Chapter 9

Lattice of $L$-closure operators

9.1 Introduction

In 1965 Zadeh [77] introduced fuzzy sets as a generalization of ordinary sets. After that Chang [13] introduced fuzzy topology and that led to the discussion of various aspects of $L$-topology by many authors. The Čech closure spaces introduced by Čech. [12] is a generalization of the topological spaces. The theory of fuzzy closure spaces has been established by Mashhour and Ghanim [39] and Srivastava et. al ([45],[54]). The definition of Mashhour and Ghanim is an analogue of Čech closure spaces and Srivastava et. al. have introduced it as an analogue of the definition of closure space given by Dikranjan et. al.[16]. Based on [54], Rekha Srivas-

* Some results of this chapter are included in the following paper. 
tava and Manjari Srivastava studied the subspace of a fuzzy closure space. The notion of $T_0$-fuzzy closure spaces and $T_1$ fuzzy closure spaces were also introduced in [45]. In [43] Ramachandran studied some properties of lattice of closure operators. In [28] Johnson studied some properties of the lattice $L(X)$ of all fuzzy closure operators on a fixed set $X$. In [76] Wu-Neng Zhou introduced the concept of $L$-closure spaces and the convergence in $L$-closure spaces. In this chapter we study properties of the lattice $LC(X)$ of $L$-closure operators and $L$-closure spaces which is a generalization of the concept of fuzzy closure spaces. Here we proved that the complete lattice $LC(X)$ is not modular. Also we identify the infra $L$-closure operator and ultra $L$-closure operator and establish the relation between ultra $L$-topology and ultra $L$-closure operator. We proved that an $L$-closure operator is an ultra $L$-closure operator if and only if it is the $L$-closure operator associated with an ultra $L$-topology. Also proved that infra $L$-closure operators are less than or equal to any non principal ultra $L$-closure operator and no non principal ultra $L$-closure operator has a complement so that the lattice of $L$-closure operators is not complemented in general.

9.2 Preliminaries

A completely distributive lattice $L$ is called a $F$-lattice, if there is an order reversing involution from $L$ to $L$. Let $X$ be any nonempty set and $L$ is a $F$-lattice. The fundamental definition of $L$-fuzzy set theory and $L$-fuzzy topology are assumed to be familiar to the reader as in [34]. Here we call $L$-fuzzy subsets as $L$ subsets and $L$-fuzzy topology as $L$-topology.
9.2. Preliminaries

**Definition 9.2.1.** [39] A Čech fuzzy closure operator on a set $X$ is a function $\chi : I^X \rightarrow I^X$, satisfying the following three axioms

(i). $\chi(0) = 0$,

(ii). $f \leq \chi(f)$ for every $f$ in $I^X$,

(iii). $\chi(f \lor g) = \chi(f) \lor \chi(g)$ where $I = [0, 1]$.

For convenience it is called fuzzy closure operator on $X$ and $(X, \chi)$ is called fuzzy closure space. In [76] Wu-Neng Zhou defined $L$-closure operator as follows.

**Definition 9.2.2.** A mapping $C : L^X \rightarrow L^X$ is called an $L$-closure operator or an $L$-closure, if it satisfies the following conditions for any $A, B \in L^X$:

(i). $C(0_X) = 0_X$,

(ii). $A \leq C(A)$,

(iii). $A \leq B$ implies $C(A) \leq C(B)$,

(iv). $C(C(A)) = C(A)$.

But in this chapter we take the definition of $L$-closure operator as a generalization of fuzzy closure operator in [39]
Definition 9.2.3. Let $X$ be a non empty set and $L$ be a $F$ lattice. An \( L \)-closure operator on \( L^X \) is a mapping \( \psi : L^X \to L^X \) satisfying the following conditions:

(i) \( \psi(\emptyset) = \emptyset \),

(ii) \( f \leq \psi(f) \),

(iii) \( \psi(f \lor g) = \psi(f) \lor \psi(g) \) for every \( f, g \in L^X \).

The pair \((X, \psi)\) is called an \( L \)-closure space. An \( L \)-subset \( f \) of \( X \) is said to be an \( L \)-closed set in \((X, \psi)\) if \( \psi(f) = f \). An \( L \)-subset \( f \) of \( X \) is open if its complement is closed in \((X, \psi)\). The set of all open \( L \)-subsets of \((X, \psi)\) form an \( L \)-topology on \( X \) called the \( L \)-topology associated with the \( L \)-closure operator \( \psi \).

Let \( F \) be an \( L \)-topology on a set \( X \). Then a function \( \psi : L^X \to L^X \) defined by \( \psi(f) = \overline{f} \) for all \( f \in L^X \), where \( \overline{f} \) denotes the closure of \( f \) with respect to \( F \) is called the \( L \)-closure operator associated with the \( L \)-topology \( F \).

An \( L \)-closure operator on a set \( X \) is called \( L \)-topological if it is the \( L \)-closure operator associated with an \( L \)-topology on \( X \). That is \( \psi(\psi(f)) = \psi(f) \) for all \( f \in L^X \). Note that different \( L \)-closure operators can have the same associated \( L \)-topology. But different \( L \)-topologies cannot have the same associated \( L \)-closure operator.

Example 9.2.1. Let \( X = \{a, b, c\}, L = \{0, \alpha, \beta, 1\} \). Let \( \psi_1 : L^X \to \)}
9.3. Lattice of $L$-closure operators

$L^X$ defined by

$$\psi_1(f) = \begin{cases} 0 & \text{if } f = 0 \\ \beta & \text{if } f(x) < \beta, \forall x \in X \\ 1 & \text{otherwise} \end{cases}$$

Then $\psi_1$ is a fuzzy closure operator.

$\psi_2 : L^X \rightarrow L^X$ defined by

$$\psi_2(f) = \begin{cases} 0 & \text{if } f = 0 \\ 1 & \text{otherwise} \end{cases}$$

Then $\psi_2$ is a fuzzy closure operator.

Associated fuzzy topologies of $\psi_1$ and $\psi_2$ are same, which is the indiscrete fuzzy topology.

9.3 Lattice of $L$-closure operators

Let $\psi_1$ and $\psi_2$ be $L$-closure operators on $X$. Then $\psi_1 \leq \psi_2$ if and only if $\psi_2(f) \leq \psi_1(f)$ for every $f$ in $L^X$. The relation $\leq$ defined above is a partial order on the set of all $L$-closure operators on $L^X$. We denote the poset by $LC(X)$. Then $LC(X)$ is a lattice. The $L$-closure operator $D$ on $X$ defined by $D(f) = f$ for every $f$ in $L^X$ is called the discrete $L$-closure operator.

The $L$-closure operator $I$ on $X$ defined by $I(f) = \begin{cases} 0 & \text{if } f = 0 \\ 1 & \text{otherwise} \end{cases}$ is called the indiscrete $L$-closure operator.

Remark 9.3.1. $D$ and $I$ are the $L$-closure operators associated with
the discrete and indiscrete $L$-topologies on $X$ respectively. Moreover $D$ is the unique $L$-closure operator whose associated $L$-topology is discrete. Also $I$ and $D$ are the smallest and the largest elements of $LC(X)$ respectively.

**Theorem 9.3.1.** $LC(X)$ is a complete lattice.

**Proof.** It is enough to show that every subset of $LC(X)$ has greatest lower bound in $LC(X)$. Let $S = \{\chi_j | j \in J\}$ be a subset of $LC(X)$. Then $\sup_{j \in J}\{\chi_j(f)\} = \inf_{j \in J}\{\chi_j\}$ is an $L$-closure operator and is the greatest lower bound of $S$ in $LC(X)$.

**Definition 9.3.1.** [20] A lattice $L$ is called modular if it satisfies the condition $x \geq z$ implies that $(x \land y) \lor z = x \land (y \lor z), \forall x, y, z \in L$.

Lattice of $L$ closure operators $LC(X)$ is modular if and only if $\chi \geq \eta \Rightarrow \chi \land (\psi \lor \eta) = (\chi \land \psi) \lor \eta, \forall \chi, \psi, \eta \in LC(X)$.

**Theorem 9.3.2.** $LC(X)$ is not modular.

**Proof.** Let $X$ be any set and $x \in X$. Define $\psi_x, \chi_x, \eta_x$ from $L^X \rightarrow L^X$ by

\[
\psi_x(0) = 0 \\
\psi_x(f)(y) = \begin{cases} 
  f(y) & \text{if } y \neq x \\
  1 & \text{if } y = x
\end{cases}
\]

\[
\chi_x(0) = 0 \\
\chi_x(f)(y) = \begin{cases} 
  1 & \text{if } y \neq x \\
  f(y) & \text{if } y = x
\end{cases}
\]

\[
\eta_x(0) = 0 \\
\eta_x(f)(y) = \begin{cases} 
  f(y) & \text{if } y \neq x \\
  1 & \text{if } y = x
\end{cases}
\]
\[ \eta_x(0) = 0 \]
\[ \eta_x(f)(y) = \begin{cases} 
1 & \text{if } y \neq x \\
\beta & \text{if } y = x \quad \text{and } \beta \geq f(y)
\end{cases} \]

Then \( \chi_x(f)(y) \leq \eta_x(f)(y) \), \( \forall y \). Hence \( \chi_x \geq \eta_x \).

\[ \chi_x \land \psi_x = \inf(\chi_x, \psi_x) \]
\[ = \sup(\chi_x(f)(y), \psi_x(f)(y)) \]
\[ = 1 \]
\[ (\chi_x \land \psi_x) \lor \eta_x = \inf(1, \eta_x(f)(y)) \]
\[ = f(y) \]
\[ \psi_x \lor \eta_x = \sup(\psi_x, \eta_x) \]
\[ = \inf(\psi_x(f)(y), \eta_x(f)(y)) \]
\[ = f(y) \]
\[ \chi_x \land (\psi_x \lor \eta_x) = \sup(\chi_x(f)(y), f(y)) \]
\[ = 1 \]

Therefore \( \chi_x \land (\psi_x \lor \eta_x) \neq (\chi_x \land \psi_x) \lor \eta_x \)

So \( LC(X) \) is not modular. \( \square \)

**Definition 9.3.2.** An \( L \)-closure operator on \( X \) is called an infra \( L \)-closure operator if the only \( L \)-closure operator on \( X \) strictly smaller than it is \( I \).

Let \( X \) be any set and \( a, b \in X \) such that \( a \neq b \). Define \( \psi_{a,b} : L^X \rightarrow L^X \) by \( \psi_{a,b}(f) = \begin{cases} 
f & \text{if } f = 0 \\
g_{a,b} & \text{if } f = a_a \\
1 & \text{otherwise} \end{cases} \).
\( \alpha \) is a dual atom in \( L \) and \( g_{\alpha,b} \) is defined by 
\[
g_{\alpha,b}(a) = \begin{cases} 
1 & \text{if } a \neq b \\
\alpha & \text{if } a = b
\end{cases}
\]
In the topological context Ramachandran [43] proved that a closure operator on \( X \) is an infra closure operator if and only if it is of the form \( V_{a,b} \) for some \( a, b \) in \( X, a \neq b \), where \( V_{a,b} \) is defined by
\[
V_{a,b}(A) = \begin{cases} 
\phi & \text{if } A = \phi \\
X - \{b\} & \text{if } A = \{a\} \\
X & \text{otherwise}
\end{cases}
\]

Analogously in the \( L \)-topological context we prove the following theorem.

**Theorem 9.3.3.** An \( L \)-closure operator is an infra \( L \)-closure operator if and only if it is of the form \( \psi_{a,b} \) for some \( a, b \in X, a \neq b \).

**Proof.** Let \( \psi \) be an \( L \)-closure operator on \( X \) strictly smaller than \( \psi_{a,b} \), then \( \psi(a_\alpha) \) will be strictly greater than \( \psi_{a,b}(a_\alpha) = g_{a,b} \) and hence equal to \( 1 \) so that \( \psi(f) = 1, \forall f \in L^X \) other than \( 0 \). Hence \( \psi = I \). Thus all \( L \)-closure operators of the form \( \psi_{a,b} \) are infra \( L \)-closure operators.

Conversely let \( \psi \) be any \( L \) closure operator other than \( I \). Then we can find a non zero \( L \)-subset \( f \) such that \( \psi(f) \neq I(f) = 1 \) (ie \( \psi(f) \neq 1 \)) and elements \( a_\alpha, b_\beta \) such that \( a_\alpha \leq f \) and \( b_\beta \) not in \( \psi(f) \). Then \( b_\beta \) is not an element of \( \psi(a_\alpha) \). That is \( b_\beta \notin \psi(a_\alpha) \Rightarrow g_{a,b} \notin \psi(a_\alpha) \). That is \( \psi_{a,b}(a_\alpha) \notin \psi(a_\alpha) \). Also \( \psi_{a,b}(k) = 1 \) for every nonzero \( L \)-subset \( k \) other than \( a_\alpha \). So \( \psi_{a,b}(f) \geq \psi(f), \forall f \). That is \( \psi_{a,b} \leq \psi \). Thus all infra \( L \)-closure operators are of the form \( \psi_{a,b} \) for \( a, b \in X \) such that \( a \neq b \).

**Remark 9.3.2.** When \( L = I \) there is no infra \( L \)-closure operator.
Definition 9.3.3. An $L$-topology $F$ on $X$ is an ultra $L$-topology if the only $L$-topology on $X$ strictly finer than $F$ is the discrete $L$-topology.

Let $X$ be a non empty set and $L$ is a finite pseudo complemented chain. If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_\lambda)) = \{f | f(a) = 0\} \cup \{f | f \geq b_\lambda\}$, then a principal ultra $L$-topology $\mathcal{G}(a, \mathcal{U}(b_\lambda), a_\beta) = \mathcal{G}(a_\beta)$, which is the simple extension of $\mathcal{G}$ by $a_\beta$, i.e., $\mathcal{G}(a_\beta) = \{f \lor (g \land a_\beta), f, g \in \mathcal{G}, a_\beta \notin \mathcal{G}\}$, where $a, b \in X$, $\lambda$ and $\beta$ are the atom and dual atom in $L$ respectively.

Let $X$ be a nonempty set and $L$ is a finite Boolean lattice. If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_\lambda)) = \{f | f(a) = 0\} \cup \mathcal{U}$ where $\mathcal{U}$ is a non principal ultra $L$-filter not containing $a_\lambda, 0 \neq \lambda \in L$. Then the non principal ultra $L$-topology $\mathcal{G}(a, \mathcal{U}, a_\beta) = \mathcal{G}(a_\beta)$, is the simple extension of $\mathcal{G}$ by $a_\beta$, where $a \in X$, $\beta$ is the dual atom in $L$.

Let $X$ be an infinite set and $L$ is a finite pseudo complemented chain. If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}), a \in X$, then a non principal ultra $L$-topology $\mathcal{G}_{\beta j} = L$-topology generated by any $(m - 1) \mathcal{G}(a_{\beta i})$ among $m \mathcal{G}(a_{\beta i}), i = 1, 2, \ldots, m, j = 1, 2, \ldots, m, i \neq j$ if there are $m$ dual atoms $\beta_1, \beta_2, \ldots, \beta_m$, where $\mathcal{G}(a_{\beta i})$ is simple extension of $\mathcal{G}$ by $a_{\beta i}$.

If $X$ is a non empty set and $L$ is a diamond lattice $\{0, \alpha, \beta, 1\}$ then the
Theorem 9.3.4. Let $X$ be a non-empty set and $L$ be a diamond lattice $\{0, \alpha, \beta, 1\}$. Then an $L$-closure operator on $X$ is an ultra $L$-closure operator if and only if it is the $L$-closure operator associated with some ultra $L$-topology on $X$.

Proof. Let $\mathcal{G}(a, \mathcal{U}, a_\beta)$ be an ultra $L$-topology on $X$ and $\psi$ be the associated $L$-closure operator. Let $\psi'$ be an $L$-closure operator on $X$ strictly larger than $\psi$. Then there exists an $L$ subset $f$ of $X$ such that $\psi'(f) < \psi(f)$. But $\psi'(f) \neq \psi(f)$. Then $\psi(f) = f \lor a_\alpha$ and $\psi'(f) = f$, which means that complement of $f$ is open in $(X, \psi')$ and not open in $(X, \psi)$. Also every open set in $(X, \psi)$ is open in $(X, \psi')$. Thus the associated $L$-topology of $\psi'$ is strictly larger than the ultra $L$-topology and hence is discrete. Thus $\psi' = D$. Hence the $L$-closure operator associated with an ultra $L$-topology is an ultra $L$-closure operator.

Next to prove that every ultra $L$-closure operator is the $L$-closure operator associated with an ultra $L$-topology.

Let $\psi$ be an $L$-closure operator on $X$ other than $D$. It suffices to prove...
that there exists an \(L\)-closure operator associated with an ultra \(L\)-topology larger than \(\psi\). Since \(\psi \neq D\) there exists an element \(a\) of \(X\) such that \(a_{\alpha}\) is not open in \((X, \psi)\). Now consider \(\mathcal{G} = \{f|f(a) = 0\} \cup \mathcal{U}\) where \(\mathcal{U}\) is an ultra \(L\)-filter not containing \(a_{\lambda}, 0 \neq \lambda \in L\). Then \(a_{\alpha}\) is not an element of \(\mathcal{G}\). Now consider the ultra \(L\)-topology \(\mathcal{G}(a, \mathcal{U}, a_{\lambda}) = \) simple extension of \(\mathcal{G}\) by\(a_{\alpha}\). Let \(\psi'\) be the \(L\)-closure operator associated with it. Then \(\psi \leq \psi'\). Otherwise if \(\psi' \leq \psi\), then every open set in \(\psi'\) is open in \(\psi\). But \(a_{\alpha}\) is open in \(\psi'\). So it must be open in \(\psi\), which is a contradiction. \(\square\)

**Remark 9.3.3.** In a similar way we can prove the above theorem when \(L\) is a finite pseudo complemented chain or other Boolean lattices.

**Definition 9.3.4.** Let \(x \in X, \lambda \in L\). An \(L\) point \(x_\lambda\) is defined by
\[
x_\lambda(y) = \begin{cases} 
\lambda & \text{if } y = x \\
0 & \text{if } y \neq x 
\end{cases} \quad \text{where } 0 < \lambda \leq 1
\]

**Definition 9.3.5.** An \(L\)-closure operator \(\psi\) on \(X\) is \(T_1\) if every \(L\) point is closed. That is \(\psi(x_\lambda) = x_\lambda, \forall x \in X, \lambda \in L\).

**Definition 9.3.6.** [62] Let \(\psi_1 = \{f|\psi(f) = f\}\). A fuzzy closure space \((X, \psi)\) is called quasi-separated if and only if for any two fuzzy points \(x_\lambda\) and \(y_\gamma\) with \(x_\lambda \in C(y_\gamma)\), there exist \(f, g \in \psi_1\) such that \(x_\lambda \in f \leq C(y_\gamma)\) and \(y_\gamma \in g \leq C(x_\lambda)\).

**Theorem 9.3.5.** [62]

A fuzzy closure space is quasi-separated if and only if every fuzzy point in \(X\) is Čech-fuzzy closed.

**Proposition 9.3.1.**

Let \(\psi_1 = \{f \in L^X|\psi(f) = f\}\). An \(L\)-closure space \((X, \psi)\) is said to be \(T_1\)
if for every pair of distinct $L$ points $x_{\lambda}$ and $y_{\gamma}$, there exist $f, g \in \psi_1$ such that $x_{\lambda} \in f \leq C(y_{\gamma})$ and $y_{\gamma} \in g \leq C(x_{\lambda})$.

**Proof.** Necessary part
Suppose that the $L$-closure operator $\psi$ is $T_1$. Then by definition $\psi(x_{\lambda}) = x_{\lambda}$ Then by theorem 9.3.5 the $L$-closure space $(X, \psi)$ is quasi separated. Hence for every pair of distinct $L$ points $x_{\lambda}$ and $y_{\gamma}$, there exist $f, g \in \psi_1$ such that $x_{\lambda} \in f \leq C(y_{\gamma})$ and $y_{\gamma} \in g \leq C(x_{\lambda})$.

Sufficient part
Suppose that for every pair of distinct $L$ points $x_{\lambda}$ and $y_{\gamma}$, there exist $f, g \in \psi_1$ such that $x_{\lambda} \in f \leq C(y_{\gamma})$ and $y_{\gamma} \in g \leq C(x_{\lambda})$. Then by definition $(X, \psi)$ is quasi separated. Then by theorem 9.3.5, $(X, \psi)$ is a $T_1$ $L$-closure space. \qed

**Proposition 9.3.2.** [62]
An $L$-closure space $(X, \psi)$ is $T_1$ if and only if the associated $L$ topological space $(X, F)$ is $T_1$

**Theorem 9.3.6.** Infra $L$-closure operators are less than or equal to any non principal ultra $L$-closure operator.

**Proof.** Let $\psi_{a,b}$ be an infra $L$-closure operator and $\psi$ be a non principal ultra $L$-closure operator. Since $\psi_{a,b}(f) = 1$ for all $f$ in $L^X$ other than $0$ and $a_\alpha$, it is enough to show that $\psi(a_\alpha) < \psi_{a,b}(a_\alpha) = g_{a,b}$. Since all non principal ultra $L$-topologies are $T_1$, the corresponding $L$-closure operators are $T_1$ by the above proposition. Hence by the definition $\psi(a_\alpha) = a_\alpha$. That is $a_\alpha < g_{a,b} \Rightarrow \psi(a_\alpha) < \psi_{a,b}(a_\alpha)$
That is $\psi(f) \leq \psi_{a,b}(f) \forall f \Rightarrow \psi_{a,b} \leq \psi$. \qed
Theorem 9.3.7. No non principal ultra $L$-closure operator has a complement.

Proof. Assume the contrary. Let $\psi$ be a non principal ultra $L$-closure operator with a complement $\psi'$ in the lattice $LC(X)$. Since $\psi'$ is not indiscrete there exists an infra $L$-closure operator $\psi_{a,b} \leq \psi'$ by the proof of the theorem 9.3.3. But $\psi_{a,b} \leq \psi$ by theorem 9.3.6. This contradicts the fact that $\psi$ and $\psi'$ are complements in the lattice $LC(X)$ and hence the proof of the theorem. □

Remark 9.3.4. The lattice of $L$-closure operators is not complemented in general.

If $L$ is a diamond lattice, the principal ultra $L$-closure operator associated with the principal ultra $L$-topology $\mathcal{G}(a, \mathcal{U}(b_\beta), a_\alpha)$ is given by

$$\phi_{a,b}(f) = \begin{cases} f & \text{if } f = 0 \text{ or } a_\alpha \leq f \text{ or } cf \in \mathcal{U}(b_\beta) \\ f \lor a_\alpha & \text{otherwise} \end{cases}$$

Theorem 9.3.8. An infra $L$-closure operator $\psi_{a,b}$ and $\phi_{b,a}$ are in comparable if $L$ is a diamond lattice.

Proof. We have $\psi_{a,b}(a_\alpha) = g_{a,b}$ and $\phi_{b,a}(a_\alpha) = a_\alpha \lor b_\beta$

Since $\alpha$ and $\beta$ are not comparable, $\psi_{a,b}$ and $\phi_{b,a}$ are not comparable. □

Remark 9.3.5. In a similar way, we can discuss the above theorem if $L$ is a finie pseudo complemented chain or other Boolean lattices.

Definition 9.3.7. An $L$-closure space $(X, \psi)$ is said to be $T_0$ if for
all \( x, y \in X, x \neq y, \exists \) a closed \( L \)-subset \( f \) such that \( f(x) \neq f(y) \).

**Example 9.3.1.** Let \( X = \{a, b, c\} \) and \( L = \{0, \alpha, \beta, 1\} \), a diamond lattice. Consider the \( L \)-topology

\[
\phi = \{0, a, b, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{a, b\}}, 1, \}
\]

\[
f : a \to \alpha \quad g : a \to \alpha \quad h : a \to 1
\]

\[
b \to \beta \quad b \to 1 \quad b \to \beta
\]

\[
c \to 0 \quad c \to 0 \quad c \to 0,
\]

Define \( c : L^X \to L^X \) by \( c(f) = \wedge\{g \in \phi : g \geq f\} \), for all \( f \in L^X \). Then \((X, c)\) is a \( T_0 \) \( L \)-closure space.

**Definition 9.3.8.** An \( L \)-closure space \((X, \psi)\) is said to be \( T_1 \), if every \( L \) point \( x_\lambda \) is closed.

**Example 9.3.2.** Let \( X = \{a, b, c\} \) and \( L = \{0, \alpha, \beta, 1\} \). Consider the discrete \( L \)-topology \( L^X \). Define \( c : L^X \to L^X \) by \( c(f) = \wedge\{g \in L^X : g \geq f\} \) for all \( f \in L^X \). Then \((X, c)\) is a \( T_1 \) \( L \)-closure space.

**Remark 9.3.6.** Every \( T_1 \) \( L \)-closure space is \( T_0 \). But the converse need not be true.

**Example 9.3.3.** Let \( X = \{a, b\} \) and \( L = \{0, \alpha, \beta, 1\} \), a diamond lattice. Consider the \( L \)-topology

\[
\phi = \{0, a, \mu_{\{a\}}, 1\}.
\]

Define \( c : L^X \to L^X \) by \( c(f) = \wedge\{g \in \phi : g \geq f\} \), for all \( f \in L^X \). Then \((X, c)\) is a \( T_0 \) \( L \)-closure space. But it is not a \( T_1 \) \( L \)-closure space, since \( b_\beta \) is not closed in \((X, c)\).
Concluding remarks and suggestions for further study

We have identified the principal and non principal ultra $L$-topologies and determined the number of ultra $L$-topologies on an arbitrary set. Also we have analyzed the lattice structure of some sublattices of Lattice of $L$-topologies. However it is not yet analyzed in detail that under what condition on the $F$-lattice $L$, the above lattices are dually atomic.