Chapter 2

Ultra $L$-Topologies in the Lattice of $L$-Topologies

2.1 Introduction

In the paper ‘On the combination of topologies’ [11], G.Birkhoff proved that the collection of all topologies on a given set $X$ forms a complete lattice. Birkhoff’s ordering was the natural one of set inclusion; that is, if $\tau$ and $\tau'$ are topologies on a given set $X$, $\tau$ is less than or equal to $\tau'$ if and only if $\tau$ is a subset of $\tau'$. The least element is the indiscrete topology and the greatest element is the discrete topology. In the above lattice, the least upperbound of a collection of topologies is the topology generated by

Some results of this chapter are included in the following paper.
their union and the greatest lower bound is their intersection. Since 1936, many topologists, Vaidynathaswamy [66], Otto Fröhlich [18], Hartmanis [25], Steiner [58], Van Rooij [68] have investigated several properties of this lattice.

In [30] Johnson studied the lattice structure of the set of all $L$-topologies on a given set $X$. The least upper bound of a collection of $L$-topologies is the $L$-topology generated by their union and the greatest lower bound is their intersection. In this paper Johnson proved that this lattice is complete, atomic and not complemented. Also he showed that it is neither modular nor dually atomic in general. In [18] Fröhlich determined the ultra spaces (ultra topologies) on a set $X$, and he proved that if $|X| = n$, there are $n(n-1)$ principal ultra topologies in the lattice of topologies on a set $X$. In [59] Steiner studied some topological properties of the ultra spaces. A related problem in the lattice of $L$-topologies is to identify the ultra $L$-topologies in the lattice of $L$-topologies. In this chapter we show that if $|X| = n$ and $L$ is a finite pseudocomplemented chain or a Boolean lattice, there are $n(n-1)mk$ principle ultra $L$-topologies, where $m$ and $k$ are the number of dual atoms and atoms in $L$ respectively. If $X$ is infinite, there are $|X|$ principal ultra $L$-topologies and $|X|$ nonprincipal ultra $L$-topologies. Also we study some topological properties of the ultra $L$ topologies and characterise $T_0, T_1, T_2 L$-topologies.

### 2.2 Preliminaries

Let $X$ be a non empty ordinary set and $L = L(\leq, \lor, \land, ')$ be a completely distributive lattice with the smallest element 0 and the largest element
1((0 \neq 1) and with an order reversing involution \( a \rightarrow a' \) called \( F \)-lattice [34](which is also called Hutton algebra in e.g., [47]). We denote the constant function in \( L^X \) taking the value \( \alpha \in L \) by \( \alpha \). Here we call \( L \)-fuzzy subsets as \( L \)-subsets and a subset \( F \) of \( L^X \) is called an \( L \)-topology in the sense of Chang [13] and Goguen [23] as in [34] if

(i) \( 0, 1 \in F \)

(ii) \( f, g \in F \Rightarrow f \land g \in F \)

(iii) \( f_i \in F \) for each \( i \in I \Rightarrow \bigvee_{i \in I} f_i \in F \).

In this chapter, \( L \)-filter on \( X \) are defined according to the definition given by Katsaras [33] and Srivastava and Gupta[56] by taking a \( F \)-lattice \( L \) to be the membership lattice, instead of the closed unit interval \([0, 1]\).

**Definition 2.2.1.** A non empty subset \( \mathcal{U} \) of \( L^X \) is said to be an \( L \)-filter if

(i) \( \emptyset \notin \mathcal{U} \)

(ii) \( f, g \in \mathcal{U} \) implies \( f \land g \in \mathcal{U} \) and

(iii) \( f \in \mathcal{U}, g \in L^X \) and \( g \geq f \) implies \( g \in \mathcal{U} \).

An \( L \)-filter is said to be an ultra \( L \)-filter if it is not properly contained in any other \( L \)-filter.
Definition 2.2.2. Let \( x \in X, \lambda \in L \) An \( L \)-point \( x_\lambda \) is defined by

\[
x_\lambda(y) = \begin{cases} 
\lambda & \text{if } y = x \\
0 & \text{if } y \neq x 
\end{cases} \text{ where } 0 < \lambda \leq 1 
\]

Definition 2.2.3. In a filter \( \mathcal{U} \), if there is an \( L \)-subset with finite support, then \( \mathcal{U} \) is called a principal \( L \)-filter.

Example 2.2.1. Let \( \mathcal{U} = \{ f \in L^X | f \geq x_\lambda, \text{where } x_\lambda \text{ is an } L \text{-point} \} \). Then \( \mathcal{U} \) is a principal \( L \)-filter.

Definition 2.2.4. In a filter \( \mathcal{U} \), if there is no \( L \)-subset with finite support, then \( \mathcal{U} \) is called a non principal \( L \)-filter.

Example 2.2.2. Let \( \mathcal{U} = \{ f \in L^X | f > 0 \text{ for all but finite number of points} \} \). Then \( \mathcal{U} \) is a nonprincipal \( L \)-filter.

Let \( f \) be a nonzero \( L \)-subset with finite support. Then \( \mathcal{U}(f) \subset L^X \) defined by \( \mathcal{U}(f) = \{ g \in L^X | g \geq f \} \) is an \( L \)-filter on \( X \), called the principal \( L \)-filter at \( f \). Every \( L \)-filter is contained in an ultra \( L \)-filter. From the definition it follows that on a finite set \( X \), there are only principal ultra \( L \)-filters.

2.3 Ultra \( L \)-topologies

An \( L \)-topology \( F \) on \( X \) is an ultra \( L \)-topology if the only \( L \)-topology on \( X \) strictly finer than \( F \) is the discrete \( L \)-topology.

Definition 2.3.1. [62] Let \((X, F)\) be an \( L \)-topological space and
suppose that $g \in L^X$ and $g \notin F$. Then the collection $F(g) = \{g_1 \lor (g_2 \land g) | g_1, g_2 \in F\}$ is called the simple extension of $F$ determined by $g$.

**Theorem 2.3.1.** [62] Let $(X, F)$ be an $L$-topological space and suppose that $F(g)$ be the simple extension of $F$ determined by $g$. Then $F(g)$ is an $L$-topology on $X$.

**Theorem 2.3.2.** [62] Let $F$ and $G$ be two $L$-topologies on a set $X$ such that $G$ is a cover of $F$. Then $G$ is a simple extension of $F$.

**Theorem 2.3.3.** [18] The ultraspaces on a set $E$ are exactly the topologies of the form $\mathcal{S}(x, \mathcal{U}) = \wp(E - \{x\}) \cup \mathcal{U}$ where $x \in E$ and $\mathcal{U}$ is an ultrafilter on $E$ not containing $\{x\}$.

Analogously we can define ultra $L$-topologies in the lattice of $L$-topologies according to the nature of lattices. If it contains principal ultra $L$-filter, then it is called principal ultra $L$-topology and if it contains non principal ultra $L$-filter, it is called non principal ultra $L$-topology.

**Theorem 2.3.4.** [3] A principal $L$-filter at $x_\lambda$ on $X$ is an ultra $L$-filter iff $\lambda$ is an atom in $L$.

**Theorem 2.3.5.** Let $a$ be a fixed point in $X$ and $\mathcal{U}$ be an ultra $L$-filter not containing $a_\alpha, 0 \neq \alpha \in L$. Define $\mathcal{F}_a = \{f \in L^X | f(a) = 0\}$. Then $\mathcal{S} = \mathcal{S}(a, \mathcal{U}) = \mathcal{F}_a \cup \mathcal{U}$ is an $L$-topology.

**Proof.** Can be easily proved.

**Theorem 2.3.6.** If $X$ is a finite set having $n$ elements and $L$ is a finite pseudo complemented chain or a Boolean lattice, there are $n(n - 1)mk$
principal ultra $L$-topologies, where $m$ and $k$ are the number of dual atoms and atoms in $L$ respectively. If $k = m$ there are $n(n - 1)m^2$ ultra $L$-topologies.

Illustration:

1. Let $X = \{a, b, c\}, L = \{0, \alpha, \beta, 1\}$, a pseudo complemented chain. Here $\alpha$ is the atom and $\beta$ is the dual atom. (Refer figure 2.1)

\[
\begin{array}{c}
\text{1} \\
\beta \\
\alpha \\
0
\end{array}
\]

Figure 2.1:

Let $\mathcal{S} = \mathcal{S}(a, \mathcal{U}(b_\alpha)) = \{f | f(a) = 0\} \cup \{f | f \geq b_\alpha\}$, $\mathcal{S}$ does not contain the $L$-points $a_\alpha, a_\beta, a_1$. Then $\mathcal{S}(a, \mathcal{U}(b_\alpha), a_\beta) = \mathcal{S}(a_\beta) = \text{simple extension of } \mathcal{S} \text{ by } a_\beta = \{f \lor (g \land a_\beta) | f, g \in \mathcal{S}, a_\beta \notin \mathcal{S}\}$ is an ultra $L$-topology, since $\mathcal{S}(a_1)$ is the discrete $L$-topology. Similarly

if $\mathcal{S} = \mathcal{S}(a, \mathcal{U}(c_\alpha))$, then $\mathcal{S}(a_\beta)$ is an ultra $L$-topology.

if $\mathcal{S} = \mathcal{S}(b, \mathcal{U}(a_\alpha))$, then $\mathcal{S}(b_\beta)$
if $\mathcal{G} = \mathcal{G}(b, U(c))$, then $\mathcal{G}(b_\beta)$

if $\mathcal{G} = \mathcal{G}(c, U(a))$, then $\mathcal{G}(c_\beta)$

if $\mathcal{G} = \mathcal{G}(c, U(b))$, then $\mathcal{G}_{c_\beta}$

Number of ultra $L$-topologies $= 6 = 3 \times 2 \times 1 = n(n - 1)m^2$, where $n = 3, k = m = 1$.

2. Let $X = \{a, b, c\}, L = \text{Diamond lattice} \{0, \beta_1, \beta_2, 1\}$. (Refer figure 2.2)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.2.png}
\caption{Figure 2.2:}
\end{figure}

Here $\beta_1$ and $\beta_2$ are the atoms as well as the dual atoms. Let $\mathcal{G} =$
\[ \mathcal{G}(a, \mathcal{U}(b_{\beta})) = \{ f | f(a) = 0 \} \cup \{ f | f \geq b_{\beta} \} , \]
does not contain the \( L \)-points \( a_{\beta 1}, a_{\beta 2}, a_1 \). Then the simple extension \( \mathcal{G}(a_{\beta 1}) \) contains the \( L \)-point \( a_{\beta 1} \) also. Let \( \mathcal{G}_1 = \mathcal{G}(a_{\beta 1}) \). Then the simple extension \( \mathcal{G}_1(a_{\beta 2}) \) contains all \( L \)-points and hence it is discrete. So \( \mathcal{G}(a_{\beta 1}) = \mathcal{G}(a, \mathcal{U}(b_{\beta 1}), a_{\beta 1}) \) is an ultra \( L \)-topology. Similarly the simple extension \( \mathcal{G}(a_{\beta 2}) = \mathcal{G}(a, \mathcal{U}(b_{\beta 1}), a_{\beta 2}) \) is an ultra \( L \)-topology. If \( \mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_{\beta 1})) \), then the simple extensions \( \mathcal{G}(a_{\beta 1}) \) and \( \mathcal{G}(a_{\beta 2}) \) are ultra \( L \)-topologies. That is corresponding to the elements \( a \) and \( b \) there are 4 ultra \( L \)-topologies. Similarly corresponding to the elements \( a \) and \( c \), there are 4 ultra \( L \)-topologies. So there are 8 ultra \( L \)-topologies corresponding to \( a \). Similarly there are 8 ultra \( L \)-topologies corresponding to \( b \) and 8 ultra \( L \)-topologies corresponding to \( c \). Hence total number of ultra \( L \)-topologies

\[ = 8 + 8 + 8 = 24 = 3 \times 2 \times 2 \times 2 = n(n - 1)m^2, \] where \( n = 3, k = m = 2 \).

3. Let \( X = \{ a, b, c \}, L = \mathcal{V}(X) = \{ \phi, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, X \}. \alpha_1 = \{ a \}, \alpha_2 = \{ b \}, \alpha_3 = \{ c \}, \beta_1 = \{ a, b \}, \beta_2 = \{ a, c \}, \beta_3 = \{ b, c \}. \) Atoms are \( \alpha_1, \alpha_2, \alpha_3 \) and dual atoms are \( \beta_1, \beta_2, \beta_3 \). (Refer figure 2.3)

Let \( \mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_{\alpha})) = \{ f | f(a) = 0 \} \cup \{ f | f \geq b_{\alpha} \} \), does not contain the \( L \)-points \( a_{\alpha 1}, a_{\alpha 2}, a_{\alpha 3}, a_{\beta 1}, a_{\beta 2}, a_{\beta 3}, a_1 \). Let \( \mathcal{G}_1 = \) Simple extension of \( \mathcal{G} \) by \( a_{\alpha 1} \) denoted by \( \mathcal{G}(a_{\alpha 1}) \). Then \( \mathcal{G}_1 \) contains more \( L \)-subsets than \( \mathcal{G} \), but not discrete \( L \)-topology. Let \( \mathcal{G}_2 = \mathcal{G}_1(a_{\beta 2}) \), simple extension of \( \mathcal{G}_1 \) by \( a_{\beta 2} \). Then \( \mathcal{G}_2 \) contain more \( L \) subsets than \( \mathcal{G}_1 \) but not discrete \( L \)-topology.

Let \( \mathcal{G}_3 = \mathcal{G}_2(a_{\beta 3}) \), simple extension of \( \mathcal{G}_2 \) by \( a_{\beta 3} \), which is a discrete \( L \)-topology. Hence \( \mathcal{G}_2 = \mathcal{G}_1(a_{\beta 2}) \) is an ultra \( L \)-topology, which is the \( L \)-topology generated by \( \mathcal{G}(a_{\alpha 1}) \) and \( \mathcal{G}(a_{\beta 2}) \). Also \( L \)-topology generated by \( \mathcal{G}(a_{\alpha 1}) \) and \( \mathcal{G}(a_{\beta 3}) \) and \( L \)-topology generated by \( \mathcal{G}(a_{\beta 2}) \) and \( \mathcal{G}(a_{\beta 3}) \) are ultra \( L \)-topologies. That is if \( \mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_{\alpha 1})) \), there are 3 ultra \( L \)-topologies. Similarly if \( \mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_{\alpha 2})) \), there are 3 ultra \( L \)-topologies.
2.3. Ultra L-topologies

and if $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_{o3}))$, there are 3 ultra $L$-topologies. So corresponding to the elements $a, b$ there are 9 ultra $L$-topologies. Similarly corresponding to the elements $a, c$ there are 9 ultra $L$-topologies. Hence there are 18 ultra $L$-topologies corresponding to the element $a$. Similarly corresponding to each element $b$ and $c$ there are 18 ultra $L$-topologies. So total number of ultra $L$-topologies $= 54 = 3*2*3*3 = n(n-1)m^2, n = 3, k = m = 3$.

4. Let $X = \{a, b, c, d\}, L = \varphi(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{c, d, a\}, X\}$. Let $\{a\} = \alpha_1, \{b\} = \alpha_2, \{c\} = \alpha_3, \{d\} = \alpha_4, \{a, b\} = \gamma_1, \{a, c\} = \gamma_2, \{a, d\} = \gamma_3, \{b, c\} = \gamma_4, \{b, d\} = \gamma_5, \{c, d\} = \gamma_6, \{a, b, c\} = \beta_1, \{a, b, d\} = \beta_2, \{b, c, d\} = \beta_3, \{c, d, a\} = \beta_4$. (Refer figure 2.4)

If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_{o1}))$, there are 4 ultra $L$-topologies.
Figure 2.4:
If $S = S(a, U(b_2))$, 

If $S = S(a, U(b_3))$, 

If $S = S(a, U(b_4))$, 

So corresponding to the elements $a, b$, there are 16 ultra $L$-topologies. Similarly corresponding to the elements $a, c$, there are 16 ultra $L$-topologies and corresponding to the elements $a, d$, there are 16 ultra $L$-topologies. Hence there are 48 ultra $L$-topologies corresponding to the element $a$. Similarly corresponding to each elements $b, c$ and $d$, there are 48 ultra $L$-topologies. So total number of ultra $L$-topologies = $48 \times 4 = 192 = n(n - 1)m^2$, $n = 4, k = m = 4$. In general if $|X| = n$ and $L$ is a finite pseudo complemented chain or a Boolean lattice, there are $n(n - 1)m^2$ ultra $L$-topologies where $m$ and $k$ are the number of dual atoms and number of atoms respectively. If $k = m$, it is equal to $n(n - 1)m^2$.

**Remark 2.3.1.** If $L$ is neither a finite pseudo complemented chain nor a Boolean lattice, we cannot identify the principal ultra $L$-topologies in this way. But we can identify ultra $L$-topology in certain cases.

**Example 2.3.1.** Let $X = \{a, b, c\}, L = D_{12} = \{1, 2, 3, 4, 6, 12\}$

Here the atoms are $\alpha_1 = 2, \alpha_2 = 3$ and dual atoms are $\beta_1 = 4, \beta_2 = 6$. If $S = S(a, U(b_{\alpha_1})) = \{f|f(a) = 0\} \cup \{f|f \geq b_{\alpha_1}\}$, $L$-topology generated by $S(a_{\beta_1})$ and $S(a_{\beta_2})$ does not contain the $L$-point $a_{\alpha_2}$. It is not a discrete $L$-topology. So we cannot say that $S(a_{\beta_1})$ is a principal ultra $L$-topology. But $L$-topology generated by $S(a_{\beta_1})$ and $S(a_{\beta_2})$ is a principal ultra $L$-topology.
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Figure 2.5:

**Theorem 2.3.7.** If $X$ is infinite and $L$ is a finite pseudo complemented chain or a Boolean lattice, there are $|X|$ principal ultra $L$-topologies and $|X|$ non principal ultra $L$-topologies.

**Illustration:**

If $X$ is countably infinite, we have $|X|$, cardinality of $X = \aleph_0$ and If $X$ is uncountable, we have $|X| > \aleph_0$

**Case 1.**

$X$ is infinite and $L$ is finite

Let $X = \{a, b, \ldots\}$, $L = \{0, \alpha, \beta, 1\}$ a pseudo complemented chain.
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Let $\mathcal{S} = \mathcal{S}(a, \mathcal{U}(b_a)) = \{ f | f(a) = 0 \} \cup \{ f | f \geq b_a \}$. $\mathcal{S}$ does not contain the $L$ points $a_\alpha, a_\beta, a_1$. Here $\mathcal{S}(a_\beta) = \mathcal{S}(a, \mathcal{U}(b_a), a_\beta)$ is a principal ultra $L$-topology since $\mathcal{S}(a_1)$ is the discrete $L$-topology, where $\mathcal{S}(a_\beta)$ is the simple extension of $\mathcal{S}$ by $a_\beta$. Similarly we can identify other ultra $L$-topologies. Hence corresponding to the element $a$, there are $|X| - 1 = |X|$ principal ultra $L$-topologies. Similarly corresponding to each element $b, c, d, \ldots$ there are $|X|$ principal ultra $L$-topologies. So total number of principal ultra $L$-topologies = $|X||X| = |X|$. Let $\mathcal{S} = \mathcal{S}(a, \mathcal{U}) = \{ f | f(a) = 0 \} \cup \mathcal{U}$, where $\mathcal{U}$ is a nonprincipal ultra $L$-filter not containing $a_\lambda, 0 \neq \lambda \in L$. Then the simple extension of $\mathcal{S}$ by $a_\beta = \mathcal{S}(a_\beta) = \mathcal{S}(a, \mathcal{U}, a_\beta)$ is a nonprincipal ultra $L$-topology since $\mathcal{S}(a_1)$ is discrete $L$-topology. So there are $|X|$ non principal ultra $L$-topologies.

Case 2.

$X$ and $L$ are infinite

Let $X = \{ a, b, c, \ldots \}, L = \varnothing(X)$. There are $|X|$ atoms and $|X|$ dual atoms. Number of principal ultra $L$-topologies corresponding to one element = $|X||X|(|X| - 1) = |X|$. Hence total number of principal ultra $L$-topologies = $|X||X| = |X|$. Let $\mathcal{S} = \mathcal{S}(a, \mathcal{U}) = \{ f | f(a) = 0 \} \cup \mathcal{U}$, where $\mathcal{U}$ is a nonprincipal ultra filter not containing $a_\lambda, 0 \neq \lambda \in L$. There are $|X|$ nonprincipal ultra $L$-filters not containing $a_\lambda$ so that corresponding to $a$ there are $|X||X| = |X|$ nonprincipal ultra $L$-topologies. So total number of nonprincipal ultra $L$-topologies = $|X||X| = |X|$. 


2.4 Topological Properties

(a). Principal Ultra $L$-topologies

Let $X$ be a non empty set and $L$ is a finite pseudo complemented chain. If $S = S(a, U(b_\lambda)) = \{ f \mid f(a) = 0 \} \cup \{ f \mid f \geq b_\lambda \}$, then a principal ultra $L$-topology is $S(a, U(b_\lambda), a_\beta) = S(a_\beta)$, which is the simple extension of $S$ by $a_\beta$. i.e., $S(a_\beta) = \{ f \lor (g \land a_\beta) \mid f, g \in S, a_\beta \notin S \}$, where $a, b \in X, \lambda$ and $\beta$ are the atom and dual atom in $L$ respectively.

Let $X$ be a non empty set and $L$ is a finite Boolean lattice. If $S = S(a, U(b_\lambda)) = \{ f \mid f(a) = 0 \} \cup \{ f \mid f \geq b_\lambda \}$ where $a, b \in X, \lambda$ is an atom, then a principal ultra $L$-topology denoted by $S(\beta_j) = S(a, U(b_\lambda)) = L$-topology generated by any $(m-1)$ $S(\beta_i)$ among $m$ $S(\beta_i), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$ if there are $m$ dual atoms $\beta_1, \beta_2, ..., \beta_m$, where $S(\beta_i) = S(a, U(b_\lambda), a_\beta)$.

Definition 2.4.1. An $L$-topology $F$ is said to be a $T_0$-$L$ topology if for every two distinct $L$-points $x_\lambda$ and $y_\gamma$ with distinct support, there is an open $L$ subset containing one and not the other.

Definition 2.4.2. An $L$-topology $F$ is said to be a $T_1$-$L$ topology if for every two distinct $L$-points $x_\lambda$ and $y_\gamma$, with distinct support, there exists an $f \in F$ such that $x_\lambda \in f$ and $y_\gamma \notin f$ and another $g \in F$ such that $y_\gamma \in g$ and $x_\lambda \notin g \ \forall \lambda, \gamma \in L \setminus \{0\}$.

Definition 2.4.3. An $L$-topology $F$ is said to be a $T_2$-$L$ topology if for every two distinct $L$-points $x_\lambda$ and $y_\gamma$, with distinct support, there exists $f, g \in F$ such that $x_\lambda \in f$ and $y_\gamma \in g$ with $f \land g = 0$. 
Theorem 2.4.1. Let $X$ be a non empty set and $L$ is a finite pseudo complemented chain or a Boolean lattice. Then every principal ultra $L$-topology $\mathcal{G}_{\beta j} = \mathcal{G}_{\beta j}(a, \mathcal{U}(b_\lambda))$ is $T_0$-$L$ topology but not $T_1$-$L$ topology.

Example 2.4.1. Let $X$ be a non empty set

Suppose that $L$ is a finite pseudo complemented chain and $a, b \in X, \lambda, \beta$ are atom and dual atom in $L$ respectively. Take two distinct $L$-points $a_1, b_\lambda, b_\lambda$ is an open $L$ subset contain $b_\lambda$ but not $a_1$. Since $\mathcal{U}(b_\lambda) = \{f | f \geq b_\lambda\}$, any open set contains $a_1$ must contain $b_\lambda$. So $\mathcal{G}_{\beta j} = \mathcal{G}_{\beta j}(a, \mathcal{U}(b_\lambda))$ is a $T_0$-$L$ topology but not $T_1$-$L$ topology.

Suppose that $L$ is a finite Boolean lattice and $a, b \in X, \lambda$ is an atom and $\beta_1, \beta_2, \ldots$ are dual atoms in $L$. Take two distinct $L$-points $a_1, b_\lambda$. $b_\lambda$ is an open $L$-subset that contains $b_\lambda$ but not $a_1$. Since $\mathcal{U}(b_\lambda) = \{f | f \geq b_\lambda\}$, any open set contains $a_1$ must contain $b_\lambda$. So the principal ultra $L$-topology $\mathcal{G}_{\beta j} = \mathcal{G}_{\beta j}(a, \mathcal{U}(b_\lambda))$ is $T_0$-$L$ topology but not $T_1$-$L$ topology.

Definition 2.4.4. An $L$-topological space $(X, F), F \subseteq L^X$ is called door $L$-space if every $L$-subset $g$ of $X$ is either $L$-open or $L$-closed in $F$.

Example 2.4.2. Let $X = \{a, b\}$ and $L = \{0, .5, 1\}$. Define $f_1(a) = 0, f_1(b) = 0, f_2(a) = 0, f_2(b) = .5, f_3(a) = 0, f_3(b) = 1, f_4(a) = .5, f_4(b) = 0, f_5(a) = .5, f_5(b) = .5, f_6(a) = .5, f_6(b) = 1, f_7(a) = 1, f_7(b) = 0, f_8(a) = 1, f_8(b) = .5, f_9(a) = 1, f_9(b) = 1$. Let $F = \{f_1, f_9, f_2, f_3, f_4, f_5, f_6\}$. Then $f_7$ and $f_8$ are closed $L$-subsets. So $(X, F)$ is a door $L$-space.

Let $X = \{a, b, c\}, L = [0, 1]$ and the the $L$-topology $F = \{0, \mu_{\{a\}}, \mu_{\{b,c\}}, 1\}$. Then $(X, F)$ is not a door $L$-space since $\mu_{\{b\}}$ is neither an $L$-open set nor an $L$-closed set.
In a principal ultra $L$-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathcal{U}(b_\lambda))$ every $L$-subset of $X$ is either open or closed if $L$ is a finite pseudo complemented chain or a Boolean lattice. So every principal ultra $L$-topological space $\mathfrak{S}_{\beta j}$ is a door $L$ space.

**Definition 2.4.5.** An $L$-topological space $(X, F)$ is said to be regular at an $L$-point $a_\lambda$ if for every closed $L$ subset $h$ of $X$ not containing $a_\lambda$, there exists disjoint open sets $f, g$ such that $a_\lambda \in f$ and $h \in g$. $(X, F)$ is said to be regular $L$-topology if it is regular at each of its $L$-points.

**Theorem 2.4.2.** Let $X$ be a non empty set and $L = \wp(X)$. Then the principal ultra $L$-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathcal{U}(b_\lambda))$ is not regular if $|X| \geq 3$.

**Example 2.4.3.** Let $X = \{a, b, c\}, L = \wp(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. $\alpha_1 = \{a\}, \alpha_2 = \{b\}, \alpha_3 = \{c\}, \beta_1 = \{a, b\}, \beta_2 = \{a, c\}, \beta_3 = \{b, c\}$. Atoms are $\alpha_1, \alpha_2, \alpha_3$ and dual atoms are $\beta_1, \beta_2, \beta_3$. Take $\lambda = \alpha_1$ in the principal ultra $L$-topology $\mathfrak{S}_{\beta 3}$, which is an $L$-topology generated by $\mathfrak{S}(a, \mathcal{U}(b_\lambda), a_{\beta 1})$ and $\mathfrak{S}(a, \mathcal{U}(b_\lambda), a_{\beta 2})$. Consider the point $a_{\beta 1}$ and then $a_{\alpha 3}$ is a closed $L$ subset not containing $a_{\beta 1}$. Consider the open sets $f, g$ such that $f(a) = \beta_1, f(b) = \alpha_1, f(c) = \alpha_1, g(a) = \beta_2, g(b) = 0, g(c) = 0$. $f$ is an open set containing $a_{\beta 1}$ and $g$ is an open set containing $a_{\alpha 3}$ but $f \wedge g \neq 0$. That is $f$ and $g$ are not disjoint.

**Definition 2.4.6.** An $L$ topological space $(X, F)$ is said to be normal if for every two disjoint closed $L$ subsets $h$ and $k$, there exists two disjoint open $L$ subsets $f, g$ such that $h \in f$ and $k \in g$.

**Theorem 2.4.3.** Let $X$ be a non empty set and $L = \wp(X)$. Then the principal ultra $L$-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathcal{U}(b_\lambda))$ is not a normal $L$-topology if $|X| \geq 3$. 
Example 2.4.4. Let \( X = \{a, b, c\} \), \( L = \wp(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \). \( \alpha_1 = \{a\}, \alpha_2 = \{b\}, \alpha_3 = \{c\}, \beta_1 = \{a, b\}, \beta_2 = \{a, c\}, \beta_3 = \{b, c\} \). Atoms are \( \alpha_1, \alpha_2, \alpha_3 \) and dual atoms are \( \beta_1, \beta_2, \beta_3 \). Take \( \lambda = \alpha_1 \) in the principal ultra \( L \)-topology \( \mathcal{G}_{\beta_3} \). Then \( a_\alpha_2 \) and \( a_\alpha_3 \) are disjoint closed \( L \)-subsets. There is no disjoint open \( L \) subsets containing \( a_\alpha_2 \) and \( a_\alpha_3 \).

(b). Non Principal Ultra \( L \)-topology

Let \( X \) be an infinite set and \( L \) is a finite pseudo complemented chain. If \( \mathcal{S} = \mathcal{G}(a, \wp) = \{f \mid f(a) = 0\} \cup \wp \) where \( \wp \) is a non principal ultra \( L \)-filter not containing \( a_\lambda, 0 \neq \lambda \in L \). Then the non principal ultra \( L \)-topology = \( \mathcal{G}(a, \wp, a_\beta) = \mathcal{G}(a_\beta) \), is the simple extension of \( \mathcal{G} \) by \( a_\beta \), i.e., \( \mathcal{G}(a_\beta) = \{f \lor (g \land a_\beta), f, g, \in \mathcal{G}, a_\beta \not\in \mathcal{G}\} \), where \( a \in X, \beta \) is the dual atom in \( L \).

Let \( X \) be an infinite set and \( L \) be a Boolean lattice. If \( \mathcal{G} = \mathcal{G}(a, \wp) \), then a non principal ultra \( L \)-topology denoted by \( \mathcal{G}_{\beta_j} = \mathcal{G}_{\beta_j}(a, \wp) \) is the \( L \)-topology generated by any \( (m - 1) \mathcal{G}(a_\beta_i) \) among \( m \mathcal{G}(a_\beta_i), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j \) if there are \( m \) dual atoms \( \beta_1, \beta_2, ... \beta_m \) where \( \mathcal{G}(a_\beta_i) = \mathcal{G}(a, \wp, a_\beta_i) \).

Theorem 2.4.4. Every non principal ultra \( L \)-topology \( \mathcal{G}_{\beta_j}(a, \wp) \) is a \( T_1-L \) topology.

Proof. Let \( X \) be an infinite set and \( \mathcal{G}_{\beta_j} = \mathcal{G}_{\beta_j}(a, \wp) \) be a non principal ultra \( L \)-topology. Let \( a_\alpha, b_\beta \) be any two distinct \( L \)-points, \( a, b \in X, \alpha, \beta \in L \). Since \( \wp \) is a non principal ultra \( L \)-filter, there exists \( L \) open sets containing each \( L \)-points but not the other. \( \square \)
Theorem 2.4.5. Every non principal ultra $L$ topology $\mathcal{S}_{\beta_1}(a, \mathcal{U})$ is a $T_2$-$L$ topology.

Proof. Let $X$ be an infinite set and $\mathcal{S}_{\beta_1}(a, \mathcal{U})$ be a non principal ultra $L$-topology. Take two distinct $L$-points $a_\alpha, b_\beta$, where $a, b, \in X, \alpha, \beta \in L$. Since $\mathcal{U}$ is a non principal ultra $L$-filter, we can find disjoint open sets $f$ and $g$ such that $a_\alpha \in f, b_\beta \notin f$ and $b_\beta \in g, a_\alpha \notin g$. □

Theorem 2.4.6. Suppose that $X$ is an infinite set and $L$ is a Boolean lattice. Then every non principal ultra $L$-topology is a door $L$-space.

Proof. Let $X$ be an infinite set, $L$ be a Boolean lattice and $\mathcal{S}_{\beta_1}(a, \mathcal{U})$ be a non principal ultra $L$-topology. Since $L$ is a Boolean Lattice, it is complemented. So every $L$-subset of $X$ is either $L$-closed or $L$-open in $\mathcal{S}_{\beta_1}(a, \mathcal{U})$. Since $a$ and $\beta_1$ are arbitrary, every non principal ultra $L$-topology is door $L$-space. □

Remark 2.4.1. If $L$ is not a complemented $F$-lattice except a finite pseudo complemented chain, $\mathcal{S}_{\beta_1}(a, \mathcal{U})$ is not a door $L$-space.

Example 2.4.5. Let $X$ be a nonempty set and $L = D_{12} = \{1, 2, 3, 4, 6, 12\}$. Here the atoms are $\alpha_1 = 2, \alpha_2 = 3$ and dual atoms are $\beta_1 = 4, \beta_2 = 6$ (Refer figure 2.5). Take the non principal ultra $L$ topology $\mathcal{S}_{\beta_1}(a, \mathcal{U})$. The $L$ point $a_{\beta_2}$ is not open in $\mathcal{S}_{\beta_1}(a, \mathcal{U})$. Since $L$ is not complemented, the $L$ point $a_{\beta_2}$ is not closed also in $\mathcal{S}_{\beta_1}(a, \mathcal{U})$.

Theorem 2.4.7. If $X$ is an infinite set and $L$ is a finite pseudo complemented chain or a diamond lattice, then the non principal ultra $L$-topology $\mathcal{S}_{\beta_1}(a, \mathcal{U})$ is a regular $L$-topology.
2.4. Topological Properties

Proof. It is trivial. □

**Theorem 2.4.8.** Let $X$ be an infinite set and $L = \wp(X)$. Then the non principal ultra $L$-topology $\mathcal{S}_{\beta_j}(a, U)$ is not a regular $L$-topology.

Proof. Let $X = \{a, b, c,\ldots\}, L = \wp(X)$. Let $\alpha_1, \alpha_2, \ldots$ be atoms and $\beta_1, \beta_2, \ldots$ are dual atoms in $L$. Consider $a_{\beta_1}$. Then there exists a closed $L$ subset $a_{\alpha_i}$ for some $i$ not containing $a_{\beta_1}$. But we cannot find disjoint open $L$ subsets $f$ and $g$ such that $f$ contains $a_{\beta_1}$ and $g$ contains $a_{\alpha_i}$. □

**Theorem 2.4.9.** If $X$ is an infinite set and $L$ is a finite pseudo complemented chain or a diamond lattice, the non principal ultra $L$-topology $\mathcal{S}(a, U, a_\beta)$ is a normal $L$-topology.

Proof. It is trivial □

**Theorem 2.4.10.** If $X$ is an infinite set and $L = \wp(X)$ having dual atoms $\beta_1, \beta_2, \ldots$, then the non principal ultra $L$ topology $\mathcal{S}_{\beta_j}(a, U), a \in X$ is not a normal $L$-topology.

Proof. Let $X = \{a, b, c, \ldots\}, L = \wp(X)$. Let $\alpha_1, \alpha_2, \ldots$ be atoms and $\beta_1, \beta_2, \ldots$ are dual atoms in $L$. Then there exists two closed $L$ subsets $a_{\alpha_i}$ and $a_{\alpha_j}$ for some $i$ and $j$. But there does not exists disjoint open $L$ subsets $f$ and $g$ such that $f$ contains $a_{\alpha_i}$ and $g$ contains $a_{\alpha_j}$. □

**Theorem 2.4.11.** Let $X$ is an infinite set and $L$ is a finite pseudo complemented chain or a Boolean lattice. An ultra $L$-topology $F$ is a $T_1-L$ topology if and only if it is a non principal ultra $L$-topology.
Proof. Suppose that the ultra $L$-topology $F$ is a $T_1$-$L$ topology. We have to show that $F$ is a non principal ultra $L$-topology. $F$ is a principal ultra $L$-topology implies $F$ is not a $T_1$-$L$ topology. So we can say that $F$ is a $T_1$-$L$ topology implies $F$ is a non principal ultra $L$-topology.

Next assume that $F$ is a non principal ultra $L$-topology. Then by theorem 2.4.4 $F$ is a $T_1$-$L$ topology. \hfill \Box

**Theorem 2.4.12.** An $L$-topology $F$ on $X$ is a $T_1$-$L$ topology if and only if it is the infimum of non principal ultra $L$-topologies.

**Proof.** Necessary part

Any $L$-topology finer than a $T_1$-$L$ topology must also be a $T_1$-$L$ topology. So a $T_1$-$L$ topology can be the infimum of only non principal ultra $L$ topologies.

Sufficient part

Each non principal ultra $L$-topology on $X$ contains non principal ultra $L$-filter. So there exists distinct $L$-points $a_\lambda, b_\gamma$ where $a, b \in X; \lambda, \gamma \in L$ and $L$-open sets $f, g$ such that $a_\lambda \in f, b_\gamma \notin f$ and $a_\lambda \notin g, b_\gamma \in g$. This is also true in the infimum of any family of non principal ultra $L$-topologies since every $L$-points are closed in non principal ultra $L$-filters. So infimum of any family of non principal ultra $L$-topologies is a $T_1$-$L$ topology. \hfill \Box

**Theorem 2.4.13.** Let $X$ be an infinite set and $L$ is a finite pseudo complemented chain or a Boolean lattice. Then an ultra $L$-topology is a $T_2$-$L$ topology if and only if it is a non principal ultra $L$-topology.

**Proof.** Suppose that an ultra $L$-topology is a $T_2$-$L$ topology. This
2.5. Mixed L- topologies

implies that the ultra $L$-topology is a $T_1$-$L$ topology. Hence it is a non principal ultra $L$-topology.

Conversely suppose that the ultra $L$-topology is a non principal ultra $L$-topology. Since a non principal ultra $L$-topology contains a non principal ultra $L$-filter, for any two distinct $L$-points in the non principal ultra $L$-topology there exists disjoint $L$-open sets contains each $L$-point but not the other. So it is a $T_2$-$L$topology.

\[\square\]

2.5 Mixed L- topologies

In [59] Steiner studied the mixed topologies. Analogously we can say that a mixed $L$-topology on $X$ is not a $T_1$-$L$ topology and does not have a principal representation. Thus a mixed $L$-topology is the intersection of a $T_1$-$L$ topology and a principal $L$-topology.

The representation of a mixed $L$-topology as the infimum of a $T_1$-$L$ topology and a principal $L$-topology need not be unique.

**Example 2.5.1.** Let $\mathcal{C} = \{\mu_A|X - A \text{ is finite}\}$ together with $\emptyset$, is a $T_1$-$L$ topology and $\delta$ and $\delta'$ be the principal $L$-topologies given by $\delta = \bigwedge_{a \in X - \{b,c\}} \mathcal{G}_{\beta_j}$, $\delta' = \bigwedge_{a \in X - \{b\}} \mathcal{G}_{\beta_j}$, if $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_\lambda))$, $\lambda$ is an atom and $\beta_j$’s are dual atoms in $L$.

$\mathcal{C} \wedge \delta = \{\mu_A|b \in A, X - A \text{ is finite or } f = \emptyset\} = \mathcal{C} \wedge \delta'$ is a mixed $L$-topology. Here $c_\lambda \in \delta$ and $c_\lambda \notin \delta'$. That is the representation of a mixed topology as the infimum of $T_1$-$L$ topology and principal $L$-topology need not be unique.