Chapter 8

Lattice of Stratified Principal $L$-topologies

8.1 Introduction

In this chapter we investigate the lattice structure of the set of all stratified principal $L$-topologies on a given set $X$. In [11], Birkhoff described a technique of comparison of topologies and noted that the set of all topologies on a fixed set forms a complete lattice with the natural order of set inclusion. In [24], Aygün, Warner and Kundri introduced a new class of functions from a topological space $(X, \tau)$ to a fuzzy lattice $L$ with its Scott topology called Scott continuous functions as a generalization of lower semi continuous functions from $(X, \tau)$ to $[0, 1]$. It is known [30] that

* Some results of this chapter are included in a paper accepted for publication in International journal of Fuzzy Information and Engineering, Springer.
the lattice of \( L \)-topologies on a given set \( X \) is complete and atomic. In [32], Jose and Johnson studied the lattice structure of the set \( L(X) \) of all stratified \( L \)-topologies on a given set \( X \). A related problem is to find sub-families in \( L(X) \) having certain properties. The collection of all stratified principal \( L \)-topologies \( S_P(X) \) forms a lattice with the natural order of set inclusion. The concept of principal topology in the crisp case was studied by Steiner [58]. The lattice of principal topologies is both atomic and dually atomic. Analogously, we study the lattice structure of the set of all stratified principal \( L \)-topologies \( S_P(X) \) on a given set \( X \). This lattice has atoms if and only if the membership lattice \( L \) has atoms. If the lattice \( S_P(X) \) has dual atoms, then \( L \) has dual atoms and atoms. Also if \( L \) is a finite pseudocomplemented chain or a Boolean lattice, then \( S_P(X) \) has dual atoms. It is also complete and join-complemented.

8.2 Preliminaries

Let \( X \) be a non empty set and \( L \) be a completely distributive lattice with an order reversing involution called \( F \)-lattice [34]. We denote the constant function in \( L^X \) taking the value \( \alpha \in L \) by \( \underline{\alpha} \). Here we call \( L \)-fuzzy subsets as \( L \)-subsets and a subset \( F \) of \( L^X \) is called an \( L \)-topology in the sense of Chang [13] and Goguen [23] as in [34], if

i. \( \underline{0}, \underline{1} \in F \)

ii. \( f, g \in F \Rightarrow f \land g \in F \)

iii. \( f_i \in F \) for each \( i \in I \Rightarrow \bigvee_{i \in I} f_i \in F \).
A subset $F$ of $L^X$ is called a stratified $L$-topology, if

i. $\alpha \in F$ for all $\alpha \in L$

ii. $f, g \in F \Rightarrow f \land g \in F$

iii. $f_i \in F$ for each $i \in I \Rightarrow \bigvee_{i \in I} f_i \in F$.

(The idea goes up to Lowen [35], while the term “stratified” has appeared for the first time in [42]). Steiner [58] proved that a topology $\tau$ is a principal topology if and only if arbitrary intersections of open sets are open(such kind of spaces are also called Alexandroff spaces [1]). Analogously, we define principal $L$-topology.

**Definition 8.2.1.** An $L$-topology is called principal $L$-topology provided that arbitrary intersections of open $L$-subsets are open $L$-subsets.

**Example 8.2.1.** Let $X$ be an infinite set. Then $F = \{f \in L^X : x \leq f\}$ together with 0, where $x \in X$ and $\alpha$ is an atom in $L$, is a stratified principal $L$-topology.

**Example 8.2.2.** Let $X = R$ and $F = \{f \in L^X : f(x) > 0$ for all but finite number of points of $X\}$ together with 0. Then $F$ is a stratified $L$-topology, which is not a principal $L$-topology.

**Definition 8.2.2.** A principal $L$-topology is called stratified principal $L$-topology provided that it contains every constant $L$-subset.

**Definition 8.2.3.** An element $p$ of $L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \land b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of $L$ will be denoted $Pr(L)$.

**Definition 8.2.4.** [73] The Scott topology on $L$ is the topology $S$, generated by the sets of the form $\{t \in L : t \nleq p \text{ where } p \in Pr(L)\}$. Let
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$(X, \tau)$ be a topological space and let $L$ be a fuzzy lattice. $f : (X, \tau) \to L$ is said to be Scott continuous if $f : (X, \tau) \to (L, S)$ is continuous, i.e., if for every $p \in Pr(L), f^{-1}\{t \in L : t \not\succeq p\} \in \tau$.

Remark 8.2.1. When $L = [0, 1]$, the Scott topology coincides with the topology of topologically generated spaces of Lowen [35]. The set $\omega_L(\tau) = \{f \in LX | f : (X, \tau) \to L$ is Scott continuous $\}$ is a stratified $L$-topology. If $\tau$ is a principal topology, then $\omega_L(\tau)$ is a stratified principal $L$-topology, which is denoted $\omega_{PL}(\tau)$. A stratified principal $L$-topology $F$ on $X$ is called induced provided that there exists a principal topology $\tau$ on $X$ such that $F = \omega_{PL}(\tau)$.

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Let $S_P(X) = \{F | F$ is a stratified principal $L$-topology on $X\}$ and $\Pi$ is the lattice of principal topologies on $X$. The family $S_P(X)$ of all stratified principal $L$-topologies forms a lattice under the natural order of set inclusion. The smallest stratified $L$-topology is the indiscrete $L$-topology, with all constant $L$-subsets, is denoted 0 and the largest stratified principal $L$-topology is the discrete $L$-topology, consisting of all $L$-subsets and is denoted 1.

Definition 8.3.1. [10] A lattice $L$ is said to be join-complemented provided that for every $x$ in $L$, there exists $y$ in $L$ such that $x \lor y = 1$.

Definition 8.3.2. [10] A lattice $L$ is said to be meet-complemented provided that for every $x$ in $L$, there exists $y$ in $L$ such that $x \land y = 0$. 
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**Definition 8.3.3.** [10] A lattice $L$ is said to be complemented provided that for every $x$ in $L$, there exists $y$ in $L$ such that $x \land y = 0$ and $x \lor y = 1$.

**Definition 8.3.4.** [10] A lattice $L$ is said to be semi-complemented provided that it is either join-complemented or meet-complemented.

**Theorem 8.3.1.** [18] The Ultra spaces on a set $E$ are exactly the topologies of the form $\mathcal{G}(x, \mathcal{U}) = \varphi(E - x) \cup \mathcal{U}$, where $x \in E$, $\mathcal{U}$ is an ultrafilter on $E$ not containing $\{x\}$.

**Theorem 8.3.2.** [58] The lattice of topologies $\Sigma$ on a set $E$ is distributive if $E$ has fewer than three elements. If $E$ has three or more elements, $\Sigma$ is not even modular.

**Theorem 8.3.3.** [58] The lattice $\Pi$ of principal topologies is a complemented lattice.

**Theorem 8.3.4.** The lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ is complete.

**Proof.** Let $K$ be any subset of $S_P(X)$. Then $K$ has the greatest lower bound and the least upper bound, since arbitrary intersections of stratified principal $L$-topologies are stratified principal $L$-topologies and $S_P(X)$ has the greatest element $1$. □

**Theorem 8.3.5.** The collection $S'_p(X)$ of all induced stratified principal $L$-topologies on any set $X$ is a complete sublattice of the complete lattice $S_P(X)$.

**Proof.** Clearly $S'_p(X)$ is a subset of $S_P(X)$. Let $F, G \in S'_p(X)$. Then
there exists topologies \( \tau \) and \( \tau' \) in \( \Pi \) such that \( F = \omega_{PL}(\tau) \) and \( G = \omega_{PL}(\tau') \). Then \( F \lor G = \omega_{PL}(\tau \lor \tau') \) and \( F \land G = \omega_{PL}(\tau \land \tau') \). Hence \( F \lor G \) and \( F \land G \) are in \( S'_p(X) \) so that \( S'_p(X) \) is a sublattice of \( S_p(X) \).

Let \( H \) be any subset of \( S'_p(X) \). Then \( H \) has the greatest lower bound since arbitrary intersections of principal topologies are principal topologies so that arbitrary intersections of induced stratified principal \( L \)-topologies are induced stratified principal \( L \)-topologies.

Let \( K \) be the set of upper bounds of \( H \). Then \( K \) is nonempty, since \( 1 \in K \). Using the above argument, \( K \) has the greatest lower bound, say \( M \). Then this \( M \) is the least upper bound of \( H \). Thus every subset \( H \) of \( SP'(X) \) has the greatest lower bound and least upper bound. Hence \( S'_p(X) \) is a complete sublattice of \( S_p(X) \).

**Proposition 8.3.1** [72]
For a stratified \( L \)-topology \((X, \omega_L(\tau))\), the family \( \beta = \{f^A_\alpha|A \in \tau, \alpha \in L\} \)
where \( f^A_\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \)
is a base for \( \omega_L(\tau) \).

**Proposition 8.3.2.** [72]
For a stratified \( L \)-topology \((X, \omega_L(\tau))\), the family \( S = \{\mu_A|\mu_A \text{ is the characteristic function of the open set } A \text{ in } \tau\} \cup \{\Omega|\alpha \in L\} \) is a subbase for \( \omega_L(\tau) \).

**Theorem 8.3.6.** The collection \( S'_p(X) \) of all induced stratified principal \( L \)-topologies on any set \( X \) forms a lattice isomorphic to \( \Pi \).

**Proof.** Let \( X \) be a nonempty set and \( L \) be an \( F \)-lattice with its Scott
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Let $\tau_1$ and $\tau_2$ are two principal topologies on $X$

$\tau_1 \lor \tau_2 = $ principal topology generated by $\tau_1$ and $\tau_2$

$$\omega_{PL}(\tau_1) = \{ f | f \text{ is a Scott continuous function from } (X, \tau_1) \to L \}$$

$$= L - \text{topology generated by } \{ \mu_A | A \in \tau_1 \} \cup \{ \alpha | \alpha \in L \}$$

$$\omega_{PL}(\tau_2) = \{ f | f \text{ is a Scott continuous function from } (X, \tau_2) \to L \}$$

$$= L - \text{topology generated by } \{ \mu_A | A \in \tau_2 \} \cup \{ \alpha | \alpha \in L \}$$

$$\omega_{PL}(\tau_1 \lor \tau_2) = \{ f | f \text{ is a Scott continuous function from } (X, (\tau_1 \lor \tau_2)) \to L \}$$

$$= L - \text{topology generated by } \{ \mu_A | A \in (\tau_1 \lor \tau_2) \} \cup \{ \alpha | \alpha \in L \}$$

$$\omega_{PL}(\tau_1) \lor \omega_{PL}(\tau_2) = \text{stratified principal } L - \text{topology generated by }$$

$$\omega_{PL}(\tau_1) \text{ and } \omega_{PL}(\tau_2)$$

$$= L - \text{topology generated by } \{ \mu_A | A \in \tau_1 \} \cup \{ \mu_A | A \in \tau_2 \} \cup \{ \alpha | \alpha \in L \}$$

$$= L - \text{topology generated by } \{ \mu_A | A \in (\tau_1 \lor \tau_2) \} \cup \{ \alpha | \alpha \in L \}$$

$$= \{ f | f \text{ is a Scott continuous function from } (X, (\tau_1 \lor \tau_2)) \to L \}$$

$$= \omega_{PL}(\tau_1 \lor \tau_2)$$

Hence $\theta(\tau_1 \lor \tau_2) = \theta(\tau_1) \lor \theta(\tau_2)$
Similarly

\[ \omega_{PL}(\tau_1 \land \tau_2) = \{ f \mid f \text{ is a Scott continuous function from } (X, \tau_1 \land \tau_2) \to L \} \]

\[ = L - \text{topology generated by } \{ \mu_A \mid A \in \tau_1 \land \tau_2 \} \cup \{ \alpha \mid \alpha \in L \} \]

\[ = L - \text{topology generated by } \{ \mu_A \mid A \in \tau_1 \} \cup \{ \alpha \mid \alpha \in L \} \land \\
L - \text{topology generated by } \{ \mu_A \mid A \in \tau_2 \} \cup \{ \alpha \mid \alpha \in L \} \]

\[ = \{ f \mid f \text{ is a Scott continuous function from } (X, \tau_1) \to L \} \land \\
\{ f \mid f \text{ is a Scott continuous function from } (X, \tau_2) \to L \} \]

\[ = \omega_{PL}(\tau_1) \land \omega_{PL}(\tau_2) \]

Hence \( \theta(\tau_1 \land \tau_2) = \theta(\tau_1) \land \theta(\tau_2) \)

\( \tau_1 \neq \tau_2 \Rightarrow \{ f \mid f : (X, \tau_1) \to (L, S) \text{ is Scott continuous} \} \neq \{ f \mid f : (X, \tau_2) \to (L, S) \text{ is Scott continuous} \} \)

\[ \Rightarrow \omega_{PL}(\tau_1) \neq \omega_{PL}(\tau_2) \]

\[ \Rightarrow \theta(\tau_1) \neq \theta(\tau_2) \]

Hence \( \theta \) is one-one. Corresponding to an induced stratified principal \( L \)-
topology \( \omega_{PL}(\tau) \) in \( S'_p(X) \), there is a topology \( \tau \) in \( \Pi \) such that \( \theta(\tau) = \omega_{PL}(\tau) \). Hence \( \theta \) is on to. So \( \theta \) is an isomorphism.

**Remark 8.3.1.** Since \( S'_p(X) \) is isomorphic to \( \Pi \), \( S'_p(X) \) possesses all the properties of \( \Pi \). That is \( S'_p(X) \) is complete, atomic, dually atomic, complemented and not modular since \( \Pi \) has these properties [58].

**Theorem 8.3.7.** The lattice of stratified principal \( L \)-topologies \( S_p(X) \)
on a set \( X \) is not modular.
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Proof. Lattice of principal $L$-topologies $\Pi$ is isomorphic to $S'_P(X)$ and $\Pi$ is not modular [58]. So $S'_P(X)$ is not modular. Since $S'_P(X)$ is a complete sublattice of $S_P(X)$, $S_P(X)$ is not modular. \hfill \Box

Theorem 8.3.8. If $L$ has atoms, then the lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ has atoms.

Proof. Let $\alpha$ be an atom in $L$ and let $A$ be a proper subset of $X$. The stratified principal $L$-topology of the form $F_\alpha^A$, where $F_\alpha^A$ is generated by $0 \cup f_\alpha^A$, where $0$ is the zero element of $S_P(X)$ and $f_\alpha^A(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ for each atom $\alpha$ in $L$, is an atom in $S_P(X)$. \hfill \Box

Theorem 8.3.9. [62] Let $(X,F)$ and $(X,G)$ be two fuzzy topological spaces on $X$. Then $G$ covers $F$ if and only if $G = F(\langle g \rangle)$ for every $g \in G - F$, where $F(\langle g \rangle)$ is the simple extension of $F$ by $g$.

Theorem 8.3.10. If the lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ has atoms, then $L$ has atoms.

Proof. Assume $L$ has more than two elements. Let $F$ be an atom in $S_P(X)$. Since $F$ is an atom, $F$ is a cover of $0$ (zero element of $S_P(X)$). So by theorem 8.3.9 there exists an element $g$ in $F - 0$ such that $F = 0(\langle g \rangle)$, the simple extension of $0$ by $g$, i.e $0(\langle g \rangle) = \{ h \lor (k \land g) | h, k \in 0, g \notin 0 \}$. This $g$ must be of the form $f_\alpha^A$, where $A \subset X, \alpha$ is an atom in $L$. Otherwise we can find a stratified principal $L$-topology $G$ smaller than $F$ and greater than $0$, which is a contradiction to the hypothesis. \hfill \Box

Combining theorem 8.3.8 and theorem 8.3.10, we get the following
Theorem 8.3.11. The lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ has atoms if and only if $L$ has atoms.

Remark 8.3.2. Atoms in $S_P(X) = 0(f^a)$, where $0 = \{\lambda | \lambda \in L\}$. Atoms in $S'_P(X) = \omega_{PL}(\tau)$, where $\tau$ is an atom in $\Pi$, lattice of principal topologies. Atoms in $S'_P(X)$ and $S_P(X)$ are different. Atoms in $S'_P(X)$ is independent of atoms in $L$. But $S_P(X)$ has atoms if and only if $L$ has atoms.

Theorem 8.3.12. The lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ is not atomic in general.

Proof. Follows from theorem 8.3.11. \qed

Theorem 8.3.13. If the lattice of principal $L$-topologies $S_P(X)$ on a set $X$ has dual atoms, then $L$ has dual atoms and atoms.

Proof. Case 1.

Let $X$ be a non empty set and $L$ be a finite pseudo complemented chain.

If $\mathcal{S} = \mathcal{S}(a, \mathcal{U}(b_\lambda)) = \{f | f(a) = 0\} \cup \{f | f \geq b_\lambda\}$, then the principal ultra $L$-topology $\mathcal{S}(a, \mathcal{U}(b_\lambda), a_\beta) = \mathcal{S}(a_\beta)$ is the simple extension of $\mathcal{S}$ by $a_\beta$, i.e., $\mathcal{S}(a_\beta) = \{f \lor (g \land a_\beta) | f, g \in \mathcal{S}, a_\beta \notin \mathcal{S}\}$, where $a, b \in X, \lambda$ and $\beta$ are the atom and dual atom in $L$ respectively (from chapter 2). So $\mathcal{S}(a_\beta)$ is a dual atom in the lattice of principal $L$ topologies. Since the simple extension of $\mathcal{S}(a_\beta)$ by the $L$ point $a_1$ is 1(discrete $L$-topology), by theorem 8.3.9, 1 is a cover of $\mathcal{S}(a_\beta)$. 
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Suppose that $F$ is a dual atom in $S_P(X)$. Then $F$ is of the form $\mathcal{G}(a_\beta) = \mathcal{G}(a, \mathcal{U}(b_\lambda), a_\beta)$ and $\beta$ must be the dual atom and $\lambda$ must be the atom in $L$. Otherwise there exists an element $G$ greater than $F$ and less than 1. Which is a contradiction to the hypothesis.

**Case 2.**

Let $X$ be a non empty set and $L$ be a finite Boolean lattice.

If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_\lambda)) = \{ f | f(a) = 0 \} \cup \{ f | f \geq b_\lambda \}$, where $a, b \in X, \lambda$ is an atom, then a principal ultra $L$-topology $\mathcal{G}_\beta(a, \mathcal{U}b_\lambda) = \mathcal{G}_\beta = L$-topology generated by any $(m - 1)$ $\mathcal{G}(a_{\beta_i})$ among $m \mathcal{G}(a_{\beta_i}), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$ if there are $m$ dual atoms $\beta_1, \beta_2, ..., \beta_m$, where $\mathcal{G}(a_{\beta_i})$ is the simple extension of $\mathcal{G}$ by $(a_{\beta_i})$, i.e, $\mathcal{G}(a_{\beta_i}) = \{ f \lor (g \land a_{\beta_i}), f, g \in \mathcal{G}, a_{\beta_i} \notin \mathcal{G} \}$. $m$ can be assumed infinite value (from chapter 2). So $\mathcal{G}(\beta_j)$ is a dual atom in the lattice of principal $L$ topologies. Since the simple extension of $\mathcal{G}(\beta_j)$ by the $L$ point $a_{\beta_j}$ is 1 (discrete $L$-topology), by theorem 8.3.9, 1 is a cover of $\mathcal{G}_\beta$.

Suppose that $F$ is a dual atom in $\beta(X)$. Then $F$ is of the form $\mathcal{G}_\beta = \mathcal{G}_\beta(a, \mathcal{U}b_\lambda)$ and $\beta_1, \beta_2, ..., \beta_m$ must be dual atoms and $\lambda$ must be atom in $L$. Otherwise there exists an element $G$ greater than $F$ and less than 1. Which is a contradiction to the assumption that $F$ is a dual atom in $S_P(X)$.

So in either case if $S_P(X)$ has dual atoms, then $L$ has dual atoms and atoms. Hence the proof of the theorem is completed. □

**Theorem 8.3.14.** If $L$ is a finite pseudo complemented chain or a Boolean lattice, then $S_P(X)$ has dual atoms.

**Proof.** Case 1.
Let $X$ be a non empty set and $L$ be a finite pseudo complemented chain.

Since $L$ is a finite pseudo complemented chain, it has atom and dual atom. Let $\tau$ be a dual atom in the lattice of principal topologies on $X$. Then by theorem 8.3.1, $\tau$ must be of the form $\mathcal{S}(a, \mathcal{U}) = \varphi(X - a) \cup \mathcal{U}$, where $a \in X$, $\mathcal{U}$ is an ultrafilter not containing $\{a\}$. Since $\tau$ is a principal topology, $\mathcal{U}$ is a principal ultra filter so that $\tau = \mathcal{S}(a, \mathcal{U}(b)) = \varphi(X - a) \cup \mathcal{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geq b_\lambda | f : (X, \tau) \rightarrow L \text{ is a scott continuous function from}(X, \tau) \text{ to } L\}, b \in X$ and $\lambda$ is an atom. Then $\omega_{PL}(\tau)$ is a stratified principal $L$-topology and $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta$ be the dual atom in $L$ and $F = \omega_{PL}(\tau) \lor a_\beta$. Then $F$ is the ultra $L$-topology $\mathcal{S}(a_\beta)$ in $S_P(X)$ since the simple extension of $F$ by $a_1$ is the discrete $L$-topology.

**Case 2.**

Let $X$ be a non empty set and $L$ be a finite Boolean lattice.

Since $L$ is a Boolean lattice, it has atoms and dual atoms. Let $\tau$ be a dual atom in the lattice of principal topologies on $X$. Then by theorem 8.3.1, $\tau$ must be of the form $\mathcal{S}(a, \mathcal{U}) = \varphi(X - a) \cup \mathcal{U}$, where $a \in X$, $\mathcal{U}$ is an ultrafilter not containing $\{a\}$. Since $\tau$ is a principal topology, $\mathcal{U}$ is a principal ultra filter so that $\tau = \mathcal{S}(a, \mathcal{U}(b)) = \varphi(X - a) \cup \mathcal{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geq b_\lambda | f : (X, \tau) \rightarrow L \text{ is a scott continuous function from}(X, \tau) \text{ to } L\}, b \in X$ and $\lambda$ is an atom. Then $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in $L$ and $F(a_{\beta_1}) = \omega_{PL}(\tau) \lor a_{\beta_1}, F(a_{\beta_2}) = \omega_{PL}(\tau) \lor a_{\beta_2}, ..., F(a_{\beta_m}) = \omega_{PL}(\tau) \lor a_{\beta_m}$. Let $F_{\beta_j}$ is the $L$-topology generated by $(m - 1) F(a_{\beta_i})$ from $m F(a_{\beta_i}), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$. Then as in case 1, $F_{\beta_j}$ is the ultra $L$-topology $\mathcal{S}_{\beta_j}$ in $\beta(X)$ since the simple extension of $F_{\beta_j}$ by $a_{\beta_j}$ is the discrete $L$-topology.

In both cases, $S_P(X)$ has dual atoms. Hence the theorem. \qed
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**Theorem 8.3.15.** The lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ is not dually atomic in general.

*Proof.* Follows from theorem 8.3.14. \hfill \Box

8.4 Complementation problem in the lattice of stratified principal $L$-topologies

**Theorem 8.4.1.** If $F$ is any stratified principal $L$-topology on $X$ such that the topology corresponding to the characteristic functions in $F$ is neither discrete nor indiscrete, then $F$ has at least one join-complement.

*Proof.* Let $\tau$ be the principal topology corresponding to the characteristic functions in $F$. Since the lattice $\Pi$ is complemented [58], we can find $\tau'$ in $\Pi$ such that $\tau \land \tau' = 0$ and $\tau \lor \tau' = 1$ in $\Pi$. Then $F \lor \omega_{PL}(\tau') = 1$ and $F \land \omega_{PL}(\tau') \neq 0$ in $S_P(X)$. \hfill \Box

**Theorem 8.4.2.** The lattice of stratified principal $L$-topologies $S_P(X)$ on a set $X$ is semi-complemented.

*Proof.* Let $F$ be any stratified principal $L$-topology on $X$ and $\tau$ be the topology corresponding to the characteristic functions in $F$. Let $\tau'$ be a complement of $\tau$ in $\Pi$. Then $F \lor \omega_{PL}(\tau') = 1$ in $S_P(X)$. \hfill \Box

**Theorem 8.4.3.** If $F$ is an induced stratified principal $L$-topology $S_P(X)$ on $X$, then $F$ has at least one complement in $S_P(X)$. 
Proof. Since $F$ is induced, there exists a topology $\tau$ in $\Pi$ such that $\omega_{PL}(\tau) = F$. Since $\Pi$ is complemented, there exists at least one topology $\tau'$ on $\Pi$ such that $\tau \land \tau' = 0$ and $\tau \lor \tau' = 1$ in $\Pi$. Then $F \lor \omega_{PL}(\tau') = 1$ and $F \land \omega_{PL}(\tau') = 0$ in $S_{P}(X)$.

Remark 8.4.1. We have analyzed the lattice structure of the set of all stratified principal $L$-topologies on an arbitrary set $X$ and have obtained characterization for certain properties of it. This study reveals more about the interplay between $L$-topology and lattice theory. Also for a given principal topology $\tau$ on $X$, the family $F_{pr}$ of all stratified principal $L$-topologies defined by families of Scott continuous functions from $(X, \tau)$ to $L$, forms a lattice under the natural order of set inclusion. From this lattice, we can deduce properties of $S_{P}(X)$ and $S'_{P}(X)$. 