Chapter 1

Introduction
Chapter I

1. INTRODUCTION

1.1. Concepts and Historical Recollection of Optimal Designs

Design of experiments forms a fascinating branch of statistics and though primarily it originated from agricultural experiments, it is finding more and more applications in various other fields. In varietal trials if the number of varieties is large, the ordinary Randomized Block Designs (RBD) and Latin Square Designs (LSD) are not at all suitable, as the efficiency of varietal comparisons become very much reduced, because of lack of effective control on experimental error. To overcome this drawback, a new series of designs known as Balanced Incomplete Block Designs (BIBD) were introduced by Yates (1936), which could accommodate more types of varietal trials. The difficulty of more replications under BIBD was removed to some extent by Bose et.al. (1939) who introduced another series of designs known as Partially Balanced Incomplete Block Designs (PBIBD). Here, when all the existing Incomplete Block Designs are binary, either a treatment does occur only once or does not occur at all in a block. Tocher (1952) considered a generalization of the structure of these designs by allowing a treatment to occur more than once in a block. A brief review is done on all n-ary block designs available upto date.

Our thesis consists of two parts. First part exhibits some new research work on Generalized N-ary partially Balanced Block (GNPBB) designs. The concept of generalized or general class of n-ary designs, introduced by Shafiq et.al. (1979) for BIBD were utilized for PBIB designs of m–associate class and new theorems and results were obtained in A–, D –, and E-optimality.

The second part of our thesis discusses in detail the construction of Balanced Treatment Incomplete Block Designs (BTIBD) proposed by Bechhofer et.al. (1981). Robson (1961) and Cox (1958) had shown that BIBD may not be very appropriate for the multiple comparisons with the control because of the special role played by the control. For a brief exposition to the history of this problem and a list of references, the reader is referred to the article by Bechhofer et.al. (1981).
With definitions and preliminary results, we give in the second part of our thesis a new method of construction of minimal complete class of generator designs with examples. Here theoretical results were worked out for arriving $A$-optimal incomplete block designs for comparing treatments to a control treatment. Our research work and results have utilized the concepts and definitions of Balanced N-ary Block Designs (BNBD) and Partially Balanced N-ary Block (PBNB) design for the construction and optimality of Balanced Treatment Incomplete Block (BTIBD) designs.

The Introduction of the optimality criteria due to Smith (1918), for comparing designs in a given experimental set-up is playing a major role in both theoretical and practical lines. A good number of papers have been published by Hotelling (1941), Wald (1943, 1950), Friedenam et.al. (1947), Stein et.al. (1947), Stein (1948), Wolfowitz (1950), Hodges et.al. (1950), Robbins et.al. (1951), Kiefer et.al. (1952),Elfving (1952, 1955, 1959), Robbins (1952), Chernoff (1953), Chung (1954), De La Garza (1954, 1956), Hodges et.al. (1955), Sacks (1956), Hoel (1958), Guest (1958), Mote (1958), Kshirsagar (1958), and Roy et.al. (1957). Later on developments occurred to a greater extent due to Kiefer (1958, 1959, 1960, 1961, 1962), Takeuchi (1961, 1963), Shah (1960) and Kiefer et.al. (1959, 1960), laid the foundations for a vigorous and systematic theory of optimum experimental designs.

In 1960, Kiefer and other authors including Hoel (1964, 1979), Karlin et.al. (1966a) and Atwood (1969) have contributed a lot for the development of optimality theory. In the early 1970's the core of the theory was crystallized in Fedorov et.al. (1972), Whittle (1972), Kiefer (1974), Hedayat et.al. (1974).

Coming to the developments of optimality, Box (1957) was concerned more with developing methods for tackling applied problems than with general mathematical theory. This work was reported in the papers of Box et.al. (1951), Box et.al. (1959), Box et.al. (1959), Box et.al. (1965), Hill (1978), Murthy et.al. (1967). Although their aims were different, there was considerable overlapping in ideas between what might be called the Kiefer and the Box approaches, and this overlapping has become more apparent in recent years when there has been more
emphasis on overlapping tools for applying the Kiefer-type theory. A seminal paper in this context is that of Wynn (1970).

The other developments took place in the USSR and are associated particularly with the name of Fedorov. In it, both the mathematical theory and algorithms for applying that theory were studied. The earlier part of the work is reported in the English literature in a book by Fedorov (1972). Since the publication of this book, there have been further advances in the theory given by Eccleston et.al. (1974), Coniff et.al. (1974), Saha (1975), Harville (1975), Shah, et.al. (1976), Nigam et.al. (1977), William et.al. (1977), Hedayat et.al. (1978), Cheng (1978a, b, 1979), Mehta et.al. (1975), Morgam (1977), Bandemer (1980), Magda (1979), and Marshall et.al. (1979).

During the 1980's the optimum theory was explained and developed by many authors like Jacroux (1983b), Kageyama (1980), Smith (1984a) and other authors including Berkum (1987), Lee et.al. (1987b) and Jacroux (1987). A detailed review was given about recent developments in the methods of optimum and related experimental designs by Hedayat et.al. (1988), Atkinson (1982b, 1988) and Dodge et.al. (1988). After this review, many papers have been published by Gupta et.al. (1989), Atkinson et.al. (1989), Fedorov (1989), Mikaeli, et.al. (1989), Huda, (1991) Mukerjee (1992), Shah et.al. (1989) and Dey et.al. (1989). Apart from the above mentioned authors, many more experts have analysed and revealed the results about the optimum design of experiments.

The aim of this research work on optimality is to reduce the cost and various tests within the short period by the use of this technique were well established by the following reprints during the years 1990-95. The E-optimality designs have been explained by Bagiah (1990), Kageyama (1990), Kozlowska et.al. (1990). Recent papers on optimality in block designs have been studied in detail by the authors Duthie (1991) Das et.al. (1992), Baines (1992), Shah et.al. (1992), Uddin et.al. (1992). Das, et.al. (1992), Jansen, et.al. (1992), Jacroux (1992),
Naick et.al. (1992), Baines (1992) Nguyen, et.al. (1992), and Bhaumik (1993) in a class of E-optimal block designs.

A group of authors discussed many types of optimality in recent years. The importance of optimal design for a linear regression theory was explained by Chang, et.al. (1994), Dette, et.al. (1994).

In the year 1994, authors like Mukherjee and Sengupta, Kunert, Ting, Chao-Ping, David, El-balbawy, Ahmed and Alharbey, Wong, Fedorov, Khan and Mukherjee, Jacroux, Yang, Mathews, Eccleston and Street have discussed the optimality in different fields.

During 1995, Atkinson et.al., Happacher, Liu et.al. Mukhopadhyay et.al., Dette et.al., Pukelsheim et.al., Cheng, Dette et.al., Dey et.al., Gaffke et.al., Huang et.al. Wong, Ahlbrandt et.al., Ghosh et.al., Kherwa et.al., and in 1996 Jung et.al., Koukouvinos, Muller et.al., Balasubramanian et.al., Dette et.al., Koumias et.al., Ting et.al. Dey et.al., Draper et.al., Dette et.al., Heise et.al., Bae, Chang et.al., Dette, Heiligers, Koukouvinos, Morgem et.al., Uddin et.al. have done works on optimalities.

In the year 1997, Bae, Liu, Rasch et.al., Morgem, Liu et.al., Dette et.al., Uddin et.al., Uddin, Jacroux et.al. Nguyen, Chang et.al., Schwabe et.al., Kushnes, Dette et.al., Firth et.al., Schwabe, Sitter et.al., Uddin et.al., Parsad et.al., Sun et.al., Dovi, cheng, Das et.al., Dette, Li et.al., Sitter et.al., Atkinson et.al., Azais et.al., and lastly in the year 1998, many authors like Cham et.al., Chiu et.al. Erkanli et.al., Githinji et.al., Kao et.al., Kirlitsa et.al., Koukouvinos, Krafft et.al., Kushner, Mukerjee, Raghavarao et.al., Riccomagno, Ghosh et.al., Kageyama et.al., Uddin et.al., Dey et.al., Dette et.al. Das Dette et.al., Li et.al., Yeh et.al., Du, Jacroux, Jaggi et.al., Ponnuswamy et.al., Berger et.al. Rao et.al., Srivastav et.al., Raghavarao have done extensive works on optimality criteria.

In our thesis, A-optimality has a natural statistical meaning and it picks up really desirable designs for the problem of multiple comparisons with the control. Specially the following authors' papers were reviewed for the importance of
A-optimality. Ting, chao-ping. (1994) has discussed A-optimal block designs for comparing test treatments with a control, when k>n. Wong, werg-kee. (1994) discussed on comparing robust properties of A-, D-, E- and G- optimal designs, whereas Jaggi et.al. (1996) have discussed on A-efficient block designs for comparing two disjoint sets of treatments. On the construction of A-efficient balanced test treatment incomplete block designs, Prasad et.al. (1995) have done a good work. In contrast to the above work, Ting et.al. (1996) have analysed Bayer A-optimal designs for comparing test treatments with a control. The paper of Brzeskwiniewicz (1996) deals with the A-, D-, E- and L- efficiency of block designs, whereas Huda et.al. (1994) have discussed about A-optimal third-order symmetric product designs for hyper-cubic regions.

A efficient block designs with unequal block sizes for comparing two sets of treatments were presented by Jaggi (1996), whereas Kirlitsa et.al. (1996) have discussed on the construction of D-and A-optimal linear designs of experiments for linear model with heteroscedastical observations. In this context, Krafft et.al. (1997) paper on A-optimal connected block designs with nearly minimal number of observations, is noteworthy. A-optimality was discussed by Chang et.al. (1998), have given the exact A-optimal designs for quadratic regression, while Chang et.al. (1998) provided A-optimal designs for an additive quadratic mixture model.

Having a note on all these papers, we are proceeding to introduce n-ary concept for incomplete as well as complete block designs with an importance to A-optimality criteria for the problem of multiple comparisions with the control. Before going for recollecting the earlier works on balanced and partially balanced n-ary block designs, we present below the concepts and the different definitions of optimality available for the incomplete block designs.

1.2. Optimality-Definitions and Theorems

The general requirement of an experiment to get maximum information is not sufficient for designing the experiment. Therefore, the theory presents different optimality criteria, and the experimenter has to choose one of them, according to
his purpose. However, the choice of the optimality criterion as well as the specification of the model needs a careful confrontation of the theory with the real situation, and the practical experience of the experimenter is unavoidable here.

A design is considered to be uniformly better than another design if it guarantees a smaller variance of estimates of any linear function of the unknown parameters. This uniform ordering of designs can be expressed by an ordering of information matrices. The variance of an estimate is considered as a function of the information matrix. Since, in general, there are no uniformly best designs, admissible designs are to be considered. The restriction to admissible designs may lead to an essential reduction of the set of trials which are to be considered.

A large amount of different criteria is to be considered to give a greater flexibility when expressing this aim. An optimality criterion is usually represented by a function $\Phi$ defined on the set of all information matrices. Analytical properties of such functions as continuity, convexity, and differentiability can be discussed.

The Central theory of optimum experimental design as introduced by Kiefer (1958, 1959) and subsequently reviewed by Wynn (1984) is concerned with the linear model $E(Y) = X\beta$, where $Y$ is a vector of responses, $\beta$ is a vector of unknown parameters and $X$ is a design matrix of full rank. Unobservable random errors are assumed to be independent and to have constant variance $\sigma^2$. The parameter vector $\beta$ are estimated by least squares giving estimates with a covariance matrix given by

$$\text{Cov} (\hat{\beta}) = \sigma^2 (X^T X)^{-1}$$

The optimum design of experiments is concerned with the choice of $X$, i.e. the design points, so as to optimize various characteristics of $X^T X$, for example to maximize $\det (X^T X)$. We are concerned with low-order polynomial and we rewrite the model as $E(Y) = F\beta$ where the $i$-th row of the $n \times p$ matrix $F$ is $f_i(x_i)$, representing
functions of m (≤ p) explanatory variables. Now the least square estimates have variance matrix \( \sigma^2 (F^TF)^{-1} \), where \( F^TF = \sum f(x_i) f(x_j) \) is called the information matrix.

A good design is a choice of n points in \( \mathcal{X} \) which makes the matrix function \( F^TF \) large, in some sense to be determined. Mathematically, it is convenient to consider instead of the continuous or approximate theory in which a design is described by a measure \( \xi \) over \( \mathcal{X} \). Then the information matrix is written as

\[
M(\xi) = \int f(x) f^T(x) \xi(dx) = \int m(x) \xi(dx)
\]  
(1.2.1)

A design for which exactly n points in \( \mathcal{X} \) are chosen is called an exact design, and is represented by a discrete measure \( \xi_n \), that puts weight \( n^{-1} \) at each of the n points \( x_1, x_2, \ldots, x_n \). Then \( M(\xi_n) = n^{-1} (F^TF) \). In the theory for continuous design it is customary to consider minimization of some measure of imprecision \( \Psi(M(\xi)) \).

1.2.1 Directional derivatives

Directional derivatives will play a basic role in our theory. There are two types of derivatives, namely (a) Gateaux derivative, and (b) Frechet derivative

a) Let the measure \( \bar{\xi} \) put unit mass at the point x. The Gateaux derivative of \( \Psi \) at \( \xi \) in the direction of \( \vec{\xi} \) is

\[
\Phi(x, \xi) = \lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \Psi [M(\xi) + \alpha M(\bar{\xi})] - \Psi [M(\xi)] \right\}
\]  
(1.2.1)

b) The Frechet derivative of \( \Psi \) at \( \xi \) in the direction \( \vec{\xi} \) is

\[
\Phi(X, \xi) = \lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \Psi [(1 - \alpha) M(\xi) + \alpha M(\bar{\xi})] - \Psi [M(\xi)] \right\}
\]  
(1.2.2)

and this derivative will serve our purposes better than the previous one.

1.2.2. Equivalence Theorem

The general equivalence theorem due to Kiefer and Wolfowitz, (1960), states the equivalence of the following three conditions.
i) $\xi^*$ minimizes $\Psi(M(\xi))$

ii) $\min \Phi(x, \xi^*) \geq 0$ \hspace{1cm} (12.3)

iii) $\Phi(x, \xi^*)$ achieves its minimum at the points of the design.

1.2.3. Specific Optimality Criteria

Some commonly used optimality criteria have been explained in this section. The relationship between these optimality criteria has been explained in Kiefer's (1958, 1959, 1975a,b) papers.

1.2.4. D-Optimality The most widely used design criterion is that of D-optimality. It was introduced by Wald (1943) and it is defined by the criterion function,

$$\Psi(M(\xi)) = -\log |M(\xi)|$$ \hspace{1cm} (1.2.4)

so that the determinant of the information matrix is to be maximized or, equivalently the determinant of $M^{-1}(\xi)$ is to be minimized.

1.2.5 $D_A$-Optimality Suppose that our interest is not in all $p$ parameters of the model, but only in a linear combinations of $\beta$, which are the elements of $A^T \beta$, $s<p$. To minimize the generalized variance of this subsystem, the analogue of D-Optimality is $D_A$-Optimality, to stress the dependence of the optimum design on the particular linear combinations of interest. It was introduced by Sibson (1974) and by this criterion, $-\log |A^T M^{-1}(\xi) A|$ is maximized.

1.2.6 S-Optimality Suppose that there are $m$-models and that, for the $i$-th, the subsystem of interest is given by $A_i^T \beta_i$, with $s_i<k_i$ where $k_i$ is the rank of the $i$-th linear model. Interest in different models is represented by the non-negative weights $w_i$.

The following three requirements on the optimum measure $\xi^*$ are equivalent:
i) $\xi^*$ maximizes

$$-\sum_{i=1}^{m} w_i \log | A_i^T M_i^{-1}(\xi) A_i |;$$

ii) $\xi^*$ minimizes the maximum over $\mathcal{B}^*$ of

$$\sum_{i=1}^{m} w_i d_{A_i}(\xi, \xi) = \sum_{i=1}^{m} w_i x^T M_i^{-1}(\xi) A_i \left[ A_i^T M_i^{-1}(\xi) A_i \right]^{-1} A_i^T M_i^{-1}(\xi) x$$

iii) the maximum value of

$$\sum_{i=1}^{m} w_i s_i,$$  \hspace{1cm} (1.2.5)

is $\sum_{i=1}^{m} w_i s_i$, where the sum is over $i = 1, 2, \ldots, m.$ \hspace{1cm} (1.2.6)

This equivalence theorem is a generalization of those given by Atkinson and Cox (1974) and by Lauter (1976), who call the criterion as $S$-Optimality.

1.2.7 $D_S^*$ Optimal A special case of importance arises when there is only one model, a subset of the parameters of which is of interest. Let the model be divided as

$$E(Y_i) = f^T(\xi_i) \beta = f_1^T(\xi_i) \beta_1 + f_2^T(\xi_i) \beta_2$$ \hspace{1cm} (1.2.7)

where $\beta_1$ is the $s$-parameters of interest, $\beta_2$ being treated as nuisance parameters.

The matrix of coefficients $A$ becomes, $A = (I_s; O)$ where $I_s$ is the $s \times s$ identity matrix.

If the information matrix is correspondingly partitioned as

$$M(\xi) = \begin{bmatrix} M_{11}(\xi) & M_{12}(\xi) \\ M_{12}^T(\xi) & M_{22}(\xi) \end{bmatrix}$$ \hspace{1cm} (1.2.8)

The determinant to be maximized is

$$\left| M_{11}(\xi) - M_{12}(\xi) M_{22}^{-1}(\xi) M_{12}^T(\xi) \right|$$

This criterion, which depends on the particular subset of parameters, is customarily called $D_s^*$ Optimality. The variance to be minimized is
\[ d_a(x, \xi) = f^T(x) M^{-1}(\xi) f(x) - f_2^T(x) M_{22}^{-1}(\xi) f_2(x) \] (1.2.9)

### 1.2.8 A-Optimality

The criterion of A-Optimality is defined by the criterion function

\[ \Psi [M(\xi)] = \text{tr} \left[ M^{-1}(\xi) \right] \quad \text{if det} \left[ M(\xi) \right] \neq 0 \]
\[ = \infty \quad \text{if det} \left[ M(\xi) \right] = 0 \] (1.2.10)

This means that the trace of \( M^{-1}(\xi) \) is minimized so that the average variance of the parameter estimates is minimized.

### 1.2.9 G-Optimality

The G-Optimality criterion is defined by the criterion function.

\[ \Psi [M(\xi)] = \max_{x \in X} \text{tr} \left[ M^{-1}(\xi) f(x) f(x)^T \right] \quad \text{if det} \left[ M(\xi) \right] \neq 0 \]
\[ = \infty \quad \text{if det} \left[ M(\xi) \right] = 0 \] (1.2.11)

The experimenter optimizing the design according to the G-optimality criterion intends to get a good estimate of the whole state function \( \theta \in \Theta \).

### 1.2.10 E-Optimality

It is defined by the criterion function.

\[ \Psi [M(\xi)] = \lambda_1^{-1} \quad \text{if det} \left[ M(\xi) \right] \neq 0 \]
\[ = \infty \quad \text{if det} \left[ M(\xi) \right] = 0 \] (1.2.12)

Here the minimum eigen value of \( M(\xi) \) is given by \( \lambda_1 \). Otherwise, it is explained that E-Optimum designs minimize the maximum eigenvalue of \( M^{-1}(\xi) \), which is equivalent to minimizing the variance of the contrast \( a^T \beta \) with largest variance, subject to \( a^T a = 1 \).
The criteria of D-, A- and E- optimum designs are special cases of a power function of the eigen values of \( \mathcal{M}(\xi) \) used by Kiefer (1975) to study the variation in structure of optimum designs as the criterion changes in a smooth way.

### 1.2.11 V-Optimality

The criteria of A- and E-Optimality have been much used in the construction of block designs by Paterson (1988). The already mentioned criterion of G-optimality is concerned with the maximum of the variance of the estimated response. A second variance based criterion is that of V- optimality in which the design is found to minimize

\[
d_{\text{ave}}(\xi) = \frac{1}{r} \sum_{i=1}^{r} \text{d}(x_i^\top \xi)
\]

(1.2.13)

Interest is now in the average of the variance at r-points which need not belong to the design region \( \mathcal{R} \).

### 1.2.13 Linear optimality

This criterion is defined by criterion function of the form

\[
\psi(\mathcal{M}(\xi)) = \text{tr}(W\mathcal{M}^{-1}(\xi)), \quad \text{if } \det(\mathcal{M}(\xi)) \neq 0
\]

\[
= \infty, \quad \text{if } \det(\mathcal{M}(\xi)) = 0
\]

(1.2.14)

where \( W \) is a positive definite \( m \times m \) matrix. The A-optimality criterion corresponds to the particular case of \( W=1 \).

### 1.2.14 The Lp-class (or) Trace-class of Optimality

Many optimality criteria are particular cases of a Lp-class of optimality criteria. A criterion belonging to the class is defined by a criterion function of the form
\[ \Psi(M(\xi)) = \left[ M^{-1} \tr \{ H M^{-1}(\xi) H^T \}^{1/p} \right], \quad \text{if} \quad \det(M(\xi)) \neq 0 \]
\[ = \infty, \quad \text{if} \quad \det(M(\xi)) = 0 \]  
(1.2.15)

where \( p > 0 \) and \( H \) is a non-singular \( m \times m \) matrix.

The linear optimality criteria are obtained from the above definition if \( p = 1 \). Also, the E-optimality criteria belong to the Lp-class if \( H = 1 \) and \( p \to \infty \).

1.2.15 M.S.- Optimality

Eckolston et al. (1974) proposed an optimality criterion, called M.S.-Optimality, extending the notion of S- Optimality introduced by Shah (1960). These criteria are known for minimizing the dispersion of the latent roots of the information matrix of a design.

1.2.16 MV-Optimality

Takeuchi (1961) argued that, in a block design, we are primarily interested in paired treatment comparisons, and an optimal design may seek to minimize the maximum variance of the corresponding estimates. This criterion has been called MV-optimality criterion by Jacroux. (1983)

A design in the class \( \mathcal{D}(b,v,k) \) is said to be MV-Optimal if it minimizes the maximum variance for a paired treatment contrast among all designs in \( \mathcal{D}(b,v,k) \).

Thus, MV-Optimality is somewhat different from E-Optimality in which the comparison refers to all treatment contrasts. It is also different from A-Optimality which seeks to relate to the average variance for all paired treatment contrasts. However, unlike the A-, D- and E-Optimality criteria, this criterion is not exclusively a function of the eigen values and, as such, it needs a separate treatment.

1.2.17 Schur-Optimality

Magda (1979) introduced the notion of Schur-Optimality via Schur-convex function as follows:
For any vector $X$ of order $(n \times 1)$, is a real valued function $\Phi(x)$ satisfying

$$\Phi(S_x) \leq \Phi(x)$$

for every doubly stochastic matrix $S_x$ is called as Schur-Convex.

Such a function is permutation-invariant in the sense that

$$\Phi(x) = \Phi(P_x)$$

for every permutation matrix $P$.

Let $X(C)$ denote the vector of non-zero eigen values of $C$. Then a Schur-Optimality criterion seeks to minimize

$$\Phi(C) = \Phi\{X(C)\}$$  \hspace{1cm} (1.2.16)

among all relevant $C$-matrices for a given Schur-Convex function $\Phi$.

### 1.2.18 Universal Optimality

Kiefer (1975) introduced the notion of universal optimality in the following manner. Consider the optimality functionals $\Phi$ defined on the set of all $C$-matrices which satisfies,

i) $\Phi(C)$ is non-increasing in $t$, $t \geq 0$

ii) $\Phi(\alpha C_1 + (1-\alpha) C_2) \leq \alpha \Phi(C_1) + (1-\alpha) \Phi(C_2)$ for $0 < \alpha < 1$ and for any pair of $C$ matrices $C_1$ and $C_2$.  \hspace{1cm} (1.2.17)

If a design is optimal with respect to all such optimality function $\Phi$, it is said to be Universally optimal.

### 1.2.19 $T$-Optimum Design

Atkinson and Fedorov (1975.b) describe $T$-optimum experimental design as:

i) A necessary and sufficient condition for a design $\xi^*$ to be Bayesian $T$-optimum if it fulfills

$$\Psi(x, \xi^*) \leq \Gamma(\xi^*)$$  \hspace{1cm} for all $x \in \mathbb{R}$. 
where \( \Psi(x, \xi^*) = \sum \prod \theta_j \ E_\theta \left[ n_i(x, \theta) - n_i(x, \xi^*) \right]^2 \).

ii) At the points of Bayesian T-optimum design \( \Psi(x, \xi^*) \) achieves its upper bound.

iii) For any non-optimum design \( \xi \), that is, a design for which \( \Gamma(\xi) < \Gamma(\xi^*) \)

\[
\sup_{x \neq x^*} \Psi(x, \xi) > \Gamma(\xi^*) \tag{1.2.18}
\]

iv) The set of Bayesian T-optimum design is convex.

At a time, when all the existing incomplete block designs are binary, that is, when either a treatment does occur only once or does not occur at all in a block, Tocher (1952) considered a generalization of the structure of these designs by allowing a treatment to occur more than once in a block. In this design, giving equal accuracy for all different treatment comparisons, we allowed the elements of the incidence matrix to take the values either 0,1 or 2 and for the first time introduced the concept of balanced ternary designs. He did not stipulate that each treatment should occur in the design a constant number of times, and in fact that the replication number for each treatment was allowed to vary between two limits determined by parameters of the design. By trial and error method, he has constructed some Balanced Ternary Block (BTB) designs, with varying replications for a constant block size. Further he has also contemplated for allowing the elements of the incidence matrix of the design to take the values 0, 1, 2, 3, ... (n-1), and he named such designs as n-ary designs.

Tocher(1952) defined a balanced n-ary block (BNE) design as an arrangement of \( V \) treatments in \( B \) blocks each of size \( K \), such that the \( i \)-th treatment occurs in the \( j \)-th block \( n_{ij} \) times, and altogether \( R \) times where \( n_{ij} \) can take values 0,1,2,...,(n-1). We say the design is variance balanced if the inner product of any two row vectors of the incidence matrix \( N_{VX} \) of the n-ary design, \( \sum_{j=1}^{B} n_{ij} n_{ik} \) is a constant and equal to \( \Lambda \) (say) for all \( i \neq k = 1,2,....V \). This implies also that \( \sum_{j=1}^{B} n_{ij}^2 = \Delta \).
(another constant) for all i=1,2,...,V. According to Hedayat and Federer (1974), a n-ary block design is said to be pairwise balanced if
\[ NN^t = D(\Lambda) + \Lambda \ J, \]
where \( N^t \) is the transpose of the incidence matrix \( N \), \( D \) is a diagonal forms of the matrix with elements \( \Delta \), \( \Lambda \) a scalar and \( J \) a matrix with unit entries everywhere. A BNB design is said to be incomplete if any of the entries in the incidence matrix \( N \) is zero; otherwise it is is said to be complete.

The concept of generalized or general class of n-ary design were introduced by Shafiq and Federer (1979) by small and capital letters or small and other letters with asterisk marks to differentiate between the parameters of basic N-ary balanced or partially balanced block designs. Here \( N \) is representing the ary number and it has the same meaning as small \( n \) is used in original BNB designs given by Tocher (1952).

For a basic N-ary balanced incomplete block (BNBIB) design with parameters \((v,b,r,k; n_{ij} = 0,1,2, ..., N-1)\), and \((v x b)\) incidence matrix \( n \), Shafiq and Federer (1979) have defined a Generalized N-ary Balanced Block (GNBB) Design with parameters \((v, b, r^*, k^*, \Lambda^*; n^*_{ij})\), with \( n^*_{ij} \) in the set \( m_0, m_1, ..., m_{N-1} \) where \( m_a = a m_1 - (a-1) m_0 \) for \( a = 0, 1, 2, ..., N-1 \), to be an arrangement of \( V \) treatments in \( B \) blocks each of size \( K^* \) (\( K^* \) not necessarily less than \( V \)) such that its incidence matrix is defined by
\[ n^* = n (m_1 - m_0) + \ J \ m_0 \]  \hspace{1cm} (1.2.19)
where \( J \) is a \((v x b)\) matrix with unit elements everywhere and \( 0 \leq m_0 < m_1 \).

We can define a generalized or general N-ary partially balanced block (GNPBIB) design as follows without having any confusion between incidence matrices of balanced and partially balanced designs.
For a basic N-ary partially balanced incomplete block design with parameters \( [v, b, r, k, \lambda_0^\alpha; n_\alpha] \), \( P_\alpha = (p_{jk}) \), \( \alpha, j, k = 1, 2, \ldots, m; n_{ij} = 0, 1, 2, \ldots, (N-1) \) and \( v \times b \) incidence matrix \( \mathbf{n} \), a generalized N-ary partially balanced block design with parameters \( [v, b, r^*, k^*, \lambda^*; n_\alpha^*] \), \( P_\alpha^* = (p_{jk}^\alpha) \), \( \alpha, j, k = 1, 2, \ldots, m; n_{ij}^* = m_0, m_1, \ldots, m_{N-1} \) is defined to be an arrangement of \( v \) treatments in \( b \) blocks each of size \( k^* (k^* \text{ not necessarily less than } v) \) such that its incidence matrix is defined by

\[
\mathbf{n}^* = \mathbf{n} (m_1 - m_0) + \mathbf{J} m_0
\]

where \( \mathbf{J} \) is a \( v \times b \) matrix with unit elements everywhere and \( 0 \leq m_0 < m_1 \).

Paik and Federer (1973), for the first time, introduced the concept of Partially Balanced N-ary Block (PBNB) designs which has reduced the number of replicates required of each treatment in our BNB designs of Tocher (1952). Soundarapandian (1980a,b,c,d) has made an attempt taking the above PBNB designs as Basic N-ary Partially Balanced Block (BPNB) designs, to generalize the concept of BPNB designs to Generalized N-ary Partially Balanced Block (GNPBB) designs which have the advantage of reducing the number of replicates required by each treatment in GNB designs of Shafiq and Federer (1979). The number of frequencies of treatments in a block of our GNPBB design will be non-negative integers.

\[
m_a = a m_1 - (a-1) m_0, \quad a = 0, 1, 2, \ldots, (N-1),
\]

such that \( 0 \leq m_0 < m_1 < \ldots < m_{N-1} \).

The GNPBB designs presented herein will be useful for the situation of within-block variance is a constant for block of size \( k \) over a range of block sizes \( 0 < k < b \) in experimentation as presented by Shafiq and Federer (1979) for balanced designs.

The detailed definitions and notations of N-ary block designs are presented then and there in the respective chapters.
1.3. Analysis of Balanced and Partially Balanced n-ary Block (BNB and PBNB) Designs

Yates (1940) and Rao (1947) suggested the method of combined intra- and inter-block analysis for binary incomplete block designs and later derived expressions in the case of E-optimal n-ary block designs. Considering the BNB designs of Toccher (1952), we now for the first time suggest and extend Rao's (1956) binary method to n-ary designs which suitably admit easy estimation of block parameters in the case of E-Optimal n-ary block designs and other similar E-Optimality of partially balanced n-ary block designs.

The usual linear additive model is

\[ y_{ijk} = \mu + t_i + b_j + e_{ijk} \quad \text{where} \]

\[ y_{ijk} \] \text{the yield to the } k\text{-th plot in the } j\text{-th block of the } i\text{-th treatment}

\[ i = 1, 2, \ldots, v; \quad j = 1, 2, \ldots, b; \quad k = 1, 2, \ldots, n_{ij} \]

\[ t_i = \text{the effect of the } i\text{-th treatment}; \quad b_j = \text{the effect of the } j\text{-th block and} \]

\[ e_{ijk}'s \text{ are uncorrelated random variables having } E(e_{ijk}) = 0 \text{ and } V(e_{ijk}) = \sigma^2, \text{ for all } i, j & k. \]

In the intrablock analysis of the above BNB designs, the reduced normal equation for the estimation of treatment differences can be written, assuming equal replication for all treatments as

\[ Q_l = \frac{(R-K-L)}{K} t_l - \frac{\Lambda_{i1}}{K} t_1 - \ldots - \frac{\Lambda_{iv}}{K} t_v \quad \text{(1.2.2)} \]

with consistent equation \( \sum_{i=1}^{v} t_i = 0 \) where

\[ Q_l = \text{the total yield for the } i\text{-th treatment minus the sum of block means in which it occurs.} \]

\[ \Lambda_{ij} = \text{the number of block pairs (balanced blocks) in which the } i\text{-th and } j\text{-th treatment occur together, and} \]
**R** = (RK−Δ), where **R** = the number of replications, K=the block size, and

\[ \Delta = \sum_{i=1}^{V} a_{ij}^2 \quad \text{for every } i = 1, 2, \ldots, V. \]

The extended inter-block analysis from binary \( \text{Rao}_R \) (1947) to n-ary is easily seen to be

\[ Q_i^* = \frac{\Delta}{K} + \frac{\Delta_1}{K} t_1 + \ldots + \frac{\Delta_V}{K} t_V \]  \hspace{1cm} (1.2.3)

Adding corresponding equations in (1.2.2) and (1.2.3) with respective weights \( w \) and \( w' \), we get the equations giving the combined estimates as

\[ p_i = \mathbf{R}^* \left( \frac{t_i}{K} + \frac{\Delta_1}{K} t_1 + \ldots + \frac{\Delta_V}{K} t_V \right) \]  \hspace{1cm} (1.2.4)

together with the consistent condition \( \sum_{i=1}^{V} t_i = 0 \) where \( \mathbf{R}^* = \left[(\text{RK-D}) w + \Delta w' \right] \),

\( = [\mathbf{R} w + \Delta w'], \Delta_{ij} = \Lambda_{ij} (w' - w), \)

\[ p_i = w Q_i + w' Q_i' \]

\( Q_i \) = the sum of means of blocks in which the \( i \)-th variety occurs minus \( R \) times the grand mean,

\( w = \) the reciprocal of the estimated intrablock variance and \( w' \) that of the interblock variance \( \text{Rao}_R' \) (1947).

For balanced designs \( \Lambda_{ij} = \Lambda_{ik} \Rightarrow \Lambda_{ij} = \Lambda_{1k} \).

Solution of equations (1.2.2) as functions of \( Q, \mathbf{R} \) and distinct \( \Lambda_{ij} \) provides solution for the latter combined equation (1.2.4) by writing \( Q, \mathbf{R} \text{Q(C)}, \mathbf{R}^* A_{ij} \) for \( Q, \mathbf{R}, \Lambda_{ij} \). The same is true of the expressions for variances. If \( (\text{RK−Δ}) = \sum_i \Lambda_{ij} \)

explicitly used, then \( \mathbf{R}^* \) and \( \Lambda_{ij} \) should be defined as

\[ \mathbf{R}^* = \left[(\text{RK−Δ})w + w' \left( (V-K) / V \right) \right] \]  \hspace{1cm} (1.2.5)

and \( \Lambda_{ij} = \Lambda_{ij} (w-w') + (w'RK/V) \)  \hspace{1cm} (1.2.6)
respectively. A certain amount of care may be necessary involving the actual examination of solution and methods instead of fully depending on the available published formulae.

### 1.3.1 The Balanced n-ary Block Designs (BNBD)

The intrablock normal equations are

\[
Q_i = \frac{(RK - A)}{K} t_i - \frac{A}{K} \sum_{j \neq i} t_j , \quad i = 1, 2, \ldots, V \text{ with } \sum t_i = 0 \tag{1.2.7}
\]

On simplification, we get

\[
t_i = \frac{K}{A \Lambda^2} Q_i \quad \text{and} \quad V(t_i - t_j) = \frac{2K}{A \Lambda^2} \sigma^2
\]

From (1.2.7) the normal equations for corresponding combined intra/inter-block analysis is

\[
Q(c) = R^* \frac{t_i}{K} - \frac{A}{K} \sum_{j \neq i} t_j , \quad i = 1, 2, \ldots, V \text{ and } \sum t_i = 0 \tag{1.2.8}
\]

where \( \Lambda = \Lambda_1 (w - w^1) \) and \( R^* = [(RK - A) w + \frac{A}{K} w^1] \)

Here variance \( \sigma^2 \) has to be dropped to have difference in notation. Considering PBNB design with two associate (m=2) classes, Soundarapandian (1980-a), the intrablock equations are

\[
Q_i = \frac{(RK - A)}{K} t_i - \frac{A_1}{K} \sum_{i_1} t_{i_1} - \frac{A_2}{K} \sum_{i_2} t_{i_2} \quad \text{and} \quad \sum t_i = 0 \text{ for } i = 1, 2, \ldots, V \tag{1.3.9}
\]

where \( \sum_{i_1} \) and \( \sum_{i_2} \) indicate the summation over first and second associates, respectively, of the ith treatment.

Following the method of solving two associate PBNB designs, from Soundarapandian (1980b), we get

\[
\sum_{i_1} Q_j = A_{22} t_i + B_{22} \sum_{i_1} t_j \tag{1.3.10}
\]

where
\[ KA_{12} = (RK - \Lambda) + \Lambda_2 \]
\[ KB_{12} = \Lambda_2 - \Lambda_1 \]
\[ KA_{22} = (\Lambda_2 - \Lambda_1)B_{12}^2 \]
\[ KB_{22} = (RK - \Lambda) + \Lambda_2 + (\Lambda_2 - \Lambda_1)(B_{11}^2 - B_{12}^2) \]

Solving we get
\[ t_i = \frac{Q_{12}B_{22} + B_{12} \sum_{1i} Q_i}{\Delta^*} \]
\[ = \frac{(B_{22} + B_{12}) Q_i + B_{12} \sum_{2i} Q_i}{\Delta^*} \]

where
\[ \Delta^* = A_{12}B_{22} - A_{22}B_{12} \]

Taking (1.2.11) for \((n_2 < n_1)\), we get the variance of difference
\[ V(t_i - t_k) = \frac{2(B_{22} + B_{12}) \sigma^2}{\Delta^*} \] if \(t_i\) and \(t_k\) are first associate and
\[ V(t_i - t_k) = \frac{2 B_{22} \sigma^2}{\Delta^*} \] if \(t_i\) and \(t_k\) are second associate

Combined analysis (intra/interblock) can be obtained by changing \(Q, RA\) by \(Q'(c), R^*\). \(\Lambda\) and dropping \(\sigma^2\) as the expression for variance. More associate cases and other special cases of n-ary designs may also be considered as above. The readers can refer Soundarapandian (1980a, 1981d) for further discussion.