{t_j\})}{\partial t_j} = 0 \]

\[ \Rightarrow p_j^{1-b} \left[ e^{l_{t_j}} + I_e l_{t_j} l_{t_j - 1} \right] - p_j^{1-b} \left( \frac{(h_0 + h_0)}{m} \left( \frac{e^{m_{t_j}}}{w} (e^{-w_{t_j} - e^{-w_{t_j}}}) + e^{m_{t_j}} e^{-w_{t_j} - e^{l_{t_j}}} \right) \right] \]

\[ + c_0 \left\{ \frac{e^{m_{t_j} - w_{t_j - 1}} + I_p \left[ \frac{e^{m_{t_j}}}{w} (e^{-w_{t_j} + e^{l_{t_j}}}) + e^{m_{t_j}} e^{-w_{t_j} - e^{l_{t_j}} \right]} \right\} \]

\[ + p_{j+1}^{1-b} \left[ e^{l_{t_j}} - I_e l_{t_j} l_{t_j} \right] - p_{j+1}^{1-b} \left( \frac{(h_0 + h_0)}{m} \left( e^{-w_{t_j} + e^{l_{t_j}}} \right) \right] \]

\[ + c_0 \left\{ \frac{e^{m_{t_j} - w_{t_j - 1} - e^{l_{t_j}}} + I_p \left[ e^{-w_{t_j} + e^{l_{t_j}}} + e^{l_{t_j} + e^{l_{t_j}}} \right]} \right\} = 0 \]

\[ \Rightarrow e^{l_{t_j}} p_j^{1-b} \left[ 1 + I_e l_{t_j} l_{t_j - 1} \right] - p_j^{1-b} \left( \frac{(h_0 + h_0)}{m} \left( \frac{e^{m_{t_j}}}{w} (e^{-w_{t_j} - e^{-w_{t_j}}}) + e^{m_{t_j}} e^{-w_{t_j} - e^{l_{t_j}}} \right) \right] \]

\[ + c_0 \left\{ \frac{e^{m_{t_j} - w_{t_j - 1}} + I_p \left[ \frac{e^{m_{t_j}}}{w} (e^{-w_{t_j} + e^{l_{t_j}}}) + e^{m_{t_j}} e^{-w_{t_j} - e^{l_{t_j}}} \right]} \right\} \]

\[ - e^{l_{t_j}} p_{j+1}^{1-b} \left[ 1 + I_e l_{t_j} l_{t_j} \right] - p_{j+1}^{1-b} \left( \frac{(h_0 + h_0)}{m} \left( e^{-w_{t_j} + e^{l_{t_j}}} \right) \right] \]

\[ + c_0 \left\{ \frac{e^{m_{t_j} - w_{t_j - 1} - e^{l_{t_j}}} + I_p \left[ e^{-w_{t_j} + e^{l_{t_j}}} + e^{l_{t_j} + e^{l_{t_j}}} \right]} \right\} = 0 \]

\[ \Rightarrow p_j^{1-b} \left[ 1 + I_e l_{t_j} l_{t_j} \right] - p_j^{1-b} \left[ \frac{(h_0 + h_0)}{m} \left( \frac{e^{m_{t_j} - w_{t_j - 1} - 1}}{w} \right) \right] \]

\[ + c_0 \left\{ \frac{e^{m_{t_j} - w_{t_j - 1} - 1} + I_p \left[ \frac{e^{m_{t_j} - w_{t_j} + e^{l_{t_j}}}}{w} (e^{-w_{t_j} + e^{l_{t_j}}} - 1) \right]} \right\} \]

\[ - p_{j+1}^{1-b} \left[ 1 + I_e l_{t_j} l_{t_j} \right] - p_{j+1}^{1-b} \left[ \frac{(h_0 + h_0)}{m} \left( e^{m_{t_j} - m_{t_j} + 1} \right) \right] \]

\[ + c_0 \left\{ \frac{e^{m_{t_j} - m_{t_j} - e^{l_{t_j}} - 1} + I_p \left[ e^{-w_{t_j} + e^{l_{t_j} + e^{l_{t_j}}} + e^{l_{t_j} + e^{l_{t_j}}}} \right]} \right\} = 0 \]
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\[ \Rightarrow -e^{wT_j} \left( p_j^{-b} \left( \frac{(\theta_0 + h_0)}{w} + c_0 \left( 1 + \frac{I_p}{w} e^{-wM} \right) \right) \right) + e^{mT_{j+1}} \left( p_{j+1}^{-b} \left( \frac{(\theta_0 + h_0)}{m} + \frac{w c_0}{m} \left( 1 + I_p e^{-wM} \right) \right) \right) + \left( 1 + I_e T_j \right) \left( p_j^{1-b} - p_{j+1}^{1-b} \right) + p_j^{-b} \left( \frac{(\theta_0 + h_0)}{w} + c_0 I_p \right) - p_{j+1}^{-b} \left( \frac{(\theta_0 + h_0)}{m} + \frac{c_0}{m} \left( -l + I_p e^{lM} \right) \right) = 0 \]

\[ \Rightarrow A_1 e^{mT_{j+1}} + B_1 t_j + C_1 = A_2 e^{wT_j} + B_2 t_j + C_2, \]

where

\[ T_{j+1} = t_{j+1} - t_j \quad T_j = t_j - t_{j-1} \]

\[ A_1 = \left( p_j^{-b} \left( \frac{(\theta_0 + h_0)}{m} + \frac{w c_0}{m} \left( 1 + I_p e^{-wM} \right) \right) \right), \]

\[ A_2 = \left( p_j^{-b} \left( \frac{(\theta_0 + h_0)}{w} + c_0 \left( 1 + \frac{I_p}{w} e^{-wM} \right) \right) \right) \]

\[ B_1 = I_e p_j^{1-b}, \quad B_2 = I_e p_{j+1}^{1-b} \]

\[ C_1 = p_j^{1-b} + p_j^{-b} \left( \frac{(\theta_0 + h_0)}{w} + c_0 I_p \right), \quad C_2 = p_{j+1}^{-b} \left( \frac{(\theta_0 + h_0)}{m} + \frac{c_0}{m} \left( -l + I_p e^{lM} \right) \right) + p_{j+1}^{1-b}. \]

Now, since \( p_{j+1} \geq p_j \) and \( b > 1 \), we have \( B_1 - B_2 = I_e \left( p_j^{1-b} - p_{j+1}^{1-b} \right) \geq 0. \)

Again, since \( m \geq w \), we get \( \frac{(\theta_0 + h_0)}{w} \geq \frac{(\theta_0 + h_0)}{m} \).

And since \( p_{j+1} \geq p_j \) and \( e^{lM} \leq m \), we get \( p_j^{-b} - \frac{e^{lM}}{m} p_{j+1}^{-b} \geq 0 \), so that

\[ c_0 I_p p_j^{-b} + p_{j+1}^{-b} \left( \frac{c_0}{m} l - \frac{c_0}{m} I_p e^{lM} \right) p_{j+1}^{1-b} = p_{j+1}^{-b} \left( \frac{c_0}{m} l + c_0 I_p \right) p_{j+1}^{-b} \geq 0. \]

Hence,

\[ C_1 - C_2 = p_j^{1-b} - p_{j+1}^{1-b} + p_j^{-b} \left( \frac{(\theta_0 + h_0)}{w} + c_0 I_p \right) - p_{j+1}^{-b} \left( \frac{(\theta_0 + h_0)}{m} + \frac{c_0}{m} \left( -l + I_p e^{lM} \right) \right) \geq 0. \]
Similarly

\[ A_1 - A_2 = p_j^{-b} \left( \frac{(\theta k_0 + h_0)}{w} + c_0 \left( 1 + \frac{I_p e^{-wM}}{w} \right) \right) - p_{j+1}^{-b} \left( \frac{(\theta k_0 + h_0)}{m} + \frac{wc_0}{m} \left( 1 + I_p e^{-wM} \right) \right) \geq 0. \]

Hence \( A_1 e^{mT_{j+1}} + (B_1 - B_2) T_j + C_1 - C_2 = A_2 e^{wT_j} \)

\[ \Rightarrow A_1 e^{mT_{j+1}} \leq A_2 e^{wT_j} \Rightarrow A_1 e^{mT_{j+1}} \leq A_2 e^{wT_j} \Rightarrow mT_{j+1} \leq wT_j \Rightarrow T_{j+1} \leq T_j. \]

The above theorem gives sufficient conditions to have \( T_1 > T_2 > \ldots > T_n \). The conditions are, however, not necessary as is evident from the example in Section 4.

### 3.2.4 Numerical Example

**Example 3.2.1:** Consider the following values of the model parameters: \( f = 0.15; r = 0.08; I_e = 0.12; I_p = 0.15; \theta = 0.02; a = 200000; b = 1.5; a_0 = 200; c = 20; M = 0.2 \text{ Yrs}; h = 1/\text{unit/yr}; H = 5 \text{ Yrs}; p = \text{Rs. 25 (initial selling price).} \)

In case 1, i.e. \( T_j \leq M \), \( \forall j \),

optimal no. of cycles = 9, and total profit = Rs.137166.74.

In case 2, i.e. \( T_j < M \), \( \forall j \),

Optimal no. of cycles = 19 and total profit = Rs.147594.90.

Hence, optimal value of \( n \) is 19, and the optimal values of \( t_j, T_j \) and selling price \( p_j \) are given in Table-3.2.1.

**Table-3.2.1:** Shows that the length of the replenishment cycle decreases while the unadjusted price increases with increase in the cycle number.

<table>
<thead>
<tr>
<th>Cycle No</th>
<th>( t_j )</th>
<th>( T_j = T_{j+1} - T_j )</th>
<th>( p_j )</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>25.00</td>
</tr>
<tr>
<td>1</td>
<td>0.1754</td>
<td>0.175376</td>
<td>25.35</td>
</tr>
<tr>
<td>2</td>
<td>0.3485</td>
<td>0.173111</td>
<td>25.71</td>
</tr>
<tr>
<td>Cycle No</td>
<td>$t_j$</td>
<td>$T_{j+1} - t_j$</td>
<td>$p_j$</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>----------------</td>
<td>-------</td>
</tr>
<tr>
<td>3</td>
<td>0.5194</td>
<td>0.170915</td>
<td>26.06</td>
</tr>
<tr>
<td>4</td>
<td>0.6882</td>
<td>0.168783</td>
<td>26.41</td>
</tr>
<tr>
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<td>0.166713</td>
<td>26.77</td>
</tr>
<tr>
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<td>1.0196</td>
<td>0.164702</td>
<td>27.12</td>
</tr>
<tr>
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<td>0.162746</td>
<td>27.48</td>
</tr>
<tr>
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<td>1.3432</td>
<td>0.160844</td>
<td>27.84</td>
</tr>
<tr>
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<td>1.5022</td>
<td>0.158993</td>
<td>28.19</td>
</tr>
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<td>28.55</td>
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<tr>
<td>19</td>
<td>3.0000</td>
<td>0.142836</td>
<td>31.78</td>
</tr>
</tbody>
</table>

### 3.2.5 Sensitivity Analysis

The following figures show the behavior of the maximum profit with change in the parameter values:
Figure 3.2.2: Optimum Profit with respect to the inflation rate

Figure 3.2.3: Optimum Profit with respect to the deterioration rate

Figure 3.2.4: Optimum Profit with respect to the discount rate
Figure 3.2.5: Optimum Profit with respect to the interest earning rate

Figure 3.2.6: Optimum Profit with respect to the holding cost
The above figures show the change in the maximum profit with change in (i) inflation rate (Figure 3.2.2), deterioration rate (Figure 3.2.3), discount rate (Figure 3.2.4), interest earned (Figure 3.2.5), holding cost (Figure 3.2.6), b (Figure 3.2.7), and permissible delay period (Figure 3.2.6).
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It is observed that while the maximum profit increases with increase in inflation rate, permissible delay period and interest earned, it decreases with increase in deterioration rate, price elasticity, discount rate and holding cost.

3.2.6 Discussion

The sub-section studies a dynamic inventory model for deteriorating items with price-dependent iso-elastic demand, which is observed in common items like paper towels, detergent, soft drinks, etc. Price inflation and time value of money are taken into consideration, and the supplier is assumed to allow the inventory manager a grace period to pay his dues. The model is investigated in a finite planning horizon and the replenishment intervals are allowed to vary. It has been shown that under certain sufficient conditions, the optimal lengths of the replenishment cycles decrease as one moves towards the end of the planning horizon.

3.3 An Ordering policy for Hybrid demand under permissible delay in payments and price inflation

3.3.1 Notations

The notations used are the same as in section 3.2.1, with the following additional notations

1. $c_b = \text{backlogging cost per unit per unit time}$
2. $c_l = \text{cost of lost sales per unit}$
3. $\delta(t) = \text{backlogging rate}$.

3.3.2 Assumptions

(a) Demand rate is hybrid, that is, it is a convex combination of linear demand and iso-elastic demand rates, and is given by

$$D(t) = \tau (a_1 - b_1 pe^{rt}) + (1 - \tau) a_2 (pe^{rt})^{b_2}$$

if $I(t) \geq 0$

$$\quad = \alpha$$

if $I(t) < 0$,

where $0 \leq \tau \leq 1, \quad b_2 \geq 1, \quad \alpha > 0$.

(b) Orders are placed at $t_1 = 0 < t_2 < \ldots < t_n$. 

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(c) Lead time is zero.

(d) Shortages are allowed.

(e) During shortage, the proportion of demand backordered is \( \delta(t) = e^{-\sigma t} \), where \( t \) is the waiting time and \( \sigma \geq 0 \).

(f) The stock on hand at \( t_1 \) before the order is placed is zero.

(f) No shortage is allowed in the last replenishment period \([t_n, H]\), which ensures that the total demand over the planning period \([0, H]\) is exactly met.

### 3.3.3 Mathematical Model

The planning period \([0, H]\) is divided into \( n \) reorder intervals, not necessarily of equal lengths, and orders are placed at the beginning of each interval. The rate of replenishment of inventory is assumed to be infinite.

The following figure shows the inventory situation over the planning period \([0, H]\):

**Figure 3.3.1:** Inventory Level over time Horizon

In the above figure, \( s_i \) denotes the time point at which the stock level comes down to zero in the \( i^{th} \) replenishment cycle \([t_i, t_{i+1})\), \( 1 \leq i \leq n \).

The \( i^{th} \) replenishment made at time \( t_i \) (\( 2 \leq i \leq n \)) is used partly to meet the accumulated backorder in the previous cycle. For the last cycle \( s_n = t_{n+1} = H \).
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During the interval \([t_i, s_i]\) the inventory is depleted by the combined effect of demand and deterioration, so that the inventory level \(I(t)\) at time \(t\) satisfies the differential equation

\[
\frac{dI(t)}{dt} + \Theta(t) = -D(t) \quad t_i \leq t \leq s_i \quad i = 1(1)n
\]

During the interval \([s_i, t_{i+1})\), 1 \(\leq i \leq n-1\), the demand rate is \(\alpha\) and the backlogging rate is \(\delta(t_{i+1} - t) = e^{-\sigma(t_{i+1} - t)}\).

Hence the amount of backorder \(B(t)\) at \(t\) satisfies

\[
\frac{dB(t)}{dt} = -\alpha e^{-\sigma(t_i - t)} \quad s_i \leq t \leq t_{i+1} \quad i = 1(1)(n-1).
\]

The boundary condition are \(I(s_i) = 0\) for \(1 \leq i \leq n\), and \(B(s_i) = 0\), \(1 \leq i \leq n-1\).

Hence, the solutions to the differential equations (3.1) and (3.2) are

\[
I(t) = \frac{a_1}{\theta} \left( e^{(s_i - t)\sigma} - 1 \right) + \frac{a_2}{b_2} \left( \frac{1 - r}{\theta - b_2 r} \right) e^{s_i (\theta - b_2 r) - \alpha - e^{-b_2 r t}} + \frac{p b_1}{(\theta + r)} e^{\theta t} - e^{s_i (\theta + r) - \alpha}, \quad t_i \leq t \leq s_i, \quad 1 \leq i \leq n,
\]

\[
B(t) = \frac{\alpha}{\sigma} \left( e^{-\sigma(t_{i+1} - t)} - e^{-\sigma(t_{i+1} - s_i)} \right) \quad s_i \leq t \leq t_{i+1}, \quad 1 \leq i \leq n-1.
\]

The order quantity at \(t_i\) is, therefore,

\[
Q_i = \begin{cases} 
I(t_i) & \text{for } i = 1 \\
I(t_i) + B(t_i) & \text{for } 2 \leq i \leq n,
\end{cases}
\]

and the amount of lost sales at time \(t\) in the interval \([s_i, t_{i+1})\) is,

\[
L(t) = \int_{s_i}^{t_{i+1}} \left( 1 - e^{-\sigma(t_{i+1} - u)} \right) du \quad s_i \leq t \leq t_{i+1}, \quad 1 \leq i \leq n-1.
\]

### 3.3.4 Profit Function

The present values of the various cost components in the \(i\)-th replenishment cycle, \(1 \leq i \leq n\), are as follows:
(a) Holding cost \( I_i = h \int_{t_i}^{t_{i+1}} e^{-Rt} I(t) dt \)

\[
= h \frac{a_1}{\theta} \left( \frac{e^{\theta(s_{i+1} - t_i)} - e^{-Rt_i}}{\theta + R} + \frac{e^{-Rt_i} - e^{-Rt_{i+1}}}{R} \right) + \frac{a_2}{b_2} \frac{(1 - \tau)}{\theta - b_2 r} \left( \frac{e^{(\theta - b_2 r)(t_{i+1} - t_i)} - e^{(R + b_2 r) t_i}}{\theta + R} + \frac{e^{(R + b_2 r) t_i} - e^{(R - b_2 r) t_i}}{b_2 r - R} \right) + \frac{\pi b_2 p}{(\theta + r)} \left( \frac{e^{(r - R) t_i} - e^{(r - R) t_{i+1}}}{r - R} + \frac{e^{(R - r) t_i} - e^{(R - r) t_{i+1}}}{r + R} \right) \]

(b) Deterioration cost \( Det_i = \alpha k \int_{t_i}^{t_{i+1}} e^{-Rt} I(t) dt \)

(c) Backlogging cost \( B_i = c_b \int_{t_i}^{t_{i+1}} e^{-Rt} B(t) dt = \frac{\alpha c_b e^{R(t_{i+1})}}{R} \left( \frac{e^{(R - \sigma)(t_{i+1} - t_i)}}{R - \sigma} - \frac{1 - e^{-\sigma(t_{i+1} - t_i)}}{\sigma} \right) \) for \( 1 \leq i \leq n - 1 \)

\( 0, \) for \( i = n. \)

(d) Lost sales cost

\[
L_i = \left\{ \begin{array}{l}
  c_i \int_{t_i}^{t_{i+1}} e^{-Rt} \left( 1 - e^{-\sigma(t_{i+1} - t)} \right) dt = \frac{\alpha c_i e^{R(t_{i+1})}}{R} \left( \frac{e^{(R - \sigma)(t_{i+1} - t_i)}}{R - \sigma} - \frac{1 - e^{-\sigma(t_{i+1} - t_i)}}{\sigma} \right) \quad \text{for} \quad 1 \leq i \leq n - 1 \\
  0, \quad \text{for} \quad i = n
\end{array} \right.
\]

(e) Purchase cost + ordering cost \( P_i = Ae^{-Rt_i} + ce^{-Rt_i} Q_i \)

On the other hand, the revenue collected during the \( n \)th cycle is

\[
R_i = p \left( e^{-Rt_i} B(t) + \int_{t_i}^{t_{i+1}} e^{-Rt} D(t) dt \right), \quad \text{where} \quad D(t) \text{is demand rate}
\]

Now, the interest earned/paid by the inventory manager owing to the permissible delay in payment depends on whether the delay period ends before or after the stock on hand comes down to zero in a replenishment cycle. Thus, we have to consider the two situations, viz.

(i) \( t_i + M \leq s_i \leq t_{i+1} \) and (ii) \( s_i \leq t_i + M \leq t_{i+1} \).

Case 1: \( t_i + M \leq s_i \leq t_{i+1} \).
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If the grace period given to the inventory manager to pay his dues ends when he still
has some stock on hand, he has to pay an interest on the unsold stock, while he can earn
interest from the revenue collected by selling the stock on hand.

The present values of the interests paid and earned in the \( i \)th replenishment cycle for case
1 are, therefore, respectively given by

\[
IP_1 = c_I \int_{t_i}^{t_i+M} e^{-Rt} I(t) \, dt
\]

\[
= c_I \left[ \frac{a_1 \tau}{\theta} \left( e^{\theta(t_i-t_i-M)-Rt_i} - e^{-Rt_i} + e^{-Rt_i} - e^{-(R+R)(t_i+M)} \right) \right. \\
+ \left. \frac{a_2 (1 - \tau)}{p \theta b_2} \left( e^{(\theta-\theta)b_2}(\theta+b_2)(t_i+M) - e^{-e^{(\theta+b_2)(t_i+M)}} + e^{(R-b_2)(t_i+M)} - e^{(R-b_2)(t_i+M)} \right) \right]
\]

\[
= \frac{a_1 \tau}{\theta} \left( e^{\theta(t_i-t_i-M)-Rt_i} - e^{-Rt_i} + e^{-Rt_i} - e^{-(R+R)(t_i+M)} \right) \\
+ \frac{a_2 (1 - \tau)}{p \theta b_2} \left( e^{(\theta-\theta)b_2}(\theta+b_2)(t_i+M) - e^{-e^{(\theta+b_2)(t_i+M)}} + e^{(R-b_2)(t_i+M)} - e^{(R-b_2)(t_i+M)} \right)
\]

\[
IE_1 = p l_e \int_{t_i}^{t_i+M} e^{-Rt} D(t) \, dt
\]

\[
= pl_e \left[ \frac{a_1 \tau}{R} (e^{-Rt_i} - e^{-Rt_i}) - \frac{\tau_1 p \theta}{r-R} (e^{(R-b_2)(t_i+M)} - e^{(R-b_2)(t_i+M)}) \right]
\]

Case 2: \( s_i \leq t_i \leq t_i+M \)

In this case, since the inventory manager sells the whole stock before the grace period
ends, he does not have to pay any interest but can earn interest on the revenue collected
from selling his goods. Thus, for the \( i \)th replenishment cycle, the present values of the
interests paid and earned for case 2 are respectively given by

\[
IP_2 = 0
\]

\[
IE_2 = pl_e \left[ \int_{t_i}^{t_i+M} e^{-Rt} D(t) \, dt + (M - (s_i - t_i)) \int_{t_i}^{t_i+M} e^{-Rt} D(t) \, dt \right]
\]

The present value of the total profit during the planning horizon thus comes out to be
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\[ TP(n, \{s_i\}, \{t_i\}) = \begin{cases} \sum_{i=1}^{n} (R_i + IE1_i - P_i - I_i - Det_i - B_i - L_i - IP1_i), & \text{if } t_i + M \leq s_i, 1 \leq i \leq n, \\ \sum_{i=1}^{n} (R_i + IE2_i - P_i - I_i - Det_i - B_i - L_i), & \text{if } t_i + M > s_i, 1 \leq i \leq n, \end{cases} \]

with \(0 = t_1 < s_1 < t_2 < s_2 < \ldots < t_{n-1} < s_{n-1} < t_n < s_n = t_{n+1} = H.\)

The problem is to find the optimal values of \(n, t_i, s_i\) such that \(TP(n, \{s_i\}, \{t_i\})\) is maximized. To do so, we need to solve the following optimization problems:

(a) Maximize \(TP_1(n, \{s_i\}, \{t_i\}) = \sum_{i=1}^{n} (R_i + IE1_i - P_i - I_i - Det_i - B_i - L_i - IP1_i)\)

subject to \(0 = t_1 < s_1 < t_2 < s_2 < \ldots < t_{n-1} < s_{n-1} < t_n < s_n = t_{n+1} = H,\)

\(t_i + M \leq s_i \text{ } i = 1(1)n.\)

(b) Maximize \(TP_2(n, \{s_i\}, \{t_i\}) = \sum_{i=1}^{n} (R_i + IE2_i - P_i - I_i - Det_i - B_i - L_i)\)

subject to \(0 = t_1 < s_1 < t_2 < s_2 < \ldots < t_{n-1} < s_{n-1} < t_n < s_n = t_{n+1} = H,\)

\(t_i + M \geq s_i \text{ } i = 1(1).\)

The maximum values of \(TP_1\) and \(TP_2\) are then compared to get the optimal solution.

3.3.5 Results and Solution

For a fixed value of \(n\), the necessary conditions for \(TP_j(n, \{s_i\}, \{t_i\})\) to be maximized are

\[
\frac{\partial TP_j(n, \{s_i\}, \{t_i\})}{\partial s_i} = 0 \text{ for } i = 1, 2, \ldots, n-1 \quad \text{and} \quad \frac{\partial TP_j(n, \{s_i\}, \{t_i\})}{\partial t_i} = 0 \text{ for } i = 2, 3, \ldots, n, \ j = 1, 2,
\]

which give the following equations to be satisfied by the optimum decision variables:

(a) for \(j = 1\), from Case 1 of Section 4, we get
\[ p ae^{-Rt_1} e^{-\sigma (t_{i+1} - s_i)} - [(ae^{-Rt_1} e^{-\sigma (t_{i+1} - s_i)} + \frac{ac_1 e^{-Rt_1}}{R} (e^{(R-\sigma)(t_{i+1} - s_i)} - e^{-\sigma (t_{i+1} - s_i)})] + \frac{ac_2 e^{-Rt_1}}{R} (e^{R(t_{i+1} - s_i)} - e^{(R-\sigma)(t_{i+1} - s_i)}) \]
\[ + pe^{-Rt_1} D(s_i) (1 + I_e) - ce^{-Rt_1} e^{(s_i - t_i)} D(s_i) \]
\[ - (h + c\theta) \frac{a_1 \tau}{\theta + R} \left( e^{\theta (t_{i+1} - t_i) - R t_i} - e^{-R t_i} \right) + \frac{a_2 (1 - \tau)}{p} \left( \frac{\theta - b_2 r}{\theta - b_2 r} \right) \left( e^{\theta (t_{i+1} - t_i) - R t_i} - e^{-R t_i} \right) \]
\[ + \frac{\theta (\theta - b_2 r)}{\theta + R} \left( e^{(\theta - b_2 r)(t_{i+1} - t_i)} - e^{(\theta - b_2 r) R t_i} \right) \]
\[ = 0 \quad (3.3.1) \]

and

\[ (h + c\theta) e^{-Rt_1} I(t_1) + cl e^{-Rt_1} I(M + t_i) + ce^{-Rt_1} I(t_1) + c Re^{-Rt_1} I(t_1) \]
\[ = ce^{-Rt_1} \left[ \alpha e^{-Rt_1} \frac{(1 - e^{-\sigma (t_{i+1} - s_i)})}{\sigma} + \frac{ac_1 e^{-Rt_1}}{R} \left( e^{(R-\sigma)(t_{i+1} - s_i)} - e^{-\sigma (t_{i+1} - s_i)} \right) \right] \]
\[ - \frac{ac_2 e^{-Rt_1}}{R - \sigma} \left[ \frac{e^{(R-\sigma)(t_{i+1} - s_i) - 1}}{\sigma} - \frac{e^{-\sigma (t_{i+1} - s_i) - 1}}{\sigma} \right] + \frac{ac_2 e^{-Rt_1}}{R} \left( e^{R(t_{i+1} - t_i)} - e^{(R-\sigma)(t_{i+1} - s_i)} \right) \]
\[ - \frac{ac_2 e^{-Rt_1}}{R} \left( e^{R(t_{i+1} - s_i) - 1} \right) \left( \frac{\alpha (e^{-Rt_1} - (R + \sigma) e^{-\sigma (t_{i+1} - s_i) - R t_i})}{\sigma} + e^{-Rt_1} D(t_1) \right) \]
\[ + pI e^{-Rt_1} D(t_1) - Ae^{-Rt_1} \quad (3.3.2) \]

The left hand side of (3.2.2) is positive. Hence, the problem has a solution only if the right hand side is also positive. We shall, therefore, assume that the right hand side of (3.2.2) is also positive.

Let us write the right hand side as...
\[ RH^{(1)}(s_{j-1}, t_j) = e^{-Rt_j} \left( c\alpha + cR - \frac{ac_b - ac_c}{R - \sigma} - \frac{p\alpha(R + \sigma)}{\sigma} \right) e^{-\sigma(t_j - s_{j-1})} \]
\[ + \left( \frac{ac_b}{R} - \frac{ac_c}{R - \sigma} - \frac{ac_i}{R} \right) e^{(R - \sigma)(t_j - s_{j-1})} \]
\[ + \left( p \left( \frac{\alpha R}{\sigma} + D(t_j) \right) + pI_e D(t_j) + ac_b \left( \frac{1}{R - \sigma} + \frac{1}{\sigma} \right) - A - cR \frac{\alpha}{\sigma} + ac_i \left( \frac{1}{R} - \frac{1}{R - \sigma} \right) \right). \]

We assume that \( RH^{(1)}(s_{j-1}, t_j) \) is positive.

(b) For \( j = 2 \), from Case 2 of Section 3.3.4, we get
\[
\begin{align*}
\frac{pae^{-Rt_i} e^{-\sigma(t_{i+1} - s_i)}}{h + c \theta} & - \left( a_1 e^{-Rt_i} e^{-\sigma(t_{i+1} - s_i)} + \frac{ac_b e^{-Rt_i}}{R} \right) \\
& + \left( ac_i e^{-Rt_i} e^{-\sigma(t_{i+1} - s_i)} \right) + pe^{-Rt_i} \left( 1 + I_e \right) - ce^{-Rt_i} e^{(\theta - b_1)\sigma} D(s_i) \\
& - \left( h + c \theta \right) \frac{a_1}{\theta + R} \left( e^{(\theta - b_1)\sigma} - e^{-Rt_i} \right) + \frac{a_2}{\theta + R} \frac{(\theta - b_2)\sigma}{p b_2} \left( e^{(\theta - b_2)\sigma} - e^{-Rt_i} \right) \\
& - \frac{e^{(\theta - b_2)\sigma}}{\theta + R} + \frac{\theta}{\theta + R} \left( e^{(\theta - b_2)\sigma} - e^{-Rt_i} \right) + pI_e (M - s_i + t_j) \left( -a_1 e^{-Rt_i} - \frac{a_2}{\theta + R} \frac{(\theta - b_2)\sigma}{p b_2} \left( e^{(\theta - b_2)\sigma} - e^{-Rt_i} \right) \right) \right) = 0
\end{align*}
\]

Here, again, the left hand side of (3.3.4) is always positive. Hence, to get a solution to (3.3.4) we assume that the right hand side of (3.3.4) is also positive.

For given \( n \), let us write
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\[ TP_j(n, \{s_i\}, \{t_i\}) = K(n) + T_j(n, 0, H), j = 1, 2, \]

where
\[
K(n) = -\sum_{i=1}^{n} Ae^{-rt_i}
\]

\[
T_j(n, 0, H) = \sum_{i=1}^{n} \left( R_i + IE_{1_i} - P_i - I_i - Det_i - B_i - L_i - IP_{1_i} \right).
\]

Let \( T_j^*(n, 0, H) \) be the optimum (maximum) value of \( T_j(n, 0, H) \).

We have the following results:

**Theorem 3.3.1:** \( T_j^*(n, 0, H) \) is concave in \( n \), for \( j = 1, 2 \).

**Proof:** Following the Bellman’s principle of optimality (1957),

\[
T_j^*(n, 0, H) = \max_{t \in [0, H]} \left[ T_j^*(n-1, 0, t) + T_j(1, t, H) \right] \tag{3.3.5}
\]

Now, for \( t \leq H \), \( T_j(1, t, H) \geq 0 \), where \( T_j(1, t, H) \) is the total revenue in a cycle, and is non-negative for \( j = 1, 2 \).

Recursive applications of (3.3.5) yields the following relations between optimal values of \( t_i \)'s and \( s_i \)'s for \( j = 1, 2 \)

\[
t_i^*(n, 0, H) = t_i^*(n-i_0, 0, t_{n-i_0}^*(n, 0, H)), \quad i = 1, 2, ..., n - i - 1.
\]

\[
s_i^*(n, 0, H) = s_i^*(n-i_0, 0, s_{n-i_0}^*(n, 0, H)), \quad i = 1, 2, ..., n - i - 1.
\]

In order to prove that \( T_j^*(n, 0, H) \) is strictly concave in \( n \), let us choose \( H_1, H_2 > H \) such that

\[
s_{n, j}^*(n + 1, 0, H_1) = s_{n, j}^*(n + 2, 0, H_2) = H.
\]

Then, using Bellman’s principle of optimality, we have
\[ T_j^*(n+1, 0, H_1) = \text{Max } \{ T_j^*(n,0,t) + T_j(1,t,H_1) \} \]

\[ = T_j^*(n,0,H) + T_j(1,H,H_1), \quad (3.3.6) \]

and

\[ T_j^*(n+2,0,H_2) = \text{Max } \{ T_j^*(n+1,0,t) + T_j(1,t,H_2) \} \]

\[ = T_j^*(n+1,0,H) + T_j(1,H,H_2). \quad (3.3.7) \]

Since \( H \) is an optimal interior point in \( T_j^*(n+1,0,H_1) \) and \( T_j^*(n+2,0,H_2) \), we get

\[ \left[ \frac{\partial T_j^*(n,0,t)}{\partial t} + \frac{\partial T_j(1,t,H_1)}{\partial t} \right]_{t=H} = 0 \quad \text{and} \]

\[ \left[ \frac{\partial T_j^*(n+1,0,t)}{\partial t} + \frac{\partial T_j(1,t,H_2)}{\partial t} \right]_{t=H} = 0. \quad (3.3.8) \]

First let us consider \( j = 1 \). If \( a \) and \( b \) be two consecutive time points at which the inventory level drops to zero and \( z \) be the replenishment point in \((a, b)\), then we get

\[ T_1(1,a,b) = \alpha e^{-Rz-z-a} - p \alpha e^{-Rz-z-a} - \frac{\alpha c \ e^{-Rz}}{R} \left( e^{(R-z-a)H} + e^{-(R-z-a)H} \right) \]

\[ = \alpha c e^{-Rz-z-a} - \frac{\alpha c \ e^{-Rz}}{R} \left( e^{(R-z-a)H} + e^{-(R-z-a)H} \right) \]

From (3.3.8), we obtain

\[ \left. \frac{\partial T_1^*(n,0,t)}{\partial t} \right|_{t=H} = \left. \frac{\partial T_1(1,1,H_1)}{\partial t} \right|_{t=H} \]

\[ = pae^{-R_{n+1}e^{(n,0,H_1)}} e^{-\sigma_{n+1}e^{(n,0,H_1)-H}} - \left[ \alpha e^{-R_{n+1}e^{(n,0,H_1)}} e^{-\sigma_{n+1}e^{(n,0,H_1)-H}} \right] \]

\[ + \frac{\alpha c e^{-R_{n+1}e^{(n,0,H_1)}}}{R} \left[ e^{(R-z-a)H} - e^{-(R-z-a)H} \right] + \frac{\alpha c e^{-R_{n+1}e^{(n,0,H_1)}}}{R} \left[ e^{(R-z-a)H} - e^{-(R-z-a)H} \right] \]

\[ \times \left( e^{(R-z-a)H} - e^{-(R-z-a)H} \right) \]

\[ = \left( b \right) \left( \frac{a_1 \tau}{\theta + R} \right) \left( e^{R^{(n,0,H_1)-H}} - e^{-(R-z-a)H} \right) \]

\[ + \frac{a_2(1-b)}{p} \left( \frac{e^{-R_{n+1}e^{(n,0,H_1)}H}}{\theta - b_2H} \right) \]
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\[-e^{(R-b_2 r)H} + \frac{\tilde{b}_1 p}{\theta + R} \left( e^{(R-R)H} - e^{(\theta+b_1 r)Rt_{n+1}^*(n,0,H)} \right) \right] - pe^{-RH} D(H)(1 + I_e) \\
+ ce^{-R_{n+1}^*(n,0,H)} e^{(H-t_{n+1}^*(n,0,H))} D(H) \\
+ cl_r \left[ \frac{a_1 \tau}{\theta + R} \left( e^{(H-t_{n+1}^*(n,0,H)-M)} - MR-Rt_{n+1}^*(n,0,H) - e^{-RH} \right) \right] + \frac{a_2 (1 - \tau)}{p \frac{b_2}{(\theta - b_2 r)} \left( \frac{H - t_{n+1}^*(n+1,0,H) + e^{-RH} \right)} \\
\times e^{H(\theta-b_2 r) - (\theta+b_2 r)H} - e^{(R+b_2 r)H} \\
+ \frac{\tilde{b}_1 p}{\theta + R} \left( e^{(R-R)H} - e^{(\theta+b_1 r)H} \right) \right] \right], (3.3.9) \\

where \(t_{n+1}^*(n,0,H)\) and \(t_{n+1}^*(n+1,0,H_1)\) are the corresponding last replenishment time when \(n\) orders are placed in \([0,H]\), and \(n+1\) orders are placed in \([0,H_1]\) respectively.

Again, from (3.3.8),

\[
\frac{\partial T_1^*(n+1,0,1)}{\partial t} \bigg|_{t=H} = - \frac{\partial T_1^* (1,1,2)}{\partial t} \bigg|_{t=H} \\
= (h + c \theta) \frac{a_1 \tau}{\theta + R} \left( e^{(H-t_{n+1}^*(n+1,0,H)) - R_{n+1}^*(n+1,0,H) - e^{-RH} \right) \\
+ \frac{a_2 (1 - \tau)}{p \frac{b_2}{(\theta - b_2 r)} \left( \frac{H - t_{n+1}^*(n+1,0,H) + e^{-RH} \right)} \\
\times e^{H(\theta-b_2 r) - (\theta+b_2 r)H} - e^{(R+b_2 r)H} \\
+ \frac{\tilde{b}_1 p}{\theta + R} \left( e^{(R-R)H} - e^{(\theta+b_1 r)H} \right) \right] - pe^{-RH} D(H)(1 + I_e) \\
+ ce^{-R_{n+1}^*(n+1,0,H)} e^{(H-t_{n+1}^*(n+1,0,H))} D(H) \\
+ cl_r \left[ \frac{a_1 \tau}{\theta + R} \left( e^{(H-t_{n+1}^*(n+1,0,H)-M)} - MR-Rt_{n+1}^*(n+1,0,H) - e^{-RH} \right) \right] + \frac{a_2 (1 - \tau)}{p \frac{b_2}{(\theta - b_2 r)} \left( \frac{H - t_{n+1}^*(n+1,0,H) + e^{-RH} \right)} \\
\times e^{H(\theta-b_2 r) - (\theta+b_2 r)H} - e^{(R+b_2 r)H} \\
+ \frac{\tilde{b}_1 p}{\theta + R} \left( e^{(R-R)H} - e^{(\theta+b_1 r)H} \right) \right] \right], (3.3.10) \\

Now, subtracting (3.3.9) from (3.3.10) we get
\[
\frac{\partial}{\partial t} \left( R^*_1(n,0,t) - T^*_1(n+1,0,t) \right)_{t=H} = R^*_1(n+1,0,H) - R^*_1(n,0,H) < 0,
\]
(3.3.11)

Where

\[
R(t, H) = (h + c \theta) \frac{a_1 \tau}{\theta + R} \left( e^{\theta(H-t) - R t} - e^{-R H} \right) + \frac{a_2(1-\tau)}{p b^2} \left( \frac{\theta - b_2 r}{\theta} \right) \left( \frac{e^{H(\theta-b_2 r) - (\theta+R) t} + (R + b_2 r) e^{-(R+b_2 r)H}}{} - e^{(R-b_2 r)H} \right) + \frac{\tau b_1 p}{\theta + R} \left( e^{(r-R)H} - e^{(\theta+R)(\theta+R)H} \right),
\]

and it is a decreasing function of \( t \) for all \( t < H \), since \( \frac{\partial R(t,H)}{\partial t} < 0 \).

Hence, \( T^*_1(n,0,H) - T^*_1(n+1,0,H) \) is a strictly decreasing function of \( H \), so that

\[
T^*_1(n,0,H) - T^*_1(n+1,0,H) > T^*_1(n,0,H_1) - T^*_1(n+1,0,H_1)
\]
(3.3.12)

Again, from (3.3.5) and (3.3.6), we have

\[
T^*_1(n,0,H_1) - T^*_1(n+1,0,H_1) = \max_{t=0,H_1} \left( T^*_1(n-1,0,t) + T(1,t,H_1) \right) - T^*_1(n,0,H) - T(1,H,H_1).
\]

Let \( t=H \). Then,

\[
T^*_1(n,0,H) - T^*_1(n+1,0,H) > T^*_1(n-1,0,H) - T^*_1(n,0,H)
\]
(3.3.13)

From (3.3.12) and (3.3.13),

\[
T^*_1(n,0,H) - T^*_1(n+1,0,H) > T^*_1(n-1,0,H) - T^*_1(n,0,H).
\]
which implies that $T^*_1(n, 0, H)$ is integer concave in $n$, and hence $TP_1(n)$ is also integer concave in $n$. For the case $t_i + M \geq s_i$, $i = 1(1)n$ it can be similarly shown that $TP_2(n)$ is integer concave in $n$.

### 3.3.6 Numerical Example

**Example 3.3.1:** Consider the following costs and parameter values –

\[
\begin{align*}
A &= \text{Rs. 250}; \quad c = \text{Rs. 40}; \quad p = \text{Rs. 50}; \quad h = \text{Rs. 5}; \quad c_b = \text{Rs. 3}; \quad c_l = \text{Rs. 7}; \quad I_c = 0.12; \quad I_r = 0.15; \quad r = 0.07; \quad d = 0.1; \quad H = 7 \text{ years}; \quad M = 0.45 \text{ year}; \quad \theta = 0.1; \quad \sigma = 0.08; \quad a_1 = 20000; \quad a_2 = 500000; \quad b_1 = 0.2; \quad b_2 = 1.5; \quad \tau = 0.7; \quad \alpha = 6000.
\end{align*}
\]

The optimal number of replenishment cycles comes out to be 17 and the maximum total profit is Rs 15380464/-.

The optimum reorder points ($t_i$), reorder interval lengths ($t_{i+1} - t_i$), time points at which the stock on hand is exhausted ($s_i$) and the order quantities ($Q_i$) are shown in Table 3.3.1.

**Table 3.3.1:** The optimal replenishment policy of example 3.3.1

<table>
<thead>
<tr>
<th>Cycle No</th>
<th>$t_i$</th>
<th>$s_i$</th>
<th>$s_i - t_i$</th>
<th>$t_{i+1} - s_i$</th>
<th>$t_{i+1} - t_i$</th>
<th>$Q_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.4126</td>
<td>0.4126</td>
<td>0.0187</td>
<td>0.4313</td>
<td>660718</td>
</tr>
<tr>
<td>2</td>
<td>0.4313</td>
<td>0.8248</td>
<td>0.3935</td>
<td>0.0190</td>
<td>0.4124</td>
<td>657891</td>
</tr>
<tr>
<td>3</td>
<td>0.8438</td>
<td>1.2364</td>
<td>0.3926</td>
<td>0.0193</td>
<td>0.4119</td>
<td>657691</td>
</tr>
<tr>
<td>4</td>
<td>1.2557</td>
<td>1.6471</td>
<td>0.3914</td>
<td>0.0197</td>
<td>0.4112</td>
<td>657449</td>
</tr>
<tr>
<td>5</td>
<td>1.6669</td>
<td>2.0570</td>
<td>0.3902</td>
<td>0.0202</td>
<td>0.4103</td>
<td>657197</td>
</tr>
<tr>
<td>6</td>
<td>2.0772</td>
<td>2.4662</td>
<td>0.3890</td>
<td>0.0205</td>
<td>0.4095</td>
<td>656977</td>
</tr>
<tr>
<td>7</td>
<td>2.4867</td>
<td>2.8747</td>
<td>0.3880</td>
<td>0.0208</td>
<td>0.4088</td>
<td>656774</td>
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<tr>
<td>8</td>
<td>2.8955</td>
<td>3.2829</td>
<td>0.3875</td>
<td>0.0209</td>
<td>0.4084</td>
<td>656633</td>
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<tr>
<td>9</td>
<td>3.3038</td>
<td>3.6912</td>
<td>0.3873</td>
<td>0.0209</td>
<td>0.4082</td>
<td>656583</td>
</tr>
<tr>
<td>10</td>
<td>3.7120</td>
<td>4.0997</td>
<td>0.3877</td>
<td>0.0207</td>
<td>0.4083</td>
<td>656578</td>
</tr>
<tr>
<td>11</td>
<td>4.1204</td>
<td>4.5088</td>
<td>0.3885</td>
<td>0.0203</td>
<td>0.4088</td>
<td>656634</td>
</tr>
<tr>
<td>12</td>
<td>4.5292</td>
<td>4.9188</td>
<td>0.3896</td>
<td>0.0199</td>
<td>0.4095</td>
<td>656756</td>
</tr>
<tr>
<td>13</td>
<td>4.9387</td>
<td>5.3298</td>
<td>0.3911</td>
<td>0.0194</td>
<td>0.4105</td>
<td>656941</td>
</tr>
<tr>
<td>14</td>
<td>5.3492</td>
<td>5.7418</td>
<td>0.3926</td>
<td>0.0189</td>
<td>0.4115</td>
<td>657118</td>
</tr>
<tr>
<td>15</td>
<td>5.7607</td>
<td>6.1547</td>
<td>0.3940</td>
<td>0.0185</td>
<td>0.4125</td>
<td>657296</td>
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<tr>
<td>16</td>
<td>6.1731</td>
<td>6.5681</td>
<td>0.3950</td>
<td>0.0182</td>
<td>0.4132</td>
<td>657388</td>
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<td>6.5864</td>
<td>7.0000</td>
<td>0.4136</td>
<td>0.0183</td>
<td>0.4136</td>
<td>660100</td>
</tr>
</tbody>
</table>
3.3.7 Sensitivity Analysis

The percentage change in profit with change in the model parameters, along with the change in the number of replenishment cycles are shown in Table 3.3.1. The model is highly sensitive to change in the holding cost(h), inflation rate(r), deterioration rate(θ), mixing proportions of linear and iso-elastic components(τ) in the demand rate.

**Table 3.3.2:** Change in the number of cycles, optimum profit and the percentage change in profit with change in the purchase cost price

<table>
<thead>
<tr>
<th>c</th>
<th>n</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>27</td>
<td>16533418</td>
<td>7.5</td>
</tr>
<tr>
<td>30</td>
<td>24</td>
<td>15835826</td>
<td>2.96</td>
</tr>
<tr>
<td>35</td>
<td>20</td>
<td>15572427</td>
<td>1.25</td>
</tr>
<tr>
<td>40</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>45</td>
<td>13</td>
<td>14804718</td>
<td>-3.74</td>
</tr>
</tbody>
</table>

**Table 3.3.3:** Change in the number of cycles, optimum profit and the percentage change in profit with change in the selling price

<table>
<thead>
<tr>
<th>p</th>
<th>n</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>55</td>
<td>17</td>
<td>16454651</td>
<td>6.98</td>
</tr>
<tr>
<td>60</td>
<td>17</td>
<td>17430874</td>
<td>13.33</td>
</tr>
<tr>
<td>65</td>
<td>18</td>
<td>18998511</td>
<td>23.52</td>
</tr>
<tr>
<td>70</td>
<td>18</td>
<td>19893004</td>
<td>29.33</td>
</tr>
</tbody>
</table>

**Table 3.3.4:** Change in the number of cycles, optimum profit and the percentage change in profit with change in the delay period

<table>
<thead>
<tr>
<th>M</th>
<th>n</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.35</td>
<td>21</td>
<td>14565273</td>
<td>-5.3</td>
</tr>
<tr>
<td>0.4</td>
<td>19</td>
<td>15116678</td>
<td>-1.72</td>
</tr>
<tr>
<td>0.45</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>16</td>
<td>15993805</td>
<td>3.99</td>
</tr>
<tr>
<td>0.55</td>
<td>15</td>
<td>16450328</td>
<td>6.95</td>
</tr>
</tbody>
</table>
Table 3.3.5: Change in the number of cycles, optimum profit and the percentage change in profit with change in the planning horizon

<table>
<thead>
<tr>
<th>H</th>
<th>n</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>13</td>
<td>12056109</td>
<td>-21.61</td>
</tr>
<tr>
<td>6</td>
<td>14</td>
<td>13165905</td>
<td>-19.84</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>16424496</td>
<td>6.79</td>
</tr>
<tr>
<td>9</td>
<td>21</td>
<td>18517178</td>
<td>20.4</td>
</tr>
</tbody>
</table>

Table 3.3.6: Change in the number of cycles, optimum profit and the percentage change in profit with change in the holding cost

<table>
<thead>
<tr>
<th>h</th>
<th>n</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>16197851</td>
<td>5.31</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
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<tr>
<td>9</td>
<td>17</td>
<td>14578323</td>
<td>-5.22</td>
</tr>
<tr>
<td>13</td>
<td>17</td>
<td>13768088</td>
<td>-10.48</td>
</tr>
<tr>
<td>17</td>
<td>17</td>
<td>12952807</td>
<td>-15.78</td>
</tr>
</tbody>
</table>

Table 3.3.7: Change in the number of cycles, optimum profit and the percentage change in profit with change in the lost sale cost

<table>
<thead>
<tr>
<th>c1</th>
<th>n</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>17</td>
<td>15380489</td>
<td>0.000161</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>15380459</td>
<td>-3.00E-05</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>17</td>
<td>15380490</td>
<td>0.00017</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>15380427</td>
<td>-0.00024</td>
</tr>
</tbody>
</table>
Table 3.3.8: Change in the number of cycles, optimum profit and the percentage change in profit with change in the back ordered cost

<table>
<thead>
<tr>
<th>$c_b$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>15380481</td>
<td>0.000112</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>14818700</td>
<td>-3.65</td>
</tr>
<tr>
<td>7</td>
<td>16</td>
<td>14818707</td>
<td>-3.65</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>14818695</td>
<td>-3.65</td>
</tr>
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</table>

Table 3.3.9: Change in the number of cycles, optimum profit and the percentage change in profit with change in the inflation rate

<table>
<thead>
<tr>
<th>$r$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>13</td>
<td>15286425</td>
<td>-0.61</td>
</tr>
<tr>
<td>0.07</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>0.09</td>
<td>17</td>
<td>17100453</td>
<td>11.18</td>
</tr>
<tr>
<td>0.11</td>
<td>17</td>
<td>17392958</td>
<td>13.08</td>
</tr>
<tr>
<td>0.13</td>
<td>17</td>
<td>17557503</td>
<td>14.15</td>
</tr>
</tbody>
</table>

Table 3.3.10: Change in the number of cycles, optimum profit and the percentage change in profit with change in the net inflation rate and the intensity parameter

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>19</td>
<td>16508816</td>
<td>7.34</td>
</tr>
<tr>
<td>0.06</td>
<td>18</td>
<td>15941990</td>
<td>3.65</td>
</tr>
<tr>
<td>0.1</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>0.14</td>
<td>16</td>
<td>14818703</td>
<td>-3.65</td>
</tr>
<tr>
<td>0.18</td>
<td>15</td>
<td>14392861</td>
<td>-6.42</td>
</tr>
</tbody>
</table>
Chapter 3: Inventory Model for non-linear...

Table 3.3.11: Change in the number of cycles, optimum profit and the percentage change in profit with change in the intensity parameter

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
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<td>16507884</td>
<td>7.33</td>
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<td>0.06</td>
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<td>3.65</td>
</tr>
<tr>
<td>0.08</td>
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<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>16</td>
<td>14818592</td>
<td>-3.65</td>
</tr>
<tr>
<td>0.12</td>
<td>15</td>
<td>14392861</td>
<td>-6.42</td>
</tr>
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</table>

Table 3.3.12: Change in the number of cycles, optimum profit and the percentage change in profit with change in the deterioration rate

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>14</td>
<td>26682623</td>
<td>73.48</td>
</tr>
<tr>
<td>0.06</td>
<td>14</td>
<td>18550869</td>
<td>20.61</td>
</tr>
<tr>
<td>0.1</td>
<td>17</td>
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<td>0</td>
</tr>
<tr>
<td>0.14</td>
<td>17</td>
<td>15288943</td>
<td>-0.59</td>
</tr>
<tr>
<td>0.18</td>
<td>17</td>
<td>13861571</td>
<td>-9.87</td>
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</table>

Table 3.3.13: Change in the number of cycles, optimum profit and the percentage change in profit with change in the mixing parameter

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>13</td>
<td>1130387</td>
<td>-92.65</td>
</tr>
<tr>
<td>0.5</td>
<td>15</td>
<td>7236937</td>
<td>-52.94</td>
</tr>
<tr>
<td>0.7</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>17</td>
<td>19282160</td>
<td>25.37</td>
</tr>
<tr>
<td>0.9</td>
<td>18</td>
<td>24097146</td>
<td>56.67</td>
</tr>
</tbody>
</table>
Table 3.3.14: Change in the number of cycles, optimum profit and the percentage change in profit with change in the convex combination parameter

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>19</td>
<td>16567813</td>
<td>7.72</td>
</tr>
<tr>
<td>5500</td>
<td>18</td>
<td>15965031</td>
<td>3.8</td>
</tr>
<tr>
<td>6000</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>6500</td>
<td>16</td>
<td>14814781</td>
<td>-3.68</td>
</tr>
<tr>
<td>7000</td>
<td>15</td>
<td>14392861</td>
<td>-6.42</td>
</tr>
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</table>

Table 3.3.15: Change in the number of cycles, optimum profit and the percentage change in profit with change in the ordered cost

<table>
<thead>
<tr>
<th>$A$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>17</td>
<td>15382160</td>
<td>0.011027</td>
</tr>
<tr>
<td>200</td>
<td>17</td>
<td>15381275</td>
<td>5.27E-03</td>
</tr>
<tr>
<td>250</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>300</td>
<td>17</td>
<td>15379729</td>
<td>-0.00478</td>
</tr>
<tr>
<td>350</td>
<td>17</td>
<td>15378925</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

Table 3.3.16: Change in the number of cycles, optimum profit and the percentage change in profit with change in the interest earned

<table>
<thead>
<tr>
<th>$I_e$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>16</td>
<td>13986690</td>
<td>-9.06</td>
</tr>
<tr>
<td>0.08</td>
<td>17</td>
<td>14951453</td>
<td>-2.79</td>
</tr>
<tr>
<td>0.12</td>
<td>17</td>
<td>15380464</td>
<td>0</td>
</tr>
<tr>
<td>0.16</td>
<td>17</td>
<td>15809317</td>
<td>2.79</td>
</tr>
<tr>
<td>0.2</td>
<td>17</td>
<td>16239511</td>
<td>5.59</td>
</tr>
</tbody>
</table>
Table 3.3.17: Change in the number of cycles, optimum profit and the percentage change in profit with change in the interest charged

<table>
<thead>
<tr>
<th>$I_r$</th>
<th>$n$</th>
<th>Opt. Profit</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.07</td>
<td>18</td>
<td>15610912</td>
<td>1.5</td>
</tr>
<tr>
<td>0.11</td>
<td>18</td>
<td>15606787</td>
<td>1.47</td>
</tr>
<tr>
<td>0.15</td>
<td>17</td>
<td>15380867</td>
<td>0</td>
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<tr>
<td>0.19</td>
<td>12</td>
<td>14747892</td>
<td>-4.11</td>
</tr>
<tr>
<td>0.23</td>
<td>9</td>
<td>14142539</td>
<td>-8.05</td>
</tr>
</tbody>
</table>

From the above tables we have the following observations:

(i) The percentage changes in profit are significantly high for increase in the values of planning period ($H$), selling price ($p$) and mixing parameter ($\tau$).

(ii) Optimal profit decreases significantly with respect to increase in rate of deterioration ($\theta$).

(iii) Optimal profit decreases moderately with respect to increase in holding cost ($h$), intensity parameter ($\sigma$), shortage demand ($\alpha$) and interest charged ($I_r$).

(iv) Optimal profit increases moderately with respect to increase in permissible delay in payments ($M$), inflation rate ($r$) and interest earned ($I_e$).

3.3.8 Discussion

In this section we analyze an inventory model for deteriorating items when the demand is hybrid in nature. Such demand is observed for commodities like coffee, cotton, tin, copper, wristwatch, automobile man’s suit, snow skis, sailboat, mattress, floor lamp, refrigerator, PC game, etc. Keeping in mind that the willingness of a customer to wait during a shortage period is likely to decline with the length of waiting time, the backlogging rate has been assumed to be a function of time. Price inflation and delay in payment have also been taken into account, and a solution procedure to determine the optimal number of replenishments, cycle times and order quantities has been indicated.

The model may be extended to include items with time dependent deterioration rate, uncertain inflationary conditions and also partial payment scheme.